Strategies in Conditional Narrowing Modulo SMT Plus Axioms

Technical report 02/21 — November, 2021

Luis Aguirre, Narciso Martí-Oliet, Miguel Palomino, Isabel Pita
Facultad de Informática, Universidad Complutense de Madrid, Spain
`{luisagui,narciso,miguelpt,ipandreu}@ucm.es`

Abstract

This work presents a narrowing calculus that uses strategies to solve reachability problems in order-sorted conditional rewrite theories whose underlying equational logic is composed of some theories solvable via a satisfiability modulo theories (SMT) solver plus some combination of associativity, commutativity, and identity. Both the strategies and the rewrite rules are allowed to be parameterized, i.e., they may have a set of common constants that are given a value as part of the solution of a problem. A proof tree based interpretation of the strategy language is used to prove the soundness and weak completeness of the calculus.

Keywords: Narrowing, strategies, reachability, rewriting logic, SMT, unification

Research partially supported by projects TRACES (TIN2015-67522-C3-3-R), ProCode (PID2019-108528RB-C22), and by Comunidad de Madrid as part of the program S2018/TCS-4339 (BLOQUES-CM) co-funded by EIE Funds of the European Union.

Contents

1 Introduction **3**

2 Preliminaries **4**
2.1 Running example . . . 5
2.2 Order-sorted equational logic . . . 6
2.3 Order-sorted equational theories . . . 8
2.4 Unification . . . 8

3 Conditional Rewriting modulo built-ins and axioms **9**

4 Abstractions, B-extensions, and R, B-rewriting **11**
4.1 Abstractions . . . 11
4.2 B-extensions . . . 12

5 Strategies **21**
5.1 Open and closed goals, derivation rules and proof trees . . . 21
5.2 Strategies and their semantics . . . 21
5.2.1 `Idle` and `fail` . . . 22
5.2.2 Rule application . . . 22
5.2.3 Top . . . 23
5.2.4 Call strategy . . . 24
5.2.5 Tests . . . 25
5.2.6 If-then-else . . . 25
5.2.7 Regular expressions . . . 26
5.2.8 Rewriting of subterms . . . 26
5.3 Interpretation of the semantics. Generalization of strategies . . . 27

6 Reachability problems **29**

7 Strategies in reachability by conditional narrowing modulo SMT and axioms **30**
7.1 Reachability goals and calculus . . . 31
7.2 Soundness and weak completeness of the calculus . . . 36
7.3 Completeness of the calculus, for topmost rewrite theories . . . 37

8 Example **37**

9 Conclusions and related work **39**

A Appendix **44**

1 Introduction

Rewriting logic is a computational logic that was developed thirty years ago [Mes90]. The semantics of rewriting logic [BM06] has a precise mathematical meaning, allowing mathematical reasoning for proving properties, providing a flexible framework for the specification of concurrent systems; moreover, it can express both concurrent computation and logical deduction, allowing its application in many areas such as automated deduction, software and hardware specification and verification, security, et cetera [MM02, Mes12].

A system is specified in rewriting logic as a rewrite theory $\mathcal{R} = (\Sigma, E, R)$, with (Σ, E) an underlying equational theory, which in this work will be *order-sorted equational logic*, where terms are given as an algebraic data type, and R is a set of rules that specify how the deductive system can derive one term from another. *Many-sorted* and *unsorted* theories can be formulated as special cases of order-sorted (OS) theories.

Strategies allow modular separation between the rules that specify a system and the way that these rules are applied. They can be used both to implement and test different algorithms over a given specification or to drive the search of solutions to *reachability problems*.

A reachability problem can have the form $\exists\bar{x}(t(\bar{x}) \to^* t'(\bar{x}))$, with t, t' terms with variables in $\bar{x}$, or be a conjunction $\exists\bar{x} \bigwedge_i (t_i(\bar{x}) \to^* t'_i(\bar{x}))$. Reachability problems can be solved by model-checking methods for finite state spaces. When the initial term t has no variables, i.e., it is a ground term, and under certain admissibility conditions, rewriting can be used in a breadth-first way to traverse the state space, trying to find a suitable matching of $t'(\bar{x})$ in each traversed node. In the general case where $t(\bar{x})$ is not a ground term, a technique known as *narrowing* [Fay79] that was first proposed as a method for solving equational goals (*unification*), has been extended to cover also reachability goals [MT07], leaving equational goals as a special case.

Such E-unification algorithm can itself make use of narrowing at another level for finding the solution to its equational goals. Specific E-unification algorithms exist for a small number of equational theories, but if the equational theory (Σ, E) can be decomposed as $E_0 \cup B$, where B is a set of axioms having a unification algorithm, and the equations E_0 can be turned into a set of rules $\overrightarrow{E_0}$, by orienting them, such that the rewrite theory $\overrightarrow{E} = (\Sigma, B, \overrightarrow{E_0})$ is admissible in the sense of the previous paragraph, then narrowing can be used on $\overrightarrow{E}$ to solve the E-unification goals generated by performing narrowing on $\mathcal{R}$. For these equational goals the idea of *variants of a term* has been applied in recent years to narrowing. A strategy known as *folding variant narrowing* [ESM12], which computes a complete set of variants of any term, has been developed by Escobar, Sasse, and Meseguer, allowing unification modulo a set of unconditional equations and axioms. The strategy terminates on any input term on those systems enjoying the *finite variant property*, and it is optimally terminating. It is being used for cryptographic protocol analysis [MT07], with tools like Maude-NPA [EMM09], termination algorithms modulo axioms [DLM$^+$08], algorithms for checking confluence and coherence of rewrite theories modulo axioms [DM12], and infinite-state model checking [BM14]. Recent development in conditional narrowing has been made for order-sorted equational theories [CEM15] and also for rewriting with constraint solvers [RMM17].

Conditional narrowing without axioms for equational theories with an order-sorted type structure has been thoroughly studied for increasingly complex categories of term rewriting systems. A wide survey can be found in [MH94]. The literature is scarce when we allow for extra variables in conditions (e.g., [GM86], [Ham00]), conditional narrowing modulo axioms (e.g., [CEM15]), or conditional narrowing modulo a set of equations (e.g., [Boc93]).

Narrowing is a technique used to inspect complex concurrent and deductive systems. One of the weaknesses of narrowing is the state space explosion associated to any reachability problem where arithmetic equational theories are involved. *Satisfiability modulo theories* (SMT) solvers [dMB08], an extension of *Boolean satisfiability* (SAT) solvers that can handle a wide variety of equational theories, including integer and real numbers, may mitigate the aforementioned state space explosion.

This paper extends in two ways our previous work [AMPP17], where we developed a sound and weakly complete, i.e., complete with respect to idempotent normalized answers, narrowing calculus for conditional narrowing modulo $E_0 \cup B$, i.e., the underlying equational theory E of the admitted rewrite theories must be decomposable into $E = E_0 \cup B$ where E_0 is a subset of the theories handled by SMT solvers and B is a set of axioms for the algebraic data types not handled by the SMT solvers:

1. *Strategies.* In [AMPP17] we found several sources of state space explosion:
   (a) the order of application of the rules,
   (b) the application of unneeded rules, and
   (c) that checking a SMT restriction that applied to any state was only possible for candidate final states,

   that even prevented the state space of some problems from being finite. These problems can be addressed with the use of strategies

2. *Parameters.* We also found out that the scope of the calculus could be broadened if we included the support for parameters in the specifications, i.e., a subset of the variables in them, either SMT or not, to be considered as *common constants* that need to be given a value in the reachability problem, either as a prerequisite or as part of its solution, allowing, for instance, the fine tuning of a proposed specification.

We have defined a strategy language suitable for narrowing that can be used either to specify algorithms or to drive the search of solutions to reachability problems. This strategy language is a subset of the Maude strategy language [MOMV04, EMOMV07, RMPV18]. We have given a proof tree based interpretation of its semantics, and we have developed a completely new narrowing calculus that includes this strategy language and the use of parameters, both in the rewrite theories and in the strategies. Under certain requirements, the calculus is proven to be sound and weakly complete.

The work is structured as follows: Section 2 presents basic definitions and properties for order-sorted equational deduction and unification. Section 3 presents rewriting modulo built-in subtheories and axioms (R/E). In Section 4 the concepts of built-in subtheory, abstraction, B-extension, and rewrite theory closed under B-extensions are presented. Also, the relation $\rightarrow_{R,B}$ is introduced. This relation is closely related to the narrowing calculus to be developed in Section 7. Then the equivalence of R/E-rewriting and R,B-rewriting, for rewrite theories closed under B-extensions, is proved. In Section 5 the strategy language and its semantics are presented; then, an interpretation of this semantics is proved. In Section 6 we define the concept of parameterized reachability problem and its solution. In Section 7 the narrowing calculus for reachability is introduced. Then the soundness and weak completeness of the calculus are proved, as well as its completeness for some rewrite theories. Section 8 shows several examples of the use of the calculus. In Section 9, related work, conclusions, and future lines of investigation for this work are presented. The appendix holds the rest of the proofs of this work. The prototype, with the running example, can be found at `http://maude.ucm.es/cnarrowing`.

2 Preliminaries

Familiarity with term rewriting and rewriting logic [BM06] is assumed. Several definitions and results from [RMM17] are included in this section.

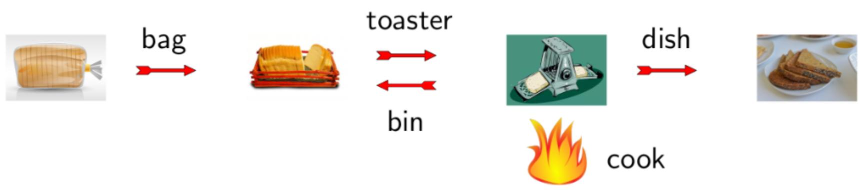

Figure 1: Running example. Toast cooking

2.1 Running example

Example 1. *Toast cooking will be used as a running example. A toast is well-cooked if both sides of the toast have been cooked for exactly* `cookTime` *(abbreviated to* `ct`*) seconds. No overcooking is allowed. Fresh toasts are taken from a toast bag, and they are cooked using a frying pan that can toast up to two toasts simultaneously, well-cooking one side of each toast in the pan. There is a bin, where fresh toasts are put when taken from the bag. A toast in the pan can be returned to the bin, being flipped in this process. Finally, there is a dish where well-cooked toasts can be output. There is a limit of* `failTime` *(*`ft`*) seconds to reach the desired final state. In this example,* `ct` *and* `ft` *will be the parameters, i.e., they are the variables that represent the common constants of the specification that must be given a value either by the conditions of the problem or by its solution.*

A `Toast` *(abbreviated to* `t`*) can be either a* `RealToast` *(*`rt`*), represented as an ordered pair of natural numbers, each one with sort* `Integer` *(*`i`*), storing the seconds that each side has already been toasted, or an* `EmptyToast` *(*`et`*) which has a constant* `zt`*, representing the absence of* `Toasts`*; a* `Pan` *(*`p`*) is an unordered pair of* `Toasts`*; a* `Kitchen` *(*`k`*) has a timer, represented by a natural number, and a* `Pan`*; a* `Bin` *(*`b`*) is a multiset of* `Toasts`*; the bag and the dish are represented by natural numbers, the number of* `RealToasts` *in each one; the* `System` *(*`s`*) has a bag, a* `Bin`*, a* `Kitchen`*, and a dish. When a* `RealToast` *is in the pan, the side being toasted is represented by the first integer of the ordered pair. We will use two auxiliary functions,* `cook` *and* `toast` *(in lowercase). The rules for* `Toast` *cooking are the following:*

1. *The function call* $\texttt{cook}(x_{\texttt{k}}, y_{\texttt{i}})$ *will return the* `Kitchen` *obtained from* `Kitchen` $x_{\texttt{k}}$ *after* $y_{\texttt{i}}$ *seconds, by calling the function* $\texttt{toast}(v_{\texttt{t}}, y_{\texttt{i}})$ *for each* `Toast` $v_{\texttt{t}}$ *in* `Kitchen` $v_{\texttt{k}}$.

2. *The function call* $\texttt{toast}(\texttt{zt}, y_{\texttt{i}})$ *will return* `zt`.

3. *The function call* $\texttt{toast}(r_{\texttt{rt}}, y_{\texttt{i}})$ *will return the* `RealToast` *obtained from* `RealToast` $r_{\texttt{rt}}$ *after toasting it for* $y_{\texttt{i}}$ *seconds, where* $y_{\texttt{i}} > 0$*, only if the side of* $r_{\texttt{rt}}$ *that is in contact with the pan gets well-cooked.*

4. *A fresh* `RealToast` *can pass from a non-empty bag to the* `Bin`.

5. *A* `RealToast` *can pass from the* `Bin` *to the* `Pan` *if there is room in the* `Pan`.

6. *A* `Kitchen` *with at least one* `RealToast` *in the* `Pan` *can* `cook` *the* `RealToasts` *that are laying on the pan any given integer number of seconds.*

7. *A* `RealToast` *in the* `Pan` *can be returned to the* `Bin`*, where it is flipped. This is the only way that a toast gets flipped.*

8. *A well cooked* `RealToast` *can be taken out to the dish.*

2.2 Order-sorted equational logic

Definition 1 (Kind completion). *A poset of* sorts $(S, \leq)$ *whose connected components are the equivalence classes corresponding to the least equivalence relation* $\equiv_{\leq}$ *containing* $\leq$ *is* kind complete *iff for each* $s \in S$ *its connected component has a top sort, denoted* $[s]$*, called the* kind *of* s*.*

Definition 2 (Order-sorted signature). *An* order-sorted (OS) signature *is a tuple* $\Sigma = (S, \leq, F)$ *where:*

- $(S, \leq)$ *is a* kind complete *poset of sorts.*
- $F = \{\Sigma_{s_1 \ldots s_n, s}\}_{(s_1 \ldots s_n, s) \in S^* \times S}$ *is an* S^*×S*-indexed family of sets of* function symbols*, where for each function symbol* f *in* $\Sigma_{s_1 \ldots s_n, s}$ *there is a function symbol* f *in* $\Sigma_{[s_1] \ldots [s_n], [s]}$*.*
- Σ *is sensible, i.e., if* f *is a function symbol in* $\Sigma_{s_1 \ldots s_n, s}$*,* f *is also a function symbol in* $\Sigma_{s'_1 \ldots s'_n, s'}$*, and* $[s_i] = [s'_i]$ *for* $i = 1, \ldots, n$ *then* $[s] = [s']$*.*

When each connected component of $(S, \leq)$ has exactly one sort, the signature is *many-sorted.*

Example 2. *In the cooking example, omitting the implied kind for each connected component of* S*,* $\Sigma = (S, \leq, F)$ *is:*
$S = \{\mathtt{Integer}, \mathtt{RealToast}, \mathtt{EmptyToast}, \mathtt{Toast}, \mathtt{Pan}, \mathtt{Kitchen}, \mathtt{Bin}, \mathtt{System}\}$,
$\leq\, = \{(\mathtt{RealToast}, \mathtt{Toast}), (\mathtt{EmptyToast}, \mathtt{Toast}), (\mathtt{Toast}, \mathtt{Bin})\}$,
$F = \{\{[\_\,,\_]\}_{\mathtt{i\,i,rt}}, \{\_\,\_\}_{\mathtt{t\,t,p}}, \{\_;\_\}_{\mathtt{b\,b,b}}, \{\_\,;\,\_\}_{\mathtt{i\,p,k}}, \{\mathtt{cook}\}_{\mathtt{k\,i,[k]}}, \{\mathtt{toast}\}_{\mathtt{t\,i,[t]}},$
$\{\_/\_/\_/\_\}_{\mathtt{i\,r\,k\,i,s}}, \{zt\}_{\mathtt{et}}\}$.
The notation used in F *has the following meaning:* $\{[\_\,,\_]\}_{\mathtt{i\,i,rt}}$ *means that there is a mix-fix function symbol* $[\_\,,\_]$ *such that if* i_1 *and* i_2 *are terms with sort* ***`Integer`*** *then* $[i_1, i_2]$ *is a term with sort* ***`RealToast`****. It is possible to use functional notation for all function symbols, but mix-fix notation will be used in order to ease the reading.*

The order $\leq$ on S is extended to S^* in the usual way: if $w = s_1 \ldots s_n$ in S^n, $w' = s'_1 \ldots s'_n$ in S^n, and $s_i \leq s'_i$ for $i = 1, \ldots, n$ then $w \leq w'$. When $f \in \Sigma_{\epsilon, s}$, ϵ being the empty word, we call f a *constant* with type s and write $f \in \Sigma_s$ instead of $f \in \Sigma_{\epsilon, s}$.

A function symbol f in $\Sigma_{s_1 \ldots s_n, s}$ is displayed as $f : s_1 \ldots s_n \to s$, its *rank* declaration. Then f is said to have *arity* n and *end type* s. *Mix-fix* notation is allowed in Σ, where the symbol _ is used to identify the position of each s_i in $s_1 \ldots s_n$. If omitted, the usual functional notation $f(s_1, \ldots, s_n)$, which is an admitted alternative notation for all functions, is assumed. An S-sorted set $\mathcal{X} = \{\mathcal{X}_s\}_{s \in S}$ of variables satisfies $s \neq s' \Rightarrow \mathcal{X}_s \cap \mathcal{X}_{s'} = \emptyset$, and the variables in $\mathcal{X}$ are disjoint from all the constants in Σ. Each variable in $\mathcal{X}$ has a subscript indicating its sort, i.e., x_s has sort s, which may be omitted when the sort of the variable is not relevant.

The sets $\mathcal{T}_{\Sigma, s}$ and $\mathcal{T}_{\Sigma}(\mathcal{X})_s$ denote, respectively, the set of ground Σ-terms with sort s and the set of Σ-terms with sort s when the variables in $\mathcal{X}$ are considered extra constants of Σ. The notations $\mathcal{T}_{\Sigma}$ and $\mathcal{T}_{\Sigma}(\mathcal{X})$ are used as a shortcut for $\bigcup_{s \in S} \mathcal{T}_{\Sigma, s}$ and $\bigcup_{s \in S} \mathcal{T}_{\Sigma}(\mathcal{X})_s$ respectively. It is assumed that Σ has non-empty sorts, i.e., $\mathcal{T}_{\Sigma, s} \neq \emptyset$ for all sorts s in S. We write *vars*(t) or V_t to denote the set of variables in a term t in $\mathcal{T}_{\Sigma}(\mathcal{X})$. This definition is extended in the usual way to any other structure, unless explicitly stated. If *vars*$(A) = \emptyset$, where A is any structure, A is said to be *ground.* A term where each variable occurs only once is said to be *linear.* For $S' \subseteq S$, a term is called S'*-linear* if no variable with sort in S' occurs in it twice.

Positions in a term t: when a term t is expressed in functional notation as $f(t_1, \ldots, t_n)$, it can be pictured as a tree with *root* f and *children* t_i at position i, for $1 \leq i \leq n$. Then the root position of t is referred as ϵ and the inner positions of t are referred as lists of nonzero natural numbers separated by dots, $i_1.i_2 \cdots i_m$, meaning the position $i_2 \cdots i_m$ of t_{i_1}, where $1 \leq i_1 \leq n$. The set of positions of a term is written *pos*(t). The set of non-variable positions of a term whose root is a function symbol in Σ is written *pos*$_{\Sigma}(t)$. The set of positions of variables from

$\mathcal{X}$ in a term is written $pos_{\mathcal{X}}(t)$. $t|_p$ is the subtree of t below position p. $t[u]_p$ is the replacement in t of the subterm at position p with a term u. $t[\,]_p$ is a *term with hole* that is equal to t except that in the position p there is a special symbol $[\,]$, the hole. As an example, if t is $f(g(a,b),c)$, then $t|_1$ is $g(a,b)$, $t|_{1.2}$ is b, $t[\,]_{1.2}$ is $f(g(a,[\,]),c)$, and $t[d]_{1.2}$ is $f(g(a,d),c)$. For any position p define $p.\epsilon = p$. For positions p and q, we write $p \leq q$ if there is a position r such that $q = p.r$, and write $p < q$ if $q = p.r$ and $r \neq \epsilon$. Trivially $p \leq p$ because $p = p.\epsilon$. $t[u_1,\ldots,u_n]_{p_1\ldots p_n}$ is the replacement in t of the subterms at the *unique* positions $p_1,\ldots,p_n$ with the terms $u_1,\ldots,u_n$, respectively, where for all $1 \leq i,j \leq n$ if $i \neq j$ then $p_i \nleq p_j$. We also write $t[\bar{u}]_{\bar{p}}$ if the *ordered lists* $\bar{u} = u_1,\ldots,u_n$ and $\bar{p} = p_1,\ldots,p_n$ are known from the context. $t[]_{\bar{p}} = t[]_{p_1}\ldots[]_{p_n}$, $t[\bar{u}[\bar{v}]_{\bar{p}}]_{\bar{q}} = t[u_1[v_1]_{p_1}]_{q_1}\ldots[u_n[v_n]_{p_n}]_{q_n}$. Given any ordered list $\bar{u}$, which may have repetitions, we call $\hat{u}$ to the set of elements of $\bar{u}$. If $\bar{p} = p_1,\ldots,p_n$ and $\hat{p} \subseteq pos(t)$ then $t|_{\bar{p}} = t|_{p_1},\ldots,t|_{p_n}$ and $t|_{\hat{p}} = \{t|_{p_1},\ldots,t|_{p_n}\}$. $vars(t[]_{\bar{p}})$ is the set of variables appearing in the term with holes $t[]_{\bar{p}}$. We also allow the use of holes and replacement in tuples, if $T = (t_1,\ldots,t_n)$ then $T|_1 = t_1$, $T[x]_1 = (x,t_2,\ldots,t_n)$, et cetera.

Definition 3 (Preregularity). *Given an order-sorted signature Σ, for each natural number n, for every function symbol f in Σ with arity n, and for every tuple $(s_1,\ldots,s_n)$ in S^n, let $S_{f,s_1\ldots,s_n}$ be the set containing all the sorts s' that appear in rank declarations in Σ of the form $f : s'_1\ldots s'_n \to s'$ such that $s_i \leq s'_i$, for $1 \leq i \leq n$. If whenever $S_{f,s_1,\ldots,s_n}$ is not empty (so a term $f(t_1,\ldots,t_n)$ where t_i has type s_i for $1 \leq i \leq n$ would be a Σ-term), it is the case that $S_{f,s_1,\ldots,s_n}$ has a least sort, then Σ is said to be* preregular.

Preregularity guarantees that every Σ-term t has a *least sort*, denoted $ls(t)$, among all the sorts that t has because of the different rank declarations that can be applied to t, which is the most accurate classification for t, i.e., for any rank declaration $f : s_1\ldots s_n \to s$ that can be applied to t it is true that $ls(t) \leq s$.

A *substitution* $\sigma : \mathcal{X} \to \mathcal{B}$, where $\mathcal{B} \subseteq \mathcal{T}_\Sigma(\mathcal{X})$, is a function that matches the identity function in all $\mathcal{X}$ except for a finite set of variables called its *domain*, $dom(\sigma)$. If $\mathcal{B} \subseteq \mathcal{T}_\Sigma$ then the substitution is *ground*. We represent the application of a substitution σ to a variable x in $\mathcal{X}$ as $x\sigma$. A substitution σ is *well-formed* if $ls(y_s\sigma) \leq s$ for each variable y_s in $dom(\sigma)$. It is assumed throughout that all substitutions are well-formed. Substitutions are written as $\sigma = \{y^1_{s_1}\mapsto t_1,\cdots,y^n_{s_n}\mapsto t_n\}$, where $dom(\sigma)$ is $\{y^1_{s_1},\ldots,y^n_{s_n}\}$ and the *range* of σ is $ran(\sigma) = \bigcup_{i=1}^n vars(t_i)$. We will write $\sigma = \{\bar{y} \mapsto \bar{t}\}$ as a shorthand if both $\bar{y}$ and $\bar{t}$ are known. We write $\sigma : \mathcal{D} \to \mathcal{B}$, where $\mathcal{D}$ is a finite set of variables, to imply that $dom(\sigma) = \mathcal{D}$. The identity substitution is displayed as *none*. A substitution σ where $dom(\sigma) = \{x^1_{s_1},\ldots,x^n_{s_n}\}$ $(n \geq 0)$, $x^i_{s_i}\sigma = y^i_{s_i} \in \mathcal{X}$, for $1 \leq i \leq n$, and $y^i_{s_i} \neq y^j_{s_j}$ for $1 \leq i < j \leq n$ is called a *renaming*, with *inverse* $\sigma^{-1} = \{y^i_{s_i} \mapsto x^i_{s_i}\}_{i=1}^n$, being *none* the trivial renaming. The restriction $\sigma_{\mathcal{V}}$ of σ to a set of variables $\mathcal{V}$ is defined as $x\sigma_{\mathcal{V}} = x\sigma$ if $x \in \mathcal{V}$ and $x\sigma_{\mathcal{V}} = x$ otherwise. The deletion $\sigma_{\setminus\mathcal{V}}$ of a set of variables $\mathcal{V}$ from σ is defined as $x\sigma_{\setminus\mathcal{V}} = x\sigma$ if $x \in dom(\sigma) \setminus \mathcal{V}$ and $x\sigma_{\setminus\mathcal{V}} = x$ otherwise. Substitutions are homomorphically extended to terms in $\mathcal{T}_\Sigma(\mathcal{X})$ and also to any other syntactic structures unless explicitly stated. The *composition* of two substitutions σ and σ' is denoted by $\sigma\sigma'$, with $x(\sigma\sigma') = (x\sigma)\sigma'$ (left associativity). Their *closed composition*, denoted by $\sigma\cdot\sigma'$, is defined as $\sigma\cdot\sigma' = (\sigma\sigma')_{\setminus ran(\sigma)}$. For a substitution σ, if $\sigma\sigma = \sigma$ we say that σ is *idempotent*. It is assumed throughout that all substitutions are idempotent, usually because $dom(\sigma) \cap ran(\sigma) = \emptyset$. For substitutions σ and σ', where $dom(\sigma) \cap dom(\sigma') = \emptyset$, we denote their union by $\sigma \cup \sigma'$. A *context* $\mathcal{C}$ is a λ-term of the form $\lambda x^1_{s_1}\cdots x^n_{s_n}.t$, with $t \in \mathcal{T}_\Sigma(\mathcal{X})$ and $\{x^1_{s_1},\ldots,x^n_{s_n}\} \subseteq vars(t)$.

A Σ-*equation* has the form $l = r$, where $l \in \mathcal{T}_\Sigma(\mathcal{X})_{s_l}$, $r \in \mathcal{T}_\Sigma(\mathcal{X})_{s_r}$, and $s_l \equiv_\leq s_r$. A *conditional* Σ-*equation* is a triple $l = r$ *if* C with $l = r$ a Σ-equation and C a conjunction of Σ-equations. We call a Σ-*equation* $l = r$: *regular* iff $vars(l) = vars(r)$; *sort-preserving* iff for each substitution σ and sort s, $l\sigma$ in $\mathcal{T}_\Sigma(\mathcal{X})_s$ implies $r\sigma$ in $\mathcal{T}_\Sigma(\mathcal{X})_s$ and vice versa; *left (or right) linear* iff l (resp. r) is linear; *linear* iff it is both left and right linear.

A set of equations E is said to be regular, or sort-preserving, or (left or right) linear, if each

$$\frac{t \in \mathcal{T}_\Sigma(\mathcal{X})}{t =_E t}\ \textsf{Reflexivity} \qquad \frac{l =_E r}{r =_E l}\ \textsf{Symmetry} \qquad \frac{l =_E t \quad t =_E r}{l =_E r}\ \textsf{Transitivity}$$

$$\frac{f \in \Sigma_{s_1\cdots s_n,s}\ \ l_i =_E r_i \ \ l_i, r_i \in \mathcal{T}_\Sigma(\mathcal{X})_{s_i}, 1 \leq i \leq n}{f(l_1,\ldots,l_n) =_E f(r_1,\ldots,r_n)}\ \textsf{Congruence}$$

$$\frac{(l = r \textit{ if } \bigwedge_{i=1}^n l_i = r_i) \in E \quad \sigma{:}\mathcal{X} \rightarrow \mathcal{T}_\Sigma(\mathcal{X}) \quad l_1\sigma =_E r_1\sigma \cdots l_n\sigma =_E r_n\sigma}{l\sigma =_E r\sigma}\ \textsf{Replacement}$$

Figure 2: Deduction rules for OS equational logic.

equation in it is so.

2.3 Order-sorted equational theories

Definition 4 (OS equational theory). *An* OS equational theory *is a pair $\mathcal{E} = (\Sigma, E)$, where Σ is an OS signature and E is a finite set of (possibly conditional) Σ-equations of the forms $l = r$ or $l = r$ if $\bigwedge_{i=1}^n\ l_i = r_i$. All the variables appearing in these Σ-equations are interpreted as universally quantified. We write $l = r$ if C as a shortcut.*

Example 3. *The OS equational theory for the toast example has $\Sigma = (S, \leq, F)$ and E is the set E_0 of equations for integer arithmetic (not displayed), together with the equations:*
$(x_{\mathtt{b}}; y_{\mathtt{b}}); z_{\mathtt{b}} = x_{\mathtt{b}}; (y_{\mathtt{b}}; z_{\mathtt{b}})$, $x_{\mathtt{b}}; y_{\mathtt{b}} = y_{\mathtt{b}}; x_{\mathtt{b}}$, $x_{\mathtt{b}}; \mathtt{zt} = x_{\mathtt{b}}$, $x_{\mathtt{t}} y_{\mathtt{t}} = y_{\mathtt{t}} x_{\mathtt{t}}$
stating that `Bin` is a multiset of `Toasts` and that the position of the `Toasts` in the `Pan` is irrelevant.

Definition 5 (Equational deduction). *Given an OS equational theory $\mathcal{E} = (\Sigma, E)$ and a Σ-equation $l = r$, $E \vdash l = r$ denotes that $l = r$ can be deduced from $\mathcal{E}$ using the rules in Figure 2 [BM06, BM12]. We write $l \leftrightarrow_E r$ iff $E \vdash l = r$ can be deduced in a single step.*

Definition 6 (Equational equivalence of substitutions). *Given two substitutions γ and δ, we write $\gamma =_E \delta$ iff (i) $dom(\gamma) = dom(\delta)$ and (ii) for each variable $x \in dom(\gamma)$, $x\gamma =_E x\delta$ and $vars(x\gamma) = vars(x\delta)$.*

An OS equational theory $\mathcal{E} = (\Sigma, E)$ has an *initial algebra* ($\mathcal{T}_{\Sigma/E}$ or $\mathcal{T}_\mathcal{E}$), whose elements are the equivalence classes $[t]_\mathcal{E}$ of ground terms in $\mathcal{T}_\Sigma$ identified by the equations in E.

We denote by $\mathcal{T}_{\Sigma/E}(\mathcal{X})$, or $\mathcal{T}_\mathcal{E}(\mathcal{X})$, the algebra whose elements are the equivalence classes of terms in $\mathcal{T}_\Sigma(\mathcal{X})$ identified by the equations in E.

The deduction rules for OS equational logic specify a sound and complete calculus, i.e., for all Σ-equations $l = r$, $E \vdash l = r$ iff $l = r$ is a logical consequence of E (written $E \vDash l = r$) [Mes97]; then we write $l =_E r$.

Proposition 1 (Instance deduction). *Let (Σ, E) be an OS equational theory. For each Σ-equation $l = r$ in Σ and each substitution σ, if $E \vdash l = r$ then $E \vdash l\sigma = r\sigma$ using the same number of deduction steps.*

Proof. Immediate by induction. □

A theory inclusion $(\Sigma, E) \subseteq (\Sigma', E')$ is called *protecting* iff the unique Σ-homomorphism $\mathcal{T}_{\Sigma/E} \longrightarrow \mathcal{T}_{\Sigma'/E'}|_\Sigma$ to the Σ-*reduct of the initial algebra* $\mathcal{T}_{\Sigma'/E'}$, i.e., the elements of $\mathcal{T}_{\Sigma'/E'}$ that consist only in function symbols from Σ, is a Σ-isomorphism, written $\mathcal{T}_{\Sigma/E} \simeq \mathcal{T}_{\Sigma'/E'}|_\Sigma$.

2.4 Unification

Given an OS equational theory (Σ, E), the *E-subsumption* preorder $\ll_E$ on $\mathcal{T}_\Sigma(\mathcal{X})$ is defined by $t \ll_E t'$ if there is a substitution σ such that $t =_E t'\sigma$. For substitutions σ, ρ and a set of variables $\mathcal{V}$ we write $\rho_\mathcal{V} \ll_E \sigma_\mathcal{V}$, and say that σ is more general than ρ with respect to $\mathcal{V}$, if there is a substitution η such that $dom(\sigma) \cap dom(\eta) = \emptyset$, $ran(\rho_\mathcal{V}) = ran((\sigma\eta)_\mathcal{V})$, and $\rho_\mathcal{V} =_E (\sigma\eta)_\mathcal{V}$.

When $\mathcal{V}$ is not specified, it is assumed that $\mathcal{V} = dom(\rho)$ and $\rho =_E \sigma{\cdot}\eta$. Then σ is said to be more general than ρ. When E is not specified, it is assumed that $E = \emptyset$.

Given an OS equational theory (Σ, E), a *system of equations* F is a conjunction $\bigwedge_{i=1}^{n} l_i = r_i$ where, for $1 \leq i \leq n$, $l_i = r_i$ is a Σ-equation. An *E-unifier* for F is a substitution σ such that $dom(\sigma) \subseteq V_{l_i,r_i}$ and $l_i\sigma =_E r_i\sigma$, for $1 \leq i \leq n$. If *none* is an *E-unifier* for F then we say that F is *trivial*. The condition in a conditional equation is a system of equations.

Definition 7 (Complete set of unifiers)**.** *For F a system of equations and $vars(F) \subseteq \mathcal{W}$, a set of substitutions $CSU_E^{\mathcal{W}}(F)$ is said to be a complete set of E-unifiers of F away from $\mathcal{W}$ iff each substitution σ in $CSU_E^{\mathcal{W}}(F)$ is an E-unifier of F, for any E-unifier ρ of F there is a substitution σ in $CSU_E^{\mathcal{W}}(F)$ such that $\rho_{\mathcal{W}} \ll_E \sigma_{\mathcal{W}}$, and for each substitution σ in $CSU_E^{\mathcal{W}}(F)$, $dom(\sigma) \subseteq vars(F)$ and $ran(\sigma) \cap \mathcal{W} = \emptyset$.*

The notation CSU_E is used when $\mathcal{W}$ is the set of all the variables that have already appeared in the current calculation, preventing the collision between new variables from the E-unifier and variables already used in the calculation. A substitution σ in $CSU_E(F)$ is always idempotent because $dom(\sigma) \cap ran(\sigma) = \emptyset$.

This notion of complete set of E-unifiers was introduced by Plotkin [Plo72]. An E-unification algorithm is *complete* if for any given system of equations it generates a complete set of E-unifiers, which may not be finite. An E-unification algorithm is said to be *finitary* and complete if it terminates after generating a finite and complete set of solutions.

3 Conditional Rewriting modulo built-ins and axioms

This section introduces the concept of signature with built-ins. Then, rewriting and rewriting modulo, both with built-ins, are defined.

Definition 8 (Signature with Built-ins [RMM17])**.** *An OS signature $\Sigma = (S, \leq, F)$ has* built-in subsignature *$\Sigma_0 = (S_0, \leq, F_0)$ iff:*

- *$\Sigma_0 \subseteq \Sigma$,*
- *Σ_0 is many-sorted,*
- *S_0 is a set of minimal elements in $(S, \leq)$, and*
- *if $f : w \to s \in F_1$, where $F_1 = F \setminus F_0$, then s is a sort not in S_0 and f has no other typing in Σ_0.*

We let $\mathcal{X}_0 = \{\mathcal{X}_s\}_{s \in S_0}$, $\mathcal{X}_1 = \mathcal{X} \setminus \mathcal{X}_0$, $S_1 = S \setminus S_0$, $\Sigma_1 = (S, \leq, F_1)$, $\mathcal{H}_\Sigma(\mathcal{X}) = \mathcal{T}_\Sigma(\mathcal{X}) \setminus \mathcal{T}_{\Sigma_0}(\mathcal{X}_0)$, and $\mathcal{H}_\Sigma = \mathcal{T}_\Sigma \setminus \mathcal{T}_{\Sigma_0}$.

If Σ has a built-in subsignature Σ_0, then the restriction of $\mathcal{T}_{\Sigma/E}$ to the terms in $\mathcal{H}_\Sigma$ is denoted by $\mathcal{H}_{\Sigma/E}$ or $\mathcal{H}_\mathcal{E}$, and the restriction of $\mathcal{T}_{\Sigma/E}(\mathcal{X})$ to the terms in $\mathcal{H}_\Sigma(\mathcal{X})$ is denoted by $\mathcal{H}_{\Sigma/E}(\mathcal{X})$ or $\mathcal{H}_\mathcal{E}(\mathcal{X})$.

Definition 9 (Rule)**.** *Given an OS signature $(\Sigma, S, \leq)$ with built-in subsignature (Σ_0, S_0), a* rule *is an expression with the form $c : l \to r$ if $\bigwedge_{i=1}^{n} l_i \to r_i \mid \phi$, written $c : l \to r$ if $\bar{l} \to \bar{r} \mid \phi$ or $c : l \to r$ if C as a shortcut, where:*

- *c is the alphanumeric* label *of the rule,*
- *l, the* head *of the rule, and r are terms in $\mathcal{H}_\Sigma(\mathcal{X})$, with $ls(l) \equiv_\leq ls(r)$,*
- *for each pair l_i, r_i, $1 \leq i \leq n$, l_i is a term in $\mathcal{H}_\Sigma(\mathcal{X}) \setminus \mathcal{X}$ and r_i is a term in $\mathcal{H}_\Sigma(\mathcal{X})$, with $ls(l_i) \equiv_\leq ls(r_i)$, and*

- $\phi \in QF(\mathcal{X}_0)$, *the set of* quantifier free *formulas made up with terms in* $\mathcal{T}_{\Sigma_0}(\mathcal{X}_0)$, *the comparison function symbols* $=$ *and* $\neq$, *and the connectives* $\vee$ *and* $\wedge$.

The symbol $\neg$ (that can be defined with respect to $=$, $\neq$, $\vee$, and $\wedge$) will also appear in this work. All the variables appearing in a rule c, $vars(c)$, are interpreted as universally quantified. Three particular cases of the general form are admitted: $c : l \to r \text{ if } \bigwedge_{i=1}^{n} l_i \to r_i$, $c : l \to r \text{ if } \phi$, and the unconditional case $c : l \to r$. We will use the label of a rule alone, as a reference of the whole rule, when there is no need to make the full rule explicit.

Definition 10 (Subterms, holes, and replacement in a formula). *We extend the use of subterms and holes to formulas. If ϕ is a formula from $QF(\mathcal{X}_0)$, i is a positive integer, p is a position, and t is a term, then $\phi|_{i.p}$ is the subterm that appears at position p in the term i of $\bar{\phi}$, the tuple formed by all terms that appear in ϕ, taken from left to right, $\phi[]_{i.p}$ consists in the replacement in $\phi|_i$ of its subterm at position p with $[]$, and $\phi[t]_{i.p}$ consists in the replacement in $\phi|_i$ of its subterm at position p with t.*

Definition 11 (B-preregularity). *Given a set of Σ-equations B, a preregular OS signature Σ is called B-*preregular *iff for each Σ-equation $u = v$ in B and substitution σ, $ls(u\sigma) = ls(v\sigma)$.*

Definition 12 (Conditional rewrite theory with built-in subtheory). *A conditional rewrite theory $\mathcal{R} = (\Sigma, E, R)$ with built-in subtheory and axioms (Σ_0, E_0) consists of:*

1. *an OS equational theory (Σ, E) where:*
   - *$\Sigma = (S, \leq, F)$ is an OS signature with built-in subsignature $\Sigma_0 = (S_0, \leq, F_0)$,*
   - *$E = E_0 \cup B$, where E_0 is the set of Σ_0-equations in E, the theory inclusion $(\Sigma_0, E_0) \subseteq (\Sigma, E)$ is protecting, B is a set of regular and linear equations, called* axioms, *each equation having only function symbols from F_1 and kinded variables,*
   - *there is a procedure that can compute $CSU_B(F)$ for any system of equations F,*
   - *Σ is B-preregular, and*
2. *a finite set of uniquely labeled alphanumerical rules R.*

Under this definition of E_0 and B, if u and v are terms in $\mathcal{T}_{\Sigma_0}$ and $u =_B v$ then $u = v$. Condition number 2 will be relaxed, but not removed, later in this work. From now on we will write "rewrite theory" as a shortcut for "conditional rewrite theory with built-in subtheory and axioms".

The transitive (resp. transitive and reflexive) closure of the relation $\to_R^1$, inductively defined below, is denoted $\to_R^+$ (resp. $\to_R^*$).

Definition 13 (R-rewriting). *Given a rewrite theory $\mathcal{R} = (\Sigma, E_0 \cup B, R)$, a term t in $\mathcal{H}_\Sigma$, a position p in $pos(t)$, a rule $c : l \to r \text{ if } \bigwedge_{i=1}^{n} l_i \to r_i \mid \phi$ in R, and a substitution $\sigma : vars(c) \to \mathcal{T}_\Sigma$, the* one-step transition *$t \to_R^1 t[r\sigma]_p$ holds iff $t = t[l\sigma]_p$, $l_i\sigma \to_R^* r_i\sigma$, for $1 \leq i \leq n$, and $E_0 \vDash \phi\sigma$. Given a rewrite theory $\mathcal{R}$, we call u reachable from t in $\to_R^1$ iff $t \to_R^* u$.*

We write $t \xrightarrow[c,p,\sigma]{}{}_R^1 t[r\sigma]_p$ when we need to make explicit the rule, position, and substitution. Any of these items can be omitted when it is irrelevant. We write $t \xrightarrow[c\sigma]{}{}_R^1 v$ to express that there exists a substitution δ such that $t \xrightarrow[c,\sigma\cdot\delta]{}{}_R^1 v$. For every rewrite step $t \to_R^1 v$ there exists a closed proof tree witnessing it, in the sense of [LMM05].

Example 4. *In the cooking example, E_0 is the theory for integer arithmetic, B is the set of axioms in Example 3, and R is the following translation of the rules for cooking, shown in Example 1, where the used abbreviations, as established before, are* `i` – `Integer`, `p` – `Pan`,

$\texttt{rt} - \texttt{RealToast}$, $\texttt{t} - \texttt{Toast}$, $\texttt{k} - \texttt{Kitchen}$, $\texttt{b} - \texttt{Bin}$, $\texttt{s} - \texttt{System}$, *and* $\texttt{ct}_\texttt{i} - \texttt{cookTime}$. *The subindex* i *will be omitted from now on, for a better readability of the examples:*
$[kitchen] : y; h_{\texttt{rt}}\, v_{\texttt{t}} \;\rightarrow\; \texttt{cook}(y; h_{\texttt{rt}}\, v_{\texttt{t}}, z)$ *if* $z > 0$
$[cook] :\; \texttt{cook}(y; h_{\texttt{rt}}\, v_{\texttt{t}}, z) \;\rightarrow\; y + z; h'_{\texttt{rt}}\, v'_{\texttt{t}}$ *if* $\texttt{toast}(h_{\texttt{rt}}, z) \;\rightarrow\; h'_{\texttt{rt}} \wedge \texttt{toast}(v_{\texttt{t}}, z) \;\rightarrow\; v'_{\texttt{t}}$
$[toast1] :\; \texttt{toast}(\texttt{zt}, z) \;\rightarrow\; \texttt{zt}$
$[toast2] :\; \texttt{toast}([a, b], z) \;\rightarrow\; [a + z, b]$ *if* $a \geq 0 \wedge a + z = \texttt{ct}$
$[bag] : n/x_{\texttt{b}}/g_{\texttt{k}}/ok \;\rightarrow\; (n-1)/[0,0]; x_{\texttt{b}}/g_{\texttt{k}}/ok$ *if* $n > 0$
$[pan] : n/h_{\texttt{rt}}; x_{\texttt{b}}/y; \texttt{zt}\, v_{\texttt{t}}/ok \;\rightarrow\; n/x_{\texttt{b}}/y; h_{\texttt{rt}}\, v_{\texttt{t}}\,/ok$
$[bin] : n/x_{\texttt{b}}/y; [a, b]\, v_{\texttt{t}}/ok \;\rightarrow\; n/[b, a]; x_{\texttt{b}}/y; \texttt{zt}\, v_{\texttt{t}}\,/ok$
$[dish] : n/x_{\texttt{b}}/y; [\texttt{ct}_\texttt{i}, \texttt{ct}_\texttt{i}]\, v_{\texttt{t}}/ok \;\rightarrow\; n/x_{\texttt{b}}/\texttt{zt}\, v_{\texttt{t}}/ok + 1$

The transitive closure of the relation $\rightarrow^1_{R/E}$, inductively defined below, is denoted $\rightarrow^+_{R/E}$. The relation $\rightarrow_{R/E}$ is defined as $\rightarrow_{R/E} = (\rightarrow^+_{R/E} \cup =_E)$.

Definition 14 (R/E-rewriting)**.** *Given a rewrite theory* $\mathcal{R} = (\Sigma, E_0 \cup B, R)$*, terms* t, v *in* $\mathcal{H}_\Sigma$*, and a rule* $c : l \rightarrow r$ *if* $\bigwedge_{i=1}^n l_i \rightarrow r_i \mid \phi$ *in* R*, where* $vars(c) \cap vars(t) = \emptyset$*, if there exist a term* u *in* $\mathcal{H}_\Sigma$*, a position* p *in* $pos_{\Sigma_1}(u)$*, and a substitution* $\sigma : vars(c) \rightarrow \mathcal{T}_\Sigma$ *such that* $t =_E u = u[l\sigma]_p$, $u[r\sigma]_p =_E v$, $l_i\sigma \rightarrow_{R/E} r_i\sigma$*, for* $1 \leq i \leq n$*, and* $E_0 \vDash \phi\sigma$ *then we say that the* one-step modulo transition $t \rightarrow^1_{R/E} v$ *holds and we write* $(t, v) \in \rightarrow^1_{R/E}$.

The position p cannot belong to $pos_{\Sigma_0}(u)$, because as l is a term in $\mathcal{H}_\Sigma(\mathcal{X})$ then $l\sigma$ is a term in $\mathcal{H}_\Sigma$, hence not in $\mathcal{T}_{\Sigma_0}$. We write $t \xrightarrow[c,u,p,\sigma]{}{}^1_{R/E} v$ when we need to make explicit the rule, matching term, position, and substitution. Any of these items can be omitted when it is irrelevant.

Rewriting modulo is *more expressive* than rewriting ($\rightarrow^1_R \subsetneq \rightarrow^1_{R/E}$): from Definitions 13 and 14 it is clear that $\rightarrow^1_R \subseteq \rightarrow^1_{R/E}$; in the next example we prove that $\rightarrow^1_{R/E} \not\subseteq \rightarrow^1_R$.

Example 5. *Let us assume a rewrite theory* $\mathcal{R} = (\Sigma, E_0 \cup B, R)$*, where* $S_0 = \{\texttt{n}\}$*,* Σ_0 *has constants* 0*,* 1*,* 2*, and a binary function symbol* $\_ + \_ : \texttt{n}\ \texttt{n} \rightarrow \texttt{n}$*;* $E_0 = \{x + y = y + x\}$*;* f *and* g *are function symbols in* Σ_1*;* $B = \{f(x, y) = f(y, x)\}$*; and the only rule in* R *is* $c : f(2 + x, 0) \rightarrow g(x)$*. Then* $f(0, 1 + 2)$ *cannot be rewritten in* R *because* $f(0, 1 + 2) \neq f(2 + x, 0)\sigma$ *for any substitution* σ*, but* $f(0, 1 + 2) \rightarrow^1_{R/E} g(1)$ *with* $\sigma = \{x \mapsto 1\}$*, because* $1 + 2 =_{E_0} 2 + 1$*, so* $f(0, 1 + 2) =_{E_0} f(0, 2 + 1) =_B f(2 + 1, 0) = f(2 + x, 0)\sigma$*.*

4 Abstractions, B-extensions, and R, B-rewriting

Although rewriting modulo is more expressive than rewriting, whether a one-step modulo transition $t \rightarrow^1_{R/E} v$ holds is undecidable, in general, since E-congruence classes can be infinite. We address the issue in this section, where two simpler relations, $\rightarrow^1_{R,B}$ and $\rightarrow_{R,B}$ [GK01] are now defined. Under several requirements, rewriting with these new relations is equivalent to rewriting modulo E, i.e., $\rightarrow^1_{R,B} = \rightarrow^1_{R/E}$ and $\rightarrow_{R,B} = \rightarrow_{R/E}$. The main difference between $\rightarrow^1_{R/E}$ and $\rightarrow^1_{R,B}$ is that while the first one uses matching modulo E, the second one uses matching modulo B, which is computable. Also the concepts of *abstraction of built-in* and B*-extension* are presented.

Most of the definitions and results presented in this section can be found in [Mes17,RMM17], or in our previous work [AMPP17]. As these definitions and results are key to the narrowing calculus shown in Section 7, they are recalled here.

4.1 Abstractions

Definition 15 (Abstraction of built-in [RMM17])**.** *If* $\Sigma \supseteq \Sigma_0$ *is a signature with built-in subsignature, then an* abstraction of built-in *is a context* $\mathcal{C} = \lambda x^1_{s_1} \cdots x^n_{s_n}.t^\circ$*, with* $n \geq 0$*, such that* $t^\circ \in \mathcal{T}_{\Sigma_1}(\mathcal{X})$ *and* $\{x^1_{s_1}, \ldots, x^n_{s_n}\} = vars(t^\circ) \cap \mathcal{X}_0$.

Lemma 1 shows that there exists an abstraction that provides a canonical decomposition of any term in $\mathcal{T}_\Sigma(\mathcal{X})$, in particular for any term in $\mathcal{H}_\Sigma(\mathcal{X})$, since $\mathcal{H}_\Sigma(\mathcal{X}) \subset \mathcal{T}_\Sigma(\mathcal{X})$.

Lemma 1 (Existence of a canonical abstraction [RMM17]). *Let Σ be a signature with built-in subsignature Σ_0. For each term t in $\mathcal{T}_\Sigma(\mathcal{X})$ there exist an abstraction of built-in $\lambda x^1_{s_1} \cdots x^n_{s_n}.t^\circ$ and a substitution $\theta^\circ : \mathcal{X}_0 \to \mathcal{T}_{\Sigma_0}(\mathcal{X}_0)$ such that (i) $t = t^\circ\theta^\circ$ and (ii) $dom(\theta^\circ) = \{x^1_{s_1}, \ldots, x^n_{s_n}\}$ are pairwise distinct and disjoint from $vars(t)$; moreover, (iii) t° can always be selected to be S_0-linear and with $\{x^1_{s_1}, \ldots, x^n_{s_n}\}$ disjoint from an arbitrarily chosen finite subset $\mathcal{Y}$ of $\mathcal{X}_0$.*

Definition 16 (Abstract function [RMM17]). *Given a term t in $\mathcal{T}_\Sigma(\mathcal{X})$ and a finite subset $\mathcal{Y}$ of $\mathcal{X}_0$, define $abstract_{\Sigma_1}(t, \mathcal{Y})$ as $\langle \lambda x^1_{s_1} \cdots x^n_{s_n}.t^\circ; \theta^\circ; \phi^\circ \rangle$ where the context $\lambda x^1_{s_1} \cdots x^n_{s_n}.t^\circ$ and the substitution θ° satisfy the properties (i)-(iii) in Lemma 1 and $\phi^\circ = \bigwedge_{i=1}^n (x^i_{s_i} = x^i_{s_i}\theta^\circ)$. If $t \in \mathcal{T}_{\Sigma_1}(\mathcal{X} \backslash \mathcal{X}_0)$ then $abstract_{\Sigma_1}(t, \mathcal{Y}) = \langle \lambda.t; none; true \rangle$. We write $abstract_{\Sigma_1}(t)$ when $\mathcal{Y}$ is the set of all the variables that have already appeared in the current calculation, so each $x^i_{s_i}$ is a* fresh *variable. For pairs of terms and pairs of lists terms we use the compact notations $abstract_{\Sigma_1}((u, v)) = \langle \lambda(\bar{x}, \bar{y}).(u^\circ, v^\circ); (\theta^\circ_u, \theta^\circ_v); (\phi^\circ_u, \phi^\circ_v) \rangle$ and $abstract_{\Sigma_1}((\bar{u}, \bar{v})) = \langle \lambda(\bar{\bar{x}}, \bar{\bar{y}}).(\bar{u}^\circ, \bar{v}^\circ); (\theta^\circ_{\bar{u}}, \theta^\circ_{\bar{v}}); (\phi^\circ_{\bar{u}}, \phi^\circ_{\bar{v}}) \rangle$, respectively.*

Definition 17 (Set of topmost Σ_0-positions [AMPP17]). *Let $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ be a rewrite theory with built-in subtheory (Σ_0, E_0), and t a term in $\mathcal{H}_\Sigma(\mathcal{X})$. The set of topmost Σ_0 positions of t, $top_{\Sigma_0}(t)$, is $top_{\Sigma_0}(t) = \{p \mid p \in pos(t) \wedge \exists i \in \mathbb{N}(p = q.i \wedge t|_q \in \mathcal{H}_\Sigma(\mathcal{X}) \wedge t|_p \in \mathcal{T}_{\Sigma_0}(\mathcal{X}_0))\}$.*

We extend the definition to lists of terms: $top_{\Sigma_0}(t_1, \ldots t_n) = \{i.p \mid 1 \le i \le n \wedge p \in top_{\Sigma_0}(t_i)\}$.

Proposition 2 (Relation between Σ-terms and abstractions [AMPP17]). *Let $\mathcal{R} = (\Sigma, E, R)$ be a rewrite theory with built-in subtheory (Σ_0, E_0), and t be a term in $\mathcal{H}_\Sigma(\mathcal{X})$, with $abstract_{\Sigma_1}(t) = \langle \lambda\bar{x}.t^\circ; \theta^\circ; \phi^\circ \rangle$. If σ is a substitution such that $E_0 \vDash \phi^\circ\sigma$, then $t^\circ\sigma =_{E_0} t\sigma$.*

Proposition 3 (Invariants of top_{Σ_0} under E_0-equality [AMPP17]). *Let $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ be a rewrite theory with built-in subtheory (Σ_0, E_0). If t and t' are two terms in $\mathcal{H}_\Sigma(\mathcal{X})$ such that $t =_{E_0} t'$ then:*

1. *$top_{\Sigma_0}(t) = top_{\Sigma_0}(t')$,*
2. *$ls(t|_q) = ls(t'|_q)$ and $t|_q =_{E_0} t'|_q$ for all positions q in $top_{\Sigma_0}(t)$,*
3. *$t|_{q'} =_{E_0} t'|_{q'}$ for all positions q' such that $t|_{q'} \in \mathcal{H}_\Sigma(\mathcal{X})$, and*
4. *if $top_{\Sigma_0}(t) = \{q_1, \ldots, q_n\}$ then $t' = t[t'|_{q_1}]_{q_1} \cdots [t'|_{q_n}]_{q_n}$.*

Proposition 4 (Relation between $abstract_{\Sigma_1}$ and top_{Σ_0} [AMPP17]). *Let $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ be a rewrite theory with built-in subtheory (Σ_0, E_0). If t is a term in $\mathcal{H}_\Sigma(\mathcal{X})$, $abstract_{\Sigma_1}(t) = \langle \lambda\bar{x}.t^\circ; \theta^\circ; \phi^\circ \rangle$, where $\bar{x} = x_1, \ldots, x_n$ and $t^\circ = t[x_1]_{q_1} \cdots [x_n]_{q_n}$, then (i) $top_{\Sigma_0}(t) = \{q_1, \ldots, q_n\}$, and (ii) for every substitution $\sigma : \hat{x} \to \mathcal{T}_{\Sigma_0}(\mathcal{X}_0)$ it holds that $top_{\Sigma_0}(t^\circ\sigma) = top_{\Sigma_0}(t)$.*

4.2 B-extensions

The concept of *B-extension*, together with its properties, has been studied in [GK01], and [Mes17]. Now, we allow for repeated labels in rules; later we will restrict this repetition. We will use subscripts or apostrophes, e.g. c_1 or c', when we need to refer to a specific rule with label c.

Definition 18 (Rewrite theory closed under B-extensions). *Let $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ be a rewrite theory, where R may have repeated labels, and let $c : l \to r$ if C be a rule in R. Assume, without loss of generality, that $vars(B) \cap vars(c) = \emptyset$. If this is not the case,* only the variables of B *will be renamed; the variables of c* will never be *renamed. We define the set of B*-extensions *of c as the set:*

$Ext_B(c) = \{c : u[l]_p \to u[r]_p$ if $C \mid u = v \in B \cup B^{-1} \wedge p \in pos_\Sigma(u) \setminus \{\epsilon\} \wedge CSU_B(l, u|_p) \neq \emptyset\}$ where, by definition, $B^{-1} = \{v = u \mid u = v \in B\}$.

All the rules in $Ext_B(c)$ have label c. Given two rules $c : l \to r \, if \, C$ and $c_1 : l' \to r' \, if \, C$ with the same *condition C, c* subsumes c_1 *iff there is a substitution δ such that: (i) $dom(\delta) \cap vars(C) = \emptyset$, (ii) $l' =_B l\delta$, and (iii) $r' =_B r\delta$.*

We say that $\mathcal{R}$ is closed under B-extensions *iff for any rule with label c in R, each rule in $Ext_B(c)$ is subsumed by one rule with label c in R.*

Meseguer [Mes17] shows an algorithm that given a rewrite theory $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ constructs a superset R that is finite and closed under B-extensions, called a *finite closure under B-extensions of $\mathcal{R}$*. It is important to remark that the rules in $Ext_B(c)$ do not rename the variables from c.

Definition 19 (Finite closure under B-extensions of a rule)**.** *Given an equational theory $(\Sigma, E_0 \cup B)$, with built-in subtheory (Σ_0, E_0), and a rule with label c, we denote by c_B the set of rules in any finite closure under B-extensions of the rewrite theory $\mathcal{R} = (\Sigma, E_0 \cup B, \{c\})$.*

Definition 20 (Associated rewrite theory closed under B-extensions)**.** *Given a rewrite theory $\mathcal{R}_1 = (\Sigma, E_0 \cup B, R)$ with no repeated rule labels, any rewrite theory $\mathcal{R}_2 = (\Sigma, E_0 \cup B, \bigcup_{c \in R} c_B)$ is called an* associated rewrite theory closed under B-extensions *of $\mathcal{R}_1$.*

Example 6. *In the toast example, R is closed under B-extensions because the subterms of the equations in B have sorts `toast`, `tray`, or `pan`, and no head of any rule in R has any of these sorts.*

Example 7. *Consider a rewrite theory $\mathcal{R}_1 = (\Sigma, E_0 \cup B, R)$ with only one sort s, $R = \{l : f(a, b) \to c\}$, where f is associative and commutative ($E_0 = \emptyset$). Then, one possible instance of l_B is $l_B = R \cup \{l : f(x_s, f(a, b)) \to f(x_s, c)\}$, because the left side of the associative rule $f(x_s, f(y_s, z_s)) = f(f(x_s f, y_s), z_s)$ has a subterm at position 2, $f(y_s, z_s)$, that matches with $f(a, b)$, so $\mathcal{R}_2 = (\Sigma, E_0 \cup B, l_B)$ is an associated rewrite theory of $\mathcal{R}_1$ closed under B-extensions.*

By definition, associated rewrite theories closed under B-extensions are allowed to have several rules with the same alphanumerical label. The only condition is that all the rules sharing a label must conform a finite closure under B-extensions of a rule. Rewriting modulo does not change if we use a rewrite theory or any of its associated rewrite theories closed under B-extensions.

Lemma 2 (Equivalence of R/E-rewriting and R_B/E-rewriting)**.** *If $\mathcal{R}_B = (\Sigma, E_0 \cup B, R_B)$ is an associated rewrite theory of $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ closed under B-extensions, then $\to^1_{R/E} = \to^1_{R_B/E}$ and $\to_{R/E} = \to_{R_B/E}$.*

Proof. Since $R \subseteq R_B$ then $\to^1_{R/E} \subseteq \to^1_{R_B/E}$ and $\to_{R/E} \subseteq \to_{R_B/E}$.

In order to prove $\to^1_{R_B/E} \subseteq \to^1_{R/E}$ and $\to_{R/E} \subseteq \to_{R_B/E}$, we will prove a stronger pair of assertions:

(i) if $t \underset{c,u}{\longrightarrow}{}^1_{R_B/E} v$, where c in R_B, then $t \underset{c,u}{\longrightarrow}{}^1_{R/E} v$ using the same number of rewrite steps, and

(ii) if $t \to_{R_B/E} v$ then $t \to_{R/E} v$ using the same number of rewrite steps.

We use induction on the number of $\to^1_{R_B/E}$ rewrite steps of the derivations, including those in the condition of the rule.

Base cases:

(i) one rewrite step: $t \xrightarrow[c,u,p,\sigma]{}{}^{1}_{R_B/E} v$ with a rule $c : \tilde{l} \to \tilde{r}$ *if* $\bigwedge_{i=1}^{n} l_i \to r_i \mid \phi$ in R_B. As there is only one rewrite step in the derivation, it must be the case that $l_i\sigma \to_{R_B/E} r_i\sigma$ in zero rewrite steps, $1 \le i \le n$. Then $l_i\sigma =_E r_i\sigma$, so $l_i\sigma \to_{R/E} r_i\sigma$ in zero rewrite steps, $1 \le i \le n$. Also, $t =_E u = u[\tilde{l}\sigma]_p$, $u[\tilde{r}\sigma]_p =_E v$, and $E_0 \vDash \phi\sigma$.

- If the rule c belongs to R then $t \xrightarrow[c,u,p,\sigma]{}{}^{1}_{R/E} v$ using the same derivation that has only one rewrite step,
- else c belongs to $c_B \backslash R$, so there is another rule $c : l \to r$ *if* $\bigwedge_{i=1}^{n} l_i \to r_i \mid \phi$ in R such that, by definition of c_B, $\tilde{l} = w[l]_{\tilde{p}}$ and $\tilde{r} = w[r]_{\tilde{p}}$, where $w = w' \in B \cup B^{-1}$ and $\tilde{p} \in pos_{\Sigma}(w) - \{\epsilon\}$.
  Now, $t =_E u = u[\tilde{l}\sigma]_p = u[w[l]_{\tilde{p}}\sigma]_p = u[w\sigma[l\sigma]_{\tilde{p}}]_p$. Then $u_{p.\tilde{p}} = l\sigma$, so $u = u[l\sigma]_{p.\tilde{p}}$. As $u[r\sigma]_{p.\tilde{p}} = u[w\sigma[r\sigma]_{\tilde{p}}]_p = u[w[r]_{\tilde{p}}\sigma]_p = u[\tilde{r}\sigma]_p =_E v$, $l_i\sigma \to_{R/E} r_i\sigma$ in zero rewrite steps, $1 \le i \le n$, and $E_0 \vDash \phi\sigma$, then $t \xrightarrow[c,u,p.\tilde{p},\sigma]{}{}^{1}_{R/E} v$ in one rewrite step.

(ii) zero rewrite steps: $t \to_{R_B/E} v$ because $t =_E v$. Then, also $t \to_{R/E} v$.

Inductive step:

(i) $t \xrightarrow[c,u,p,\sigma]{}{}^{1}_{R_B/E} v$ in $n > 1$ rewrite steps, with a rule $c : \tilde{l} \to \tilde{r}$ *if* $\bigwedge_{i=1}^{n} l_i \to r_i \mid \phi$ in R_B. Then, $l_i\sigma \to_{R_B/E} r_i\sigma$ with less than n rewrite steps, $1 \le i \le n$ so, by I.H, $l_i\sigma \to_{R/E} r_i\sigma$, $1 \le i \le n$, using the same number of rewrite steps in each derivation.

Now, using the same proof shown in the base case, we get $t \xrightarrow[c,u,p,\sigma]{}{}^{1}_{R/E} v$ if c in R, or else $t \xrightarrow[c,u,p.\tilde{p},\sigma]{}{}^{1}_{R/E} v$ using the same number of rewrite steps.

(ii) $t \to_{R_B/E} v$ in $n > 0$ rewrite steps. We distinguish two cases:

- $t \to^{1}_{R_B/E} w \to_{R_B/E} v$. If the derivation $w \to_{R_B/E} v_1$ has no rewrite steps, then $w =_E v$, so $t \to^{1}_{R_B/E} v$ and the proof in subcase (i) holds. Else, the derivations of both $t \to^{1}_{R_B/E} w$ and $w \to_{R_B/E} v$ have less than n rewrite steps so, by I.H., $t \to^{1}_{R/E} w$ and $w \to_{R/E} v$ with derivations using the same number of rewrite steps as the original ones, and then $t \to_{R/E} v$ with a derivation that uses n rewrite steps.
- $t \xrightarrow[c,u,p,\sigma]{}{}^{1}_{R_B/E} v$ in $n > 0$ rewrite steps. This case is exactly the same as the one in the subcases (i) of the base case and the inductive step, so the same proofs hold.

□

Our definition of the relation $\to^{1}_{R,B}$ will require the use of a single representative for all the instances of each E_0-equivalence class that may appear in the top_{Σ_0} positions of the subterm that we are rewriting. We use some auxiliary definitions needed for the proofs in the Appendix.

Definition 21 (Representative of a Σ_0-term over a set of Σ_0 terms). *Let t be a term in $\mathcal{T}_{\Sigma_0}$ and let $\hat{u} = \{u_1, \ldots, u_n\} \subseteq \mathcal{T}_{\Sigma_0}$ such that $t \in \hat{u}$. We define the* Σ_0-representative *of t over $\hat{u}$ as $rep^{\circ}_{\hat{u}}(t) = u_{min(\{i \mid u_i =_{E_0} t\})}$. We homomorphically extend the definition to lists and sets of terms.*

Definition 22 (Representative of a term over a set of Σ_0 terms). *Let t be a term in $\mathcal{T}_{\Sigma}$, where $top_{\Sigma_0}(t) = \hat{p}$, and let $\hat{u} \subseteq \mathcal{T}_{\Sigma_0}$ such that $t|_{\hat{p}} \subseteq \hat{u}$. We define the* representative *of t over $\hat{u}$, as $rep_{\hat{u}}(t) = t[rep^{\circ}_{\hat{u}}(t|_{\bar{p}})]_{\bar{p}}$. We homomorphically extend the definition to lists and sets of terms.*

Then $rep_{\hat{u}}(\hat{u})$ will be a set containing one element for each E_0-equivalence class that appears in $\hat{u}$, the *representative* of the class over $\hat{u}$.

Remark 1. *From the previous definitions it is immediate that:*

- *if t is a term in $\mathcal{T}_\Sigma$ then $t =_{E_0} rep_{\hat{u}}(t)$,*
- *if t is a term in $\mathcal{T}_{\Sigma_0}$ then $rep^{\circ}_{\hat{u}}(t) = rep_{\hat{u}}(t)$,*
- *if $top_{\Sigma_0}(t) = \hat{p}$ and $t|_{\hat{p}} \subseteq \hat{u}$ then $rep^{\circ}_{\hat{u}}(t|_{\hat{p}}) = rep_{\hat{u}}(t|_{\hat{p}}) \subseteq rep_{\hat{u}}(\hat{u})$,*
- *if t is a term in $rep_{\hat{u}}(\hat{u})$ then $rep_{\hat{u}}(t) = t$, and*
- *if u_1 and u_2 are two elements of $rep_{\hat{u}}(\hat{u})$ and $u_1 =_{E_0} u_2$ then $u_1 = u_2$.*

Definition 23 (Representative of a substitution over a set of Σ_0-terms)**.** *Let σ be a ground substitution and let $\hat{u} \subseteq \mathcal{T}_{\Sigma_0}$ such that $\bigcup_{z \in dom(\sigma)} \{(z\sigma)|_{top_{\Sigma_0}(z\sigma)}\} \subseteq \hat{u}$. We define the representative of σ as $rep_{\hat{u}}(\sigma) = \{z \mapsto rep_{\hat{u}}(z\sigma) \mid z \in dom(\sigma)\}$, i.e., each top_{Σ_0}-term in σ is replaced by its representative with respect to $\hat{u}$, so $\sigma =_{E_0} rep_{\hat{u}}(\sigma)$.*

Definition 24 (Representative of a term)**.** *Let t be a term in $\mathcal{T}_\Sigma$, where $top_{\Sigma_0}(t) = \hat{p}$. We define the* representative *of t as $rep(t) = rep_{t|_{\hat{p}}}(t)$.*

The transitive closure of the relation $\rightarrow^1_{R,B}$, inductively defined below, is denoted $\rightarrow^+_{R,B}$. The relation $\rightarrow_{R,B}$ is defined as $\rightarrow_{R,B} = (\rightarrow^+_{R,B} \cup =_E)$.

Definition 25 (R, B-rewriting)**.** *Given a rewrite theory $\mathcal{R} = (\Sigma, E_0 \cup B, R)$, terms t, v in $\mathcal{H}_\Sigma$, and a rule $c : l \rightarrow r$ if $\bigwedge_{i=1}^n l_i \rightarrow r_i \mid \phi$ in R, if $abstract_{\Sigma_1}(l) = \langle \lambda\bar{x}.l^\circ; \theta^\circ; \phi^\circ \rangle$ and there exist a position p in $pos_{\Sigma_1}(t)$ and a substitution $\sigma : \bar{x} \cup vars(c) \rightarrow \mathcal{T}_\Sigma$ such that $rep(t|_p) =_B l^\circ\sigma$, $v =_E t[r\sigma]_p$, $l_i\sigma \rightarrow_{R,B} r_i\sigma$, for $1 \leq i \leq n$, and $E_0 \vDash (\phi \wedge \phi^\circ)\sigma$, then we say there is a one-step transition $t \rightarrow^1_{R,B} v$.*

We write $t \xrightarrow[c,p,\sigma]{}{}^1_{R,B} v$, when we need to make explicit the rule, position, and substitution. Any of these items can be omitted when it is irrelevant. The following examples show the motivation behind all the previous definitions.

Example 8. *We justify the need of rep: consider a rewrite theory $\mathcal{R}$ where $B = \emptyset$, E_0 is integer arithmetic, there is one non-E_0 sort s, with two function symbols $g : s \rightarrow s$ and $f : s\ s \rightarrow s$, and $R = \{c : f(y_s, y_s) \rightarrow y_s\}$, so $abstract_{\Sigma_1}(f(y_s, y_s)) = \langle \lambda.f(y_s, y_s); none; true \rangle$. Let $t = f(g(3), g(1+2))$. t does not match $f(y_s, y_s)$, but $rep(t) = f(g(3), g(3))$ does, with $\sigma = \{y_s \mapsto g(3)\}$, so $t \rightarrow^1_{R,B} g(3)$. As $t =_E rep(t)$, because $t =_{E_0} rep(t)$, and $B = \emptyset$ then also $t \rightarrow^1_{R/E} g(3)$.*

Example 9. *In example 7, $\mathcal{R}_1 = (\Sigma, B, \{l : f(a, b) \rightarrow c\})$ and $\mathcal{R}_2 = (\Sigma, B, l_B)$, as $E_0 = \emptyset$, no abstraction of terms has to be performed when rewriting with $\rightarrow^1_{R_2,B}$ ($abstract_{\Sigma_1}(l) = \langle \lambda.l; none; true \rangle$ for any left side l of a Σ-rule). Then, the term $f(f(a, a), b)$ is not a normal form in $\rightarrow^1_{R_2,B}$ because l_B has the rule $l : f(x_s, f(a, b)) \rightarrow f(x_s, c)$ that can be applied on top of the term $f(f(a, a), b)$ with matching $x_s \mapsto a$, modulo associativity and commutativity, leading to $f(f(a, a), b) \rightarrow^1_{R_2,B} f(a, c)$. Also $f(f(a, a), b) \rightarrow^1_{R_1/E} f(a, c)$ and $f(f(a, a), b) \rightarrow^1_{R_2/E} f(a, c)$, because $f(f(a, a), b) =_E f(a, f(a, b))$.*

The added rule $l : f(x_s, f(a, b)) \rightarrow f(x_s, c)$ has allowed us to imitate $\rightarrow^1_{R_1/E}$ ($=\rightarrow^1_{R_2/E}$) with $\rightarrow^1_{R_2,B}$.

Definition 26 (Normalized substitution)**.** *Given a rewrite theory $\mathcal{R} = (\Sigma, E, R)$ with built-in subtheory (Σ_0, E_0), a substitution σ is* R/E-normalized (resp. R, B-normalized) *iff for each variable x in $dom(\sigma)$ there is no term t in $\mathcal{T}_\Sigma(\mathcal{X})$ such that $x\sigma \rightarrow^1_{R/E} t$ (resp. $x\sigma \rightarrow^1_{R,B} t$).*

Theorem 1 (Equivalence of R/E and R,B-rewriting). *If $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ is an associated rewrite theory closed under B-extensions of $\mathcal{R}_0 = (\Sigma, E_0 \cup B, R_0)$, then $\rightarrow^1_{R,B} = \rightarrow^1_{R/E}$ and $\rightarrow_{R,B} = \rightarrow_{R/E}$.*

Proof. There is a special case to consider when there are no rewrite steps involved in the deductions.

(i) $\rightarrow^1_{R,B} \subseteq \rightarrow^1_{R/E}$ and $\rightarrow_{R,B} \subseteq \rightarrow_{R/E}$.

In the special case, $t \rightarrow_{R,B} v$ with no rewrite steps. As $\rightarrow_{R,B} = (\rightarrow^+_{R,B} \cup =_E)$ then $t =_E v$, so $t \rightarrow_{R/E} v$. The other cases are proved using induction in the total number of $\rightarrow^1_{R,B}$ rewrite steps in the derivation.

- Base case

  $t \rightarrow^1_{R,B} t[r\sigma]_p =_E v$ with only one $\rightarrow^1_{R,B}$ rewrite step in the derivation, where $c : l \rightarrow r$ *if* $\bigwedge_{i=1}^m l_i \rightarrow r_i \mid \phi$ in R, $abstract_{\Sigma_1}(l) = \langle \lambda \bar{x}.l^\circ; \theta^\circ; \phi^\circ \rangle$, $\bar{x} = x_1, \ldots, x_n$, $l^\circ = l[\bar{x}]_{\bar{q}}$, $\phi^\circ = \bigwedge_{j=1}^n (x_j = l|_{q_j}))$, p in $pos_{\Sigma_1}(t)$, and $\sigma : \bar{x} \cup vars(c) \rightarrow \mathcal{T}_\Sigma$ such that $rep(t|_p) =_B l^\circ \sigma$, $v =_E t[r\sigma]_p$, $\bar{l}\sigma =_E \bar{r}\sigma$, and $E_0 \vDash (\phi \wedge \phi^\circ)\sigma$.

  As $E_0 \vDash \phi^\circ \sigma$ then $l\sigma = l\sigma[l\sigma|_{q_1}]_{q_1} \cdots [\sigma_{q_n}]_{q_n} =_E l\sigma[x_1\sigma]_{q_1} \cdots [x_n\sigma]_{q_n} = l^\circ\sigma =_B rep(t|_p) =_{E_0} t|_p$, so $l\sigma =_E t|_p$.

  As $t|_p =_E l\sigma$ and $\bar{l}\sigma =_E \bar{r}\sigma$, then $t = t[t|_p]_p =_E t[l\sigma]_p \rightarrow^1_R t[r\sigma]_p =_E v$ with rule c in R, that is, $t \rightarrow^1_{R/E} v$, so $t \rightarrow_{R/E} v$.

- Induction case

  There are two subcases to consider:

  1. $t \rightarrow^1_{R,B} t[r\sigma]_p =_E v$ with several $\rightarrow^1_{R,B}$ rewrite steps in the derivation. As in the base case, $c : l \rightarrow r$ *if* $\bigwedge_{i=1}^m l_i \rightarrow r_i \mid \phi$ in R, $abstract_{\Sigma_1}(l) = \langle \lambda \bar{x}.l^\circ; \theta^\circ; \phi^\circ \rangle$, $\bar{x} = x_1, \ldots, x_n$, $l^\circ = l[\bar{x}]_{\bar{q}}$, $\phi^\circ = \bigwedge_{j=1}^n (x_j = l|_{q_j}))$, p in $pos_{\Sigma_1}(t)$, and $\sigma : \bar{x} \cup vars(c) \rightarrow \mathcal{T}_\Sigma$ such that $rep(t|_p) =_B l^\circ\sigma$, $v =_E t[r\sigma]_p$, $\bar{l}\sigma =_E \bar{r}\sigma$, and $E_0 \vDash (\phi \wedge \phi^\circ)\sigma$.
     By induction hypothesis $l_i\sigma \rightarrow_{R/E} r_i\sigma$, for $1 \leq i \leq m$. As in the base case, $E_0 \vDash \phi\sigma$ and $t|_p =_E l\sigma$, so $t = t[t|_p]_p =_E t[l\sigma]_p \rightarrow^1_R t[r\sigma]_p =_E v$, i.e., $t \rightarrow^1_{R/E} v$, so $t \rightarrow_{R/E} v$.
  2. $t \rightarrow^1_{R,B} u \rightarrow^+_{R,B} w =_E v$. By the previous subcase $t \rightarrow^1_{R/E} u \rightarrow^+_{R,B} w =_E v$, and, by I.H., $t \rightarrow^1_{R/E} u \rightarrow^+_{R/E} w =_E v$, i.e., $t \rightarrow^*_{R/E} w =_E v$, or $t \rightarrow_{R/E} v$.

(ii) $\rightarrow^1_{R/E} \subseteq \rightarrow^1_{R,B}$ and $\rightarrow_{R/E} \subseteq \rightarrow_{R,B}$.

In the special case, $t \rightarrow_{R/E} v$ with no rewrite steps because $t =_E v$. As $\rightarrow_{R,B} = (\rightarrow^+_{R,B} \cup =_E)$ then $t \rightarrow_{R,B} v$. The other cases are proved using induction in the total number of $\rightarrow^1_{R/E}$ rewrite steps in the derivation.

- Base case: $t \rightarrow^1_{R/E} v$ with only one $\rightarrow^1_{R/E}$ rewrite step in the derivation using a rule $c : l \rightarrow r$ *if* C in R, where $C = \bigwedge_{i=1}^m l_i \rightarrow r_i \mid \phi$, and a substitution σ. We can assume that c is a rule in R_0 since any $\rightarrow^1_{R/E}$ step given at position p of t'' using a rule $c_1 : w[l]_q \rightarrow w[r]_q$ *if* C in $R \setminus R_0$ can also be achieved using rule c at position $p.q$ of t'', so $t =_E t'' \rightarrow^1_R u =_E v$, $t'' = t''[l\sigma]_p$, $u = t''[r\sigma]_p$, $\bar{l}\sigma =_E \bar{r}\sigma$, and $E_0 \vDash \phi\sigma$.
  By Proposition 7 there exists a term t' in $\mathcal{H}_\Sigma$ such that $t =_B t' =_{E_0} t'' \rightarrow^1_R u =_E v$. We have $t \xleftrightarrow{ax_1}_B \cdots \xleftrightarrow{ax_l}_B t'$, where ax_i (linear and regular), for $1 \leq i \leq l$ has the form $w_i = w'_i$, let $\overline{ax_i}$ be $w'_i = w_i$, so each top_{Σ_0} subterm of t is moved by $ax_1 \cdots ax_l$ and becomes another top_{Σ_0} subterm of t'. Then, $\overline{ax_l} \cdots \overline{ax_1}$ moves the top_{Σ_0} subterms of t'' in the opposite way, so there exists a term t_0 in $\mathcal{T}_\Sigma$ such that $t'' \xleftrightarrow{\overline{ax_l}}_B \cdots \xleftrightarrow{\overline{ax_1}}_B t_0 =_{E_0} t$.

  We have $t =_{E_0} t_0 =_B t'' = t''[l\sigma]_p$, so $t''|_p = l\sigma$. The more general case, where $t_0 =_B t''|_p =_B l\sigma$ is studied in Theorem 2 and Corollary 2 in [Mes17], where it is proved that there is a position q in $pos(t_0)$, a rule $c_0 : l_0 \to r_0 \textit{ if } C$ in R, maybe the original c, and a substitution σ_0, such that $t_0|_q =_B l_0\sigma_0$, $t_0[r_0\sigma_0]_q =_B u$, and $C\sigma_0 = C\sigma$, which is also valid for our particular case where $t''|_p = l\sigma$. As, by definition of rule, $l_0 \in \mathcal{H}_\Sigma(\mathcal{X})$, then $q \in pos_{\Sigma_1}(t_0)$, so $t_0|_q =_{E_0} t|_q$. Let $top_{\Sigma_0}(t|_q) = \hat{z}$. Then $rep_{t|_{q.\hat{z}}}$ is the function that given a term in $\mathcal{T}_\Sigma$ returns the same term with each top_{Σ_0} term on it replaced with the representative for that top_{Σ_0} term in $rep(t|_q)$, if it exists, so $rep(t|_q) = rep_{t|_{q.\hat{z}}}(t_0|_q) =_B rep_{t|_{q.\hat{z}}}(l_0\sigma_0)$.

Let $abstract_{\Sigma_1}(l_0) = \langle \lambda\bar{y}.l_0^\circ; \theta_0^\circ; \phi_0^\circ \rangle$, $\bar{y} = y_1, \ldots, y_k$, $l_0^\circ = l_0[\bar{y}]_{\bar{o}}$, $\phi^\circ = \bigwedge_{j=1}^k y_j = l_0|_{o_j}$. Define $\sigma' : dom(\sigma_0) \cup \hat{y} \to \mathcal{T}_\Sigma$ as: if $z = y_j \in \hat{y}$ then $z\sigma' = rep_{t|_{q.\hat{z}}}(l_0|_{o_j}\sigma_0)$ else $z\sigma' = rep_{t|_{q.\hat{z}}}(z\sigma_0)(=_{E_0} z\sigma_0)$. As, for $1 \le j \le k$, $y_j\sigma' = rep_{t|_{q.\hat{z}}}(l_0|_{o_j}\sigma_0) =_{E_0} l_0|_{o_j}\sigma_0 =_{E_0} l_0|_{o_j}\sigma'$, because $\hat{y} \cap V_{l_0|_{o_j}} = \emptyset$, then $E_0 \vDash \phi^\circ\sigma'$. Also, as $C\sigma_0 = C\sigma$ and if $z \in dom(\sigma_0)$ then $z\sigma' =_{E_0} z\sigma_0$ then $\bar{l}\sigma' =_{E_0} \bar{l}\sigma_0 =_E \bar{r}\sigma_0 =_{E_0} \bar{r}\sigma'$, i.e., $\bar{l}\sigma' =_E \bar{r}\sigma'$, and $\phi\sigma' =_{E_0} \phi\sigma_0 = \phi\sigma$, so $E_0 \vDash \phi\sigma'$. As $\phi\sigma'$ and $\phi^\circ\sigma'$ are ground, because $rep_{t|_{q.\hat{z}}}$ is replacing each ground subterm with another ground subterm, then $E_0 \vDash (\phi \wedge \phi^\circ)\sigma'$.

As

- $l_0[]_{\bar{o}}\sigma' = l_0\sigma'[]_{\bar{o}} = rep_{t|_{q.\hat{z}}}(l_0\sigma_0[]_{\bar{o}})$, and
- $y_j\sigma' = rep_{t|_{q.\hat{z}}}(l_0|_{o_j}\sigma_0)$, for $1 \le j \le k$,

then $l_0^\circ\sigma' = l_0[\bar{y}]_{\bar{o}}\sigma' = rep_{t|_{q.\hat{z}}}(l_0\sigma_0[l_0|_{\bar{o}}\sigma_0]_{\bar{o}}) = rep_{t|_{q.\hat{z}}}(l_0[l_0|_{\bar{o}}]_{\bar{o}}\sigma_0) = rep_{t|_{q.\hat{z}}}(l_0\sigma_0) =_B rep(t|_q)$, i.e., $rep(t|_q) =_B l_0^\circ\sigma'$ so, as $t[r_0\sigma']_q =_{E_0} t[r_0\sigma_0]_q =_{E_0} t_0[r_0\sigma_0]_q =_B u =_E v$, i.e., $t[r_0\sigma']_q =_E v$, we have $t \to^1_{R,B} v$.

- Induction case:

  again, there are two subcases to consider:

  1. $t \to^1_{R/E} t[r\sigma]_p =_E v$ with several $\to^1_{R/E}$ rewrite steps in the derivation. The proof is the same as the one in the base case, except that instead of having $\bar{l}\sigma' =_E \bar{r}\sigma'$ now we have $l_i\sigma \to_{R/E} r_i\sigma$, for $1 \le i \le m$, so by I.H., as $(l_i, r_i)\sigma = (l_i, r_i)\sigma'$, also $l_i\sigma' \to_{R,B} r_i\sigma'$ hence $t \to^1_{R,B} v$.
  2. $t \to^1_{R/E} u \to^+_{R/E} w =_E v$. By the previous subcase $t \to^1_{R,B} u \to^+_{R/E} w =_E v$, and, by I.H., $t \to^1_{R,B} u \to^+_{R,B} w =_E v$, i.e., $t \to^*_{R,B} w =_E v$, or $t \to_{R,B} v$.

□

Corollary 1. *If $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ is an associated rewrite theory closed under B-extensions, then any substitution is R/E-normalized iff it is R, B-normalized.*

Proposition 5 (Decomposition of a normalized substitution). *Let $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ be a rewrite theory with built-in subtheory (Σ_0, E_0). If σ is an R/E-normalized substitution and $\sigma = \sigma_1 \cdot \sigma_2$, with $dom(\sigma_1) \cap (ran(\sigma_1) \cup dom(\sigma_2)) = \emptyset$, then σ_1 and σ_2 are R/E-normalized.*

Proof. We prove that each substitution is normalized by reductio ad absurdum:

- If σ_1 is not R/E-normalized, then there exists a variable x in $dom(\sigma_1) \subseteq dom(\sigma)$ and a term t such that $x\sigma_1$ is in $\mathcal{H}_\Sigma$, so $x\sigma_1 = x\sigma_1\sigma_2 = x\sigma$, and $x\sigma_1 \to^1_{R/E} t$. As $x\sigma_1 = x\sigma$, then also $x\sigma \to^1_{R/E} t$ hence, as x is in $dom(\sigma)$, σ is not R/E-normalized, a contradiction.

- If σ_2 is not R/E-normalized, then there exists a variable x in $dom(\sigma_2)$ and a term t such that $x\sigma_2$ is in $\mathcal{H}_\Sigma$ and $x\sigma_2 \to^1_{R/E} t$, where either x in $dom(\sigma)$ or not.

– If x is in $dom(\sigma)$ then $x\sigma_2 = x\sigma$, so also $x\sigma \rightarrow^1_{R/E} t$ hence, as x is in $dom(\sigma)$, σ is not R/E-normalized, a contradiction.
– If x is not in $dom(\sigma)$ then, as $\sigma = \sigma_1\sigma_2$, x is in $ran(\sigma_1)$, so there exists y in $dom(\sigma_1) \subseteq dom(\sigma)$ and a position p such that $y\sigma_1|_p = x$. Then $y\sigma|_p = y\sigma_1\sigma_2|_p = y\sigma_1|_p\sigma_2 = x\sigma_2$, so $y\sigma|_p \rightarrow^1_{R/E} t$, hence also $y\sigma \rightarrow^1_{R/E} t$. As y is in $dom(\sigma)$, then σ is not R/E-normalized, a contradiction.

□

Proposition 6 (Preservation of the normalized property under generalization). *Let $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ be a rewrite theory with built-in subtheory (Σ_0, E_0). If ρ is an R/E-normalized substitution and σ is a more general substitution than ρ, then σ is R/E-normalized.*

Proof. We proceed again by reductio ad absurdum. By definition of $\ll_E$, there exist a substitution η such that $\rho_{\mathcal{V}} =_E (\sigma\eta)_{\mathcal{V}}$. If σ is not R/E-normalized, then there exists a variable x in $dom(\sigma) \subseteq dom(\rho)$ and a term t such that $x\sigma$ is in $\mathcal{H}_\Sigma$, so $x\sigma = x\sigma\eta =_E x\rho$, and $x\sigma \rightarrow^1_{R/E} t$. But then, also $x\rho \rightarrow^1_{R/E} t$ so, as x is in $dom(\rho)$, ρ is not R/E-normalized, a contradiction. □

Proposition 7 (Decomposition of E-equality in B-equality plus E_0-equality [AMPP17]). *Let $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ be a rewrite theory with built-in subtheory (Σ_0, E_0). If t and t'' are terms in $\mathcal{H}_\Sigma(\mathcal{X})$ and $t =_E t''$ then there exists a term t' in $\mathcal{H}_\Sigma(\mathcal{X})$ such that $t =_B t' =_{E_0} t''$.*

Rewriting with $\rightarrow^1_{R,B}$ does not depend on the chosen representative for a class of terms modulo E_0.

Lemma 3 (Independence of R,B-rewriting under E_0-equality). *Given a rewrite theory $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ with built-in subtheory (Σ_0, E_0), and terms t, u, and v in $\mathcal{H}_\Sigma$, if $t =_{E_0} u$ and $u \rightarrow^1_{R,B} v$ then there exists a term w in $\mathcal{H}_\Sigma$ such that $t \rightarrow^1_{R,B} w$ and $v =_{E_0} w$.*

Proof. As $u \rightarrow^1_{R^\circ,B} v$, there are rules $c : l \rightarrow r$ *if* $\bigwedge_{i=1}^m l_i \rightarrow r_i \mid \phi \in R$ and $c^\circ = l^\circ \rightarrow r$ *if* $\bigwedge_{i=1}^m l_i \rightarrow r_i \mid \phi \wedge \phi^\circ \in R^\circ$, where $top_{\Sigma_0}(l) = \hat{q}$ and $l^\circ = l[\bar{x}]_{\bar{q}}$, a position $p \in pos(u)$, and a substitution $\sigma : vars(c^\circ) \rightarrow \mathcal{T}_\Sigma$ such that $rep(u|_p) = l^\circ\sigma$, $l_i\sigma \rightarrow_{R,B} r_i\sigma$, for $1 \leq i \leq m$, and $E_0 \vDash (\phi \wedge \phi^\circ)\sigma$, so $v = u[r\sigma]_p$. Also $u|_p \in \mathcal{H}_\Sigma$ because $l^\circ \in \mathcal{T}_{\Sigma_1}(\mathcal{X})$ so, by Proposition 3, $t|_p =_{E_0} u|_p = l^\circ\sigma$. Let $w = t[r\sigma]_p$.

$rep(u|_p) = l^\circ\sigma$ has the form $l\sigma[\bar{x}\sigma]_{\bar{q}}$. By Proposition 4, $top_{\Sigma_0}(u|_p) = top_{\Sigma_0}(rep(u|_p)) = top_{\Sigma_0}(l^\circ\sigma) = \hat{q}$. As $rep(t|_p) =_{E_0} t|_p =_{E_0} u|_p =_{E_0} rep(u|_p)$ then, again by Proposition 3, $top_{\Sigma_0}(rep(t|_p)) = \hat{q}$, $rep(t|_p) = rep(u|_p)[rep(t|_p)|_{\bar{q}}]_{\bar{q}}$, so $rep(t|_p)[]_{\bar{q}} = rep(u|_p)[]_{\bar{q}}$, and $rep(t|_p)|_{q_i} =_{E_0} rep(u|_p)|_{q_i}$, for $1 \leq i \leq n$. Let $\sigma' = \sigma_{dom(\sigma)\setminus\hat{x}} \cup \bigcup_{j=1}^n \{x_j \mapsto rep(t|_p)|_{q_j}\}$, where if $x_i = x_j$, with $1 \leq i < j \leq n$, then $x_i\sigma = x_j\sigma =_{E_0} x_i\sigma' = x_j\sigma'$, so σ' is well defined.

As $rep(t|_p)[]_{\bar{q}} = rep(u|_p)[]_{\bar{q}}$ and $rep(u|_p) = l^\circ\sigma = l\sigma[\bar{x}\sigma]_{\bar{q}}$, then we have $l^\circ\sigma' = l\sigma[\bar{x}\sigma']_{\bar{q}} = rep(u|_p)[rep(t|_p)|_{\bar{q}}]_{\bar{q}} = rep(t|_p)[rep(t|_p)|_{\bar{q}}]_{\bar{q}} = rep(t|_p)$. Also, as $vars(c) \cap \bar{x} = \emptyset$, $r\sigma' = r\sigma$, $l_i\sigma' = l_i\sigma$, $r_i\sigma' = r_i\sigma$, for $1 \leq i \leq m$, and $\phi\sigma' = \phi\sigma$, then $l_i\sigma' = l_i\sigma \rightarrow_{R,B} r_i\sigma = r_i\sigma'$, for $1 \leq i \leq m$, and $E_0 \vDash \phi\sigma'$.

$E_0 \vDash \phi^\circ\sigma$, where $\phi^\circ\sigma = \bigwedge_{j=1}^n ((x_j\sigma = l|_{q_j}\sigma) = \bigwedge_{j=1}^n ((u|_{p.q_j}\sigma = l|_{q_j}\sigma)$. As $\phi^\circ\sigma' = \bigwedge_{j=1}^n ((x_j\sigma' = l|_{q_j}\sigma') = \bigwedge_{j=1}^n (t|_{p.q_j} = l|_{q_j}\sigma)$ and $t|_{p.q_i} =_{E_0} u|_{p.q_i}$, for $1 \leq i \leq n$, then $E_0 \vDash \phi^\circ\sigma'$, so $E_0 \vDash (\phi \wedge \phi^\circ)\sigma'$ and $t \rightarrow^1_{R^\circ,B} t[r\sigma]_p = w$. As $t =_{E_0} u$ then $t[r\sigma]_p =_{E_0} u[r\sigma]_p$, i.e., $v =_{E_0} w$. □

The following results will be used in the proof of the completeness of the calculus.

Proposition 8 (Bijection between top_{Σ_0} positions in B-equal terms). *Given an OS equational theory $\mathcal{E} = (\Sigma, E_0 \cup B)$ and two terms u and v in $\mathcal{H}_\Sigma(\mathcal{X})$ such that $u =_B v$, where $u = u_0 \xleftrightarrow{ax_1}_B \cdots \xleftrightarrow{ax_n}_B u_n = v$, call $\overline{ax} = ax_1, \ldots, ax_n$, with $\hat{ax} \subset B \cup B^{-1}$, if $top_{\Sigma_0}(u) = \hat{p}$ and $top_{\Sigma_0}(v) = \hat{q}$ then there exists a bijective function $dest_{\overline{ax}} : \hat{p} \rightarrow \hat{q}$ such that $u|_{p_i} = v|_{dest_{\overline{ax}}(p_i)}$, for each position p_i in $\hat{p}$.*

Proof. We inductively define the function $dest_l$ that tracks the final position of a subterm for a list of axioms $l = a_1, \ldots, a_m$. Given a position p':

1. $dest_{nil}(p') = p'$,

2. for a_1 in $B \cup B^{-1}$ with the form $f[\bar{x}]_{\bar{q}} = f'[\bar{x}]_{\bar{r}}$, where $vars(f[\bar{x}]_{\bar{q}}) = vars(f'[\bar{x}]_{\bar{r}}) = \hat{x}$, if $p' = q_j.s_j$, with q_j in $\hat{q}$, then $dest_{a_1}(p') = r_j.s_j$, else $dest_{a_1}(p') = p'$, and

3. for $l = a_1, \ldots, a_m$, with $m > 1$, if $dest_{a_1}(p') = p''$ then $dest_l(p') = dest_{a_2,\ldots,a_m}(p'')$.

As, by definition, the axioms in B are regular, linear, and only have function symbols from F_1, then in each step $u_{i-1} \xleftrightarrow{ax_i}_B u_i$, $1 \leq i \leq n$, if ax_i has the form $f[\bar{x}]_{\bar{q}} = f'[\bar{x}]_{\bar{r}}$, where $vars(f[\bar{x}]_{\bar{q}}) = vars(f'[\bar{x}]_{\bar{r}}) = \hat{x}$ and it is used in a subterm $u_{i-1}|_p$ then:

- if ax_i moves a subterm in a position $p.q_j$ from $top_{\Sigma_0}(u_{i-1})$, where q_j in $\hat{q}$, with parent in F_1 since ax_i has only symbols in F_1, then the subterm is moved to the position $p.r_j$, with parent also in F_1 for the same reason as before, hence it remains a top_{Σ_0} position,

- if ax_i moves a subterm t in a position $p.q_j.s_j.k_j$ from $top_{\Sigma_0}(u_{i-1})$, where q_j in $\hat{q}$, s_j may be ϵ, k_j is an integer, and the parent of t in position $p.q_j.s_j$ is a function symbol f'' from F_1, then t is moved to the position $p.r_j.s_j.k_j$, where its parent at position $p.r_j.s_j$ is the same function symbol f'' from F_1, since f'' is also moved by ax_i from $p.q_j.r_j$ to $p.q_j.s_j$, hence it remains a top_{Σ_0} position,

- the rest of positions in $top_{\Sigma_0}(u_{i-1})$ remain unchanged.

Then $dest_{\overline{ax}}$ is injective, by its definition, and it also has to be surjective, since any position in $\hat{q}$ not in the image of $dest_{\overline{ax}}$ could be always related to a single position in $\hat{p}$ just by using the list of axioms ${ax_n}^{-1}, \ldots, {ax_1}^{-1}$, all of them in $B \cup B^{-1}$, a contradiction with $dest$ being total and surjective. We will write $dest$ instead of $dest_{\overline{ax}}$ when $\overline{ax}$ is irrelevant, homomorphically extend the definition of $dest$ to lists and sets of positions, and define $orig = dest^{-1}$. ☐

Corollary 2 (Bijection between top_{Σ_0} positions in E-equal terms). *Given an OS equational theory $\mathcal{E} = (\Sigma, E_0 \cup B)$ and two terms u and v in $\mathcal{H}_\Sigma(\mathcal{X})$ such that $u =_E v$, if $top_{\Sigma_0}(u) = \hat{p}$ and $top_{\Sigma_0}(v) = \hat{q}$ then there exists a bijective function $dest : \hat{p} \to \hat{q}$, hence $\hat{q} = dest(\hat{p})$, such that $u|_{p_i} =_{E_0} v|_{dest(p_i)}$, for each position p_i in $\hat{p}$.*

Proof. As $u =_E v$ then, by Proposition 7, there exists a term w in $\mathcal{H}_\Sigma(\mathcal{X})$ such that $u =_{E_0} w =_B v$. As $u =_{E_0} w$ then, by Proposition 3, $top_{\Sigma_0}(u) = top_{\Sigma_0}(w) = \hat{p}$ and $u|_{p_i} =_{E_0} w|_{p_i}$, for each position p_i in $\hat{p}$. But, by Proposition 8, $w|_{p_i} = v|_{dest(p_i)}$, so $u|_{p_i} =_{E_0} w|_{p_i} = v|_{dest(p_i)}$, for each position p_i in $\hat{p}$. ☐

Lemma 4 (Relation between E-unifiers and B-unifiers of abstractions). *Given an OS equational theory $\mathcal{E} = (\Sigma, E_0 \cup B)$ and two terms u and v in $\mathcal{H}_\Sigma(\mathcal{X})$, if $abstract_{\Sigma_1}((u, v)) = \langle \lambda(\bar{x}, \bar{y}).(u^\circ, v^\circ); (\theta^\circ_u, \theta^\circ_v); (\phi^\circ_u, \phi^\circ_v) \rangle$ and σ' is a ground substitution such that $V_{u,v} \subseteq dom(\sigma')$, $u\sigma' =_E v\sigma'$, and $dom(\sigma') \cap (\hat{x} \cup \hat{y}) = \emptyset$ then there exists another ground substitution σ° such that $u^\circ\sigma^\circ =_B v^\circ\sigma^\circ$, $E_0 \vDash (\phi^\circ_u \wedge \phi^\circ_v)\sigma^\circ$, $dom(\sigma^\circ) = dom(\sigma') \cup \hat{x} \cup \hat{y}$, so $V_{(u^\circ, v^\circ, \phi^\circ_u, \phi^\circ_v)\sigma^\circ} = \emptyset$, and $\sigma' =_{E_0} \sigma^\circ_{dom(\sigma')}$.*

Proof. Let $\bar{x} = \{x_1, \ldots, x_{i_x}\}$ and $\bar{y} = \{y_1, \ldots, y_{i_y}\}$, so $u^\circ = u[\bar{x}]_{\bar{p}}$, $\phi^\circ_u = (\bigwedge_{i=1}^{i_x} x_i = u|_{p_i})$, $v^\circ = v[\bar{y}]_{\bar{q}}$, $\phi^\circ_v = (\bigwedge_{j=1}^{i_y} y_j = v|_{q_j})$, for proper $\bar{p}$ and $\bar{q}$ such that $\hat{p} = top_{\Sigma_0}(u)$ and $\hat{q} = top_{\Sigma_0}(v)$. Also, let $u = u[\bar{x}']_{\bar{p}'}$ and $v = v[\bar{y}']_{\bar{q}'}$, where $pos_{\mathcal{X}_1}(u) = \hat{p}'$, $V_u \cap \mathcal{X}_1 = \hat{x}'$, $pos_{\mathcal{X}_1}(v) = \hat{q}'$, and $V_v \cap \mathcal{X}_1 = \hat{y}'$, so $u^\circ = u[\bar{x}]_{\bar{p}}[\bar{x}']_{\bar{p}'}$ and $v^\circ = v[\bar{y}]_{\bar{q}}[\bar{y}']_{\bar{q}'}$. As $u\gamma'$ and $v\sigma'$ are ground terms then $\hat{x}' \cup \hat{y}' \subseteq dom(\sigma')$.

As $u^\circ = u[\bar{x}]_{\bar{p}}$ then $pos_{\mathcal{X}_0}(u^\circ) = \hat{p}$, hence $V_{u[]_{\bar{p}}} \cap \mathcal{X}_0 = \emptyset$, i.e., $V_{u[]_{\bar{p}}} = \hat{x}' \subset \mathcal{X}_1$, so $V_{u[]_{\bar{p}}[]_{\bar{p}'}} = \emptyset$, and $u[]_{\bar{p}}[]_{\bar{p}'} = u\sigma'[]_{\bar{p}}[]_{\bar{p}'}$. In the same way, $V_{v[]_{\bar{q}}} = \hat{y}' \subset \mathcal{X}_1$, $V_{v[]_{\bar{q}}[]_{\bar{q}'}} = \emptyset$, and $v[]_{\bar{q}}[]_{\bar{q}'} = v\sigma'[]_{\bar{q}}[]_{\bar{q}'}$.

Let $\hat{t}^{\sigma'}_{\Sigma_0} = \bigcup_{t \in dom(\sigma')\sigma'} t|_{top_{\Sigma_0}(t)}$, i.e., the set of all top_{Σ_0} terms that appear in $z\sigma'$, where z ranges over the variables in $dom(\sigma')$. Now, let $\hat{t} = u|_{\hat{p}}\sigma' \cup v|_{\hat{q}}\sigma' \cup \hat{t}^{\sigma'}_{\Sigma_0}$. As $\hat{x}' \cup \hat{y}' \subseteq dom(\sigma')$, then $\hat{t}$ includes all the top_{Σ_0}-terms that appear in $u\sigma'$ and $v\sigma'$, either from their top_{Σ_0} positions or as subterms of the instances of the variables in their $\mathcal{X}_1$ positions.

Define $\sigma^\circ = rep_{\hat{t}}(\sigma') \cup \{x_i \mapsto rep_{\hat{t}}(u|_{p_i}\sigma') \mid x_i \in \hat{x}\} \cup \{y_j \mapsto rep_{\hat{t}}(v|_{q_j}\sigma') \mid y_j \in \hat{y}\}$, so $rep_{\hat{t}}(\sigma) = \sigma^\circ_{dom(\sigma')}$, hence $dom(\sigma^\circ) = dom(\sigma') \cup \hat{x} \cup \hat{y}$ and $\sigma' =_{E_0} \sigma^\circ_{dom(\sigma')}$. Then:

- as $\hat{x}' \cup \bar{y}' \subseteq dom(\sigma') = dom(\sigma'_{rep})$ then $\hat{x} \cup \hat{y} \cup \hat{x}' \cup \hat{y}' \subseteq dom(\sigma^\circ)$,
- $u^\circ\sigma^\circ = u[\bar{x}]_{\bar{p}}[\bar{x}']_{\bar{p}'}\sigma^\circ = u[\bar{x}\sigma^\circ]_{\bar{p}}[\bar{x}'\sigma^\circ]_{\bar{p}'} = u[rep_{\hat{t}}(u|_{\bar{p}}\sigma')]_{\bar{p}}[\bar{x}'\sigma'_{rep}]_{\bar{p}'} =_{E_0} u[u|_{\bar{p}}\sigma']_{\bar{p}}[\bar{x}'\sigma']_{\bar{p}'} = u\sigma'[u|_{\bar{p}}\sigma']_{\bar{p}}[\bar{x}'\sigma']_{\bar{p}'} = u[u|_{\bar{p}}]_{\bar{p}}[\bar{x}]'_{\bar{p}'}\sigma' = u\sigma'$,
- $v^\circ\sigma^\circ = v[\bar{y}]_{\bar{q}}[\bar{y}']_{\bar{q}'}\sigma^\circ = v[\bar{y}\sigma^\circ]_{\bar{q}}[\bar{y}'\sigma^\circ]_{\bar{q}'} = v[rep_{\hat{t}}(v|_{\bar{q}}\sigma')]_{\bar{q}}[\bar{y}'\sigma'_{rep}]_{\bar{q}'} =_{E_0} v[v|_{\bar{q}}\sigma']_{\bar{q}}[\bar{y}'\sigma']_{\bar{q}'} = v\sigma'[v|_{\bar{q}}\sigma']_{\bar{q}}[\bar{y}'\sigma']_{\bar{q}'} = v[v|_{\bar{q}}]_{\bar{q}}[\bar{y}']_{\bar{q}'}\sigma' = v\sigma'$,
- as $u\sigma' =_E v\sigma'$, then $u^\circ\sigma^\circ =_{E_0} u\sigma' =_E v\sigma' =_{E_0} v^\circ\sigma^\circ$, i.e., $u^\circ\sigma^\circ =_E v^\circ\sigma^\circ$.

By Proposition 7, there exists a term w such that $u^\circ\sigma^\circ =_B w =_{E_0} v^\circ\sigma^\circ$, let $\hat{r} = top_{\Sigma_0}(w)$. We prove $v^\circ\sigma^\circ = w$, so $u^\circ\sigma^\circ =_B v^\circ\sigma^\circ$:

- as $u^\circ\sigma^\circ = u[rep_{\hat{t}}(u|_{\bar{p}}\sigma')]_{\bar{p}}[\bar{x}'\sigma'_{rep}]_{\bar{p}'} =_B w$ then, by Proposition 8, there exists a bijection $dest_1$ such that $dest_1(top_{\Sigma_0}(u^\circ\sigma^\circ)) = \hat{r}$ and $w|_{r_i} = u^\circ\sigma^\circ|_{orig_1(r_i)}$, for each position r_i in $\hat{r}$. As $V_{u[]_{\bar{p}}[]_{\bar{p}'}} = \emptyset$ then either:

  (i) $orig_1(r_i)$ is a position p_j in $\hat{p}$, so $w|_{r_i} = rep_{\hat{t}}(u|_{p_j}\sigma')$. As $u|_{p_j}\sigma'$ is an element of $\hat{t}$, then $w|_{r_i}$ is an element of $rep_{\hat{t}}(\hat{t})$; or

  (ii) $orig_1(r_i)$ has the form $p'_j.s_k$, where p'_j is a position in $\hat{p}'$, so s_k is a top_{Σ_0}-position of $u^\circ\sigma^\circ|_{p'_j}$. Then the variable x'_j in $\hat{x}'$, call $\hat{s} = top_{\Sigma_0}(x'_j\sigma'_{rep})$ so $s_k \in \hat{s}$, verifies $x'_j\sigma'_{rep} = rep_{\hat{t}}(x'_j\sigma')$, so $s_k \in top_{\Sigma_0}(rep_{\hat{t}}(x'_j\sigma'))$, $rep_{\hat{t}}(x'_j\sigma') = x'_j\sigma'[rep^\circ_{\hat{t}}(x'_j\sigma'|_{\bar{s}})]_{\bar{s}}$, and $w|_{r_i} = (x'_j\sigma'_{rep})|_{s_k} = rep_{\hat{t}}(x'_j\sigma')|_{s_k} = rep^\circ_{\hat{t}}(x'_j\sigma'|_{s_k}) = rep_{\hat{t}}(x'_j\sigma'|_{s_k})$. Then, as $rep_{\hat{t}}(x'_j\sigma'|_{\hat{s}}) \subseteq rep_{\hat{t}}(\hat{t})$, $w|_{r_i}$ is an element of $rep_{\hat{t}}(\hat{t})$.

  In conclusion, $w|_{\hat{r}} \subseteq rep_{\hat{t}}(\hat{t})$, hence $w = w[rep_{\hat{t}}(w|_{\bar{r}})]_{\bar{r}}$.

- $v^\circ\sigma^\circ = v[rep_{\hat{t}}(v|_{\bar{q}}\sigma')]_{\bar{q}}[\bar{y}'\sigma'_{rep}]_{\bar{q}'} =_{E_0} w = w[rep_{\hat{t}}(w|_{\bar{r}})]_{\bar{r}}$. By Proposition 3, $top_{\Sigma_0}(v^\circ\sigma^\circ) = top_{\Sigma_0}(w) = \hat{r}$. As $v = v[\bar{y}']_{\bar{q}'}$, $V_{v[]_{\bar{q}}} = \hat{y}'$, and $top_{\Sigma_0}(v) = \hat{q}$ then, for each position r_i in $\hat{r}$, either:

  (i) r_i is a position q_j in $\hat{q}$, so $rep_{\hat{t}}(w|_{r_i}) = w_{r_i} =_{E_0} v^\circ\sigma^\circ|_{q_j} = rep_{\hat{t}}(v|_{q_j}\sigma')$. As $rep_{\hat{t}}(w|_{r_i}) =_{E_0} rep_{\hat{t}}(v|_{q_j}\sigma')$ then, by Remark 1, $rep_{\hat{t}}(w|_{r_i}) = rep_{\hat{t}}(v|_{q_j}\sigma')$, i.e., $w_{r_i} = v^\circ\sigma^\circ|_{q_j} = v^\circ\sigma^\circ|_{r_i}$; or

  (ii) r_i has the form $q'_j.s_k$, where q'_j is a position in $\hat{q}'$. As $\bar{y}' \subseteq dom(\sigma'_{rep})$ then $v^\circ\sigma^\circ|_{q'_j} = y'_j\sigma'_{rep} = rep_{\hat{t}}(y'_j\sigma')$, call $\hat{s} = top_{\Sigma_0}(rep_{\hat{t}}(y'_j\sigma'))$, so $s_k \in \hat{s}$ and $rep_{\hat{t}}(w|_{r_i}) = w_{r_i} =_{E_0} v^\circ\sigma^\circ|_{q'_j.s_k} = rep_{\hat{t}}(y'_j\sigma')|_{s_k} = rep^\circ_{\hat{t}}(y'_j\sigma'|_{s_k}) = rep_{\hat{t}}(y'_j\sigma'|_{s_k})$. As $rep_{\hat{t}}(w|_{r_i}) =_{E_0} rep_{\hat{t}}(y'_j\sigma'|_{s_k})$ then, by Remark 1, $rep_{\hat{t}}(w|_{r_i}) = rep_{\hat{t}}(y'_j\sigma'|_{s_k})$, i.e., $w_{r_i} = v^\circ\sigma^\circ|_{q_j.s_k} = v^\circ\sigma^\circ|_{r_i}$.

  In conclusion, as $v^\circ\sigma^\circ[]_{\bar{r}} = w[]_{\bar{r}}$, $v^\circ\sigma^\circ = v^\circ\sigma^\circ[v^\circ\sigma^\circ|_{\bar{r}}]_{\bar{r}} = w[v^\circ\sigma^\circ|_{\bar{r}}]_{\bar{r}} = w[w|_{\bar{r}}]_{\bar{r}} = w$.

We have just proved $u^\circ\sigma^\circ =_B v^\circ\sigma^\circ$, but also:

- as $\phi^\circ_u = (\bigwedge_{i=1}^{i_x} x_i = u|_{p_i})$ and $\bar{x}\sigma^\circ = rep_{\hat{t}}(u|_{\bar{p}}\sigma') =_{E_0} u|_{\bar{p}}\sigma' = u|_{\bar{p}}\sigma^\circ$ then $E_0 \vDash \phi^\circ_u\sigma^\circ$, and
- as $\phi^\circ_v = (\bigwedge_{j=1}^{i_y} y_j = v|_{q_j})$ and $\bar{y}\sigma^\circ = rep_{\hat{t}}(v|_{\bar{q}}\sigma') =_{E_0} v|_{\bar{q}}\sigma' = v|_{\bar{q}}\sigma^\circ$ then $E_0 \vDash \phi^\circ_v\sigma^\circ$,

so $E_0 \vDash (\phi^\circ_u \wedge \phi^\circ_v)\sigma^\circ$. □

5 Strategies

In this section we present the combinators of a strategy language suitable for narrowing, which is a subset of the Maude strategy language [MOMV04, EMOMV07, RMPV18], a set-theoretic semantics for the language, and an interpretation of this semantics. We also define the set of variables of a strategy and the result of the application of a substitution to a strategy.

A *call strategy* is a name given to a strategy to simplify the development of more complex strategies. A *call strategy definition* is a user-defined association of a strategy to one call strategy.

A rewrite theory $\mathcal{R} = (\Sigma, E, R)$ and a set of call strategy definitions for $\mathcal{R}$, written $Call_{\mathcal{R}}$, have an associated set of *derivation rules* $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$ that will be used in the following.

5.1 Open and closed goals, derivation rules and proof trees

Definition 27 (Open and closed goal). *An* open goal *has the form $t \to v/ST$, where t, its* head*, and v are terms in $\mathcal{H}_\Sigma$, and ST is a strategy; a* closed goal *has the form $\overline{G}$, with G an open goal.*

Definition 28 (Derivation rule). *A derivation rule has the form $\frac{}{G}$ or $\frac{G_1 \cdots G_n}{G}$, where G and each G_i, $1 \leq i \leq n$, are open goals. In either case the* head *of the rule is G.*

Definition 29 (Proof tree). *Given a rewrite theory $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ and a set of call strategy definitions $Call_{\mathcal{R}}$,* a proof tree T, its depth, and its number of nodes *are inductively defined as either:*

- *an open or closed goal, G or $\overline{G}$, with depth 1 and number of nodes 1, or*
- *a derivation tree $\frac{T_1 \cdots T_n}{G}$, constructed by application of the derivation rules in $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$, where each T_i, $1 \leq i \leq n$, is a proof tree, we call $T_1 \cdots T_n$ a* forest*, the depth of T is 1 plus the maximum of the depths of $\overline{T}$, and the number of nodes of T is 1 plus the sum of the number of nodes in $\overline{T}$.*

The head *of T is G in all cases, and we write $head(T) = G$. T is said to be* closed *if it has no open goals on it. We denote by V_T the set of all the variables appearing in T, $V_{\mathcal{R}}$ to the set of all the variables appearing in R and B, $V_{Call_{\mathcal{R}}}$ the set of all the variables appearing in $Call_{\mathcal{R}}$, and $V_{\mathcal{R},Call_{\mathcal{R}}} = V_{\mathcal{R}} \cup V_{Call_{\mathcal{R}}}$. We will use the letter F, with or without subindex, to represent forests in a closed proof tree, c.p.t. from now on.*

Definition 30 (Application of a derivation rule to an open goal). *Given any open goal $t \to v/ST$ in a proof tree and a derivation rule with head $t' \to v'/ST$ such that $t =_E t'$ and $v =_E v'$, the application of the rule to the open goal consists in putting the derivation rule in place of the open goal, but replacing t' with t and v' with v anywhere in the rule.*

5.2 Strategies and their semantics

We present now the semantics that defines the result of the application of a strategy to the equivalence class of a term, which is based on the construction of closed proof trees. It is given by a function (in mix-fix notation)

$$\_@\_ \ : \ Strat_{\mathcal{R},Call_{\mathcal{R}}} \times \mathcal{H}_{\Sigma/E} \longrightarrow \mathcal{P}(\mathcal{H}_{\Sigma/E}),$$

with $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ and $E = E_0 \cup B$, where $[v]_E$ is an element of $ST \mathbin{@} [t]_E$ if and only if a c.p.t. with head $t \to v/ST$ can be constructed using the derivation rules in $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$, also defined below.

If $[v]_E \in ST \mathbin{@} [t]_E$, as any subtree of a c.p.t. for $t \to v/ST$, with head say $t' \to v'/ST'$, is closed then also $[v']_E \in ST' \mathbin{@} [t']_E$.

The set $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$ does not need to be computable. We will prove in this work that if a c.p.t. can be formed from an instance $G\sigma$ of a goal G (i.e., σ is a *solution* of G), then the narrowing calculus that we present can find a more general solution to the goal G, i.e., one that can be instantiated to σ.

In this work we also assume, without loss of generality, that $vars(B) \cap vars(ST) = \emptyset$ for any strategy ST in $Call_{\mathcal{R}}$, by renaming the variables in B. Now, we define $Call_{\mathcal{R}}$, $Strat_{\mathcal{R},Call_{\mathcal{R}}}$, and $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$.

We will use the following set of strategies for narrowing, which is a subset of the Maude strategy language for rewriting [MOMV04, EMOMV07, RMPV18]:

5.2.1 `Idle` and `fail`

These are constant strategies that always belong to $Strat_{\mathcal{R},Call_{\mathcal{R}}}$. While the first always succeeds, the second always fails. For each equivalence class $[t]_E \in \mathcal{H}_{\Sigma/E}$ there is a derivation rule $\overline{t \to t/\mathtt{idle}}$ in $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$. There are no derivation rules for `fail`. Then, $\mathtt{idle} \mathbin{@} [t]_E = \{[t]_E\}$ and $\mathtt{fail} \mathbin{@} [t]_E = \emptyset$. We define $vars(\mathtt{idle}) = vars(\mathtt{fail}) = \emptyset$. For any substitution δ we define $\mathtt{idle}\,\delta = \mathtt{idle}$, and $\mathtt{fail}\,\delta = \mathtt{fail}$.

Example 10. *Suppose that $t =_E v$ and we have the open goal $t \to v/\mathtt{idle}$ in a derivation tree. There is a term t' and a derivation rule $\overline{t' \to t'/\mathtt{idle}}$ in $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$ such that $t =_E t'$. As $t =_E v$ then also $v =_E t'$, so we can apply this rule to the open goal. Thus, we replace the first t' in the rule with t and the second one with v, yielding $\overline{t \to v/\mathtt{idle}}$, a c.p.t. that we put in place of the open goal, so $[v]_E \in \mathtt{idle} \mathbin{@} [t]_E$. The result $[v]_E \in \mathtt{idle} \mathbin{@} [t]_E$ was expected, since $\mathtt{idle} \mathbin{@} [t]_E = \{[t]_E\}$ and $t =_E v$ imply $[v]_E = [t]_E$.*

5.2.2 Rule application

A rule of R that has no rewrite conditions and a substitution form a *rule application.*

⟨AlphaNum⟩	::=	$A \mid \cdots \mid Z \mid a \mid \cdots \mid z \mid 0 \mid \cdots \mid 9$
⟨Label⟩	::=	⟨AlphaNum⟩
	|	⟨AlphaNum⟩⟨Label⟩
⟨Assignment⟩	::=	⟨Variable⟩ $\mapsto$ ⟨$\mathcal{T}_\Sigma(\mathcal{X})$-term⟩
⟨Assignment List⟩	::=	⟨Assignment⟩
	|	⟨Assignment⟩ ; ⟨Assignment List⟩
⟨Substitution⟩	::=	*none*
	|	⟨Assignment List⟩
⟨RuleApplic⟩	::=	⟨Label⟩ [⟨Substitution⟩]
⟨Strat⟩	::=	⟨RuleApplic⟩

If $c : l \to r \ \mathit{if}\ \psi$ is a rule in R, and $\gamma : \mathcal{X} \to \mathcal{T}_\Sigma(\mathcal{X} \setminus V_{\mathcal{R},Call_{\mathcal{R}}})$ is a substitution such that $dom(\gamma) \subseteq vars(c)$, then $c[\gamma]$ is a rule application in $Strat_{\mathcal{R},Call_{\mathcal{R}}}$. For each pair of terms t, v in $\mathcal{H}_\Sigma$, if $t \underset{c\gamma}{\longrightarrow}{}^1_R\, v$ then there is a derivation rule

$$\overline{t \to v/c[\gamma]}$$

in $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$.

We define $vars(c[\gamma]) = ran(\gamma)$. The application of $\delta : \mathcal{X} \to \mathcal{T}_\Sigma(\mathcal{X} \setminus V_{\mathcal{R},Call_{\mathcal{R}}})$ to $c[\gamma]$ is defined as $c[\gamma]\delta = c[(\gamma\delta)_{dom(\gamma)}]$.

Example 11. *The set $Call_{\mathcal{R}}$ for the running example contains the rule application* `kitchen`$[none]$.

For rules with rewrite conditions, a strategy must be supplied for each rewrite condition.

$$
\begin{array}{lcl}
\langle \text{StratList} \rangle & ::= & \langle \text{Strat} \rangle \\
& | & \langle \text{Strat} \rangle \text{ , } \langle \text{StratList} \rangle \\
\langle \text{RuleApplic} \rangle & ::= & \langle \text{Label} \rangle \text{ [} \langle \text{Substitution} \rangle \text{] \{ } \langle \text{StratList} \rangle \text{ \}}
\end{array}
$$

If $c : l \to r \textit{ if } \bigwedge_{j=1}^{m} l_j \to r_j \mid \psi$ is a rule in R, $\gamma : \mathcal{X} \to \mathcal{T}_\Sigma(\mathcal{X} \setminus V_{\mathcal{R},Call_\mathcal{R}})$ is a substitution such that $dom(\gamma) \subseteq vars(c)$, and $\overline{ST} = ST_1, \ldots, ST_m$ is an ordered list of strategies such that $dom(\gamma) \cap vars(\overline{ST}) = \emptyset$, then $RA = c[\gamma]\{\overline{ST}\}$ is a rule application in $Strat_{\mathcal{R},Call_\mathcal{R}}$.

We define $vars(RA) = ran(\gamma) \cup vars(\overline{ST})$. The application of $\delta : \mathcal{X} \to \mathcal{T}_\Sigma(\mathcal{X} \setminus V_{\mathcal{R},Call_\mathcal{R}})$ to RA is defined as $RA\delta = c[(\gamma\delta)_{dom(\gamma)}]\{\overline{ST}\delta\}$. For each substitution $\delta : vars(c\gamma) \to \mathcal{T}_\Sigma$ such that $E_0 \vDash \psi\gamma\delta$, each term u in $\mathcal{H}_\Sigma$, and each position p in $pos(u)$ such that $u|_p = l\gamma\delta$ there is a derivation rule

$$
\frac{l_1\gamma\delta \to r_1\gamma\delta/ST_1\delta \cdots l_m\gamma\delta \to r_m\gamma\delta/ST_m\delta}{u \to u[r\gamma\delta]_p/RA}
$$

in $\mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$.

$[t]_E \in c[\gamma]@[u|_p]_E$ implies $[u[t]_p]_E \in c[\gamma]@[u]_E$, and $[t]_E \in c[\gamma]\{\overline{ST}\}@[u|_p]_E$ implies $[u[t]_p]_E \in c[\gamma]\{\overline{ST}\}@[u]_E$ because no specific position is required for rewriting using a rule application.

Example 12. *The set $Call_\mathcal{R}$ for the running example contains and enhanced version of the rule application* $\texttt{cook}[none]\{(\texttt{toast1}[none] \mid \texttt{toast2}[none]), (\texttt{toast1}[none] \mid \texttt{toast2}[none])\}$*, where the symbol* $\mid$ *represents the* `or` *strategy (defined below). Rule* $[cook] : \texttt{cook}(y; h_{\texttt{rt}}\, v_{\texttt{t}}, z) \to y + z; h'_{\texttt{rt}}\, v'_{\texttt{t}} \textit{ if } \texttt{toast}(h_{\texttt{rt}}, z) \to h'_{\texttt{rt}} \wedge \texttt{toast}(v_{\texttt{t}}, z) \to v'_{\texttt{t}}$*, will be applied only if we can apply either the rule application* $\texttt{toast1}[none]$ *or the rule application* $\texttt{toast2}[none]$ *to each condition in the rule.*

5.2.3 Top

It is possible to restrict the application of a rule in R only to the top of the term. This is useful for structural rules, that are applied to the whole state, or for the strategies applied on the conditional part of a rule, as will be shown in our running example.

$$
\langle \text{Strat} \rangle \;::=\; \texttt{top(} \langle \text{RuleApplic} \rangle \texttt{)}
$$

If $c : l \to r \textit{ if } \psi$ is a rule in R and $\gamma : \mathcal{X} \to \mathcal{T}_\Sigma(\mathcal{X} \setminus V_{\mathcal{R},Call_\mathcal{R}})$ is a substitution such that $dom(\gamma) \subseteq vars(c)$, then $\texttt{top}(c[\gamma])$ is a strategy in $Strat_{\mathcal{R},Call_\mathcal{R}}$. We define $vars(\texttt{top}(c[\gamma])) = vars(c[\gamma])$ and $\texttt{top}(c[\gamma])\delta = \texttt{top}(c[\gamma]\,\delta)$. For each substitution $\delta : vars(c\gamma) \to \mathcal{T}_\Sigma$ such that $E_0 \vDash \psi\gamma\delta$ there is a derivation rule

$$
\frac{}{l\gamma\delta \to r\gamma\delta/\texttt{top}(c[\gamma])}
$$

in $\mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$.

If $c : l \to r \textit{ if } \bigwedge_{j=1}^{m} l_j \to r_j \mid \psi$ is a rule in R, $\gamma : \mathcal{X} \to \mathcal{T}_\Sigma(\mathcal{X} \setminus V_{\mathcal{R},Call_\mathcal{R}})$ is a substitution such that $dom(\gamma) \subseteq vars(c)$, $\overline{ST} = ST_1, \ldots, ST_m$ is an ordered list of strategies such that $dom(\gamma) \cap vars(\overline{ST}) = \emptyset$ and we call $RA = c[\gamma]\{\overline{ST}\}$, then $\texttt{top}(RA)$ is a strategy in $Strat_{\mathcal{R},Call_\mathcal{R}}$. We define $vars(\texttt{top}(RA)) = vars(RA)$ and $\texttt{top}(RA)\delta = \texttt{top}(RA\,\delta)$, for $\delta : \mathcal{X} \to \mathcal{T}_\Sigma(\mathcal{X} \setminus V_{\mathcal{R},Call_\mathcal{R}})$. For each substitution $\delta : vars(c\gamma) \to \mathcal{T}_\Sigma$ such that $E_0 \vDash \psi\gamma\delta$, there is a derivation rule

$$
\frac{l_1\gamma\delta \to r_1\gamma\delta/ST_1\delta \cdots l_m\gamma\delta \to r_m\gamma\delta/ST_m\delta}{l\gamma\delta \to r\gamma\delta/\texttt{top}(RA)}
$$

in $\mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$.

Example 13. *Whenever a rule application appears in the set $Call_\mathcal{R}$ for the running example, it is as part of a* `top` *strategy, e.g.,* $\texttt{top}(\texttt{kitchen}[none])$.

5.2.4 Call strategy

Call strategy definitions allow the use of parameters and the implementation of recursive strategies. A call strategy definition can be either unconditional or conditional.

⟨VarList⟩	::=	⟨Variable⟩
	|	⟨Variable⟩ , ⟨VarList⟩
⟨Equational Condition⟩	::=	⟨$\mathcal{H}_\Sigma(\mathcal{X})$-term⟩ = ⟨$\mathcal{H}_\Sigma(\mathcal{X})$-term⟩
	|	⟨Equational Condition⟩ ∧ ⟨Equational Condition⟩
⟨Strat Condition⟩	::=	⟨quantifier-free formula⟩
	|	⟨Equational Condition⟩ ∧ ⟨quantifier-free formula⟩
⟨Call Strat⟩	::=	`sd` ⟨Label⟩ ::= ⟨Strat⟩
	|	`sd` ⟨Label⟩(⟨VarList⟩) := ⟨Strat⟩
		`csd` ⟨Label⟩(⟨VarList⟩) := ⟨Strat⟩ `if` ⟨Strat Condition⟩
⟨Arguments⟩	::=	⟨$\mathcal{H}_\Sigma(\mathcal{X})$-term⟩
	|	⟨$\mathcal{H}_\Sigma(\mathcal{X})$-term⟩ , ⟨Arguments⟩
⟨Strat⟩	::=	⟨Label⟩
	|	⟨Label⟩ (⟨Arguments⟩)

The semantics for *call strategy invocations*, given a pair of terms t and v in $\mathcal{H}_\Sigma$ such that $ls(t) \equiv_\leq ls(v)$ is:

- If `sd` CS := $ST \in Call_\mathcal{R}$ then the call strategy invocation CS is a strategy in $Strat_{\mathcal{R},Call_\mathcal{R}}$. We define $vars(CS) = \emptyset$ and, for any substitution δ, $CS\delta = CS$. For every renaming γ such that $dom(\gamma) \subseteq vars(ST)$ and $ran(\gamma)$ is away from any known variable, there is a derivation rule
$$\frac{t \to v/ST\gamma}{t \to v/CS}$$
in $\mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$.

- If `sd` $CS(\bar{x})$:= $ST \in Call_\mathcal{R}$, where $\bar{x} = x^1_{s_1}, \ldots, x^n_{s_n}$ are the *parameters* of CS, $\hat{x} \subseteq vars(ST)$, $t_1, \ldots, t_n$ are terms in $\mathcal{T}_\Sigma(\mathcal{X} \setminus V_{\mathcal{R},Call_\mathcal{R}})$, with sorts $s_1, \ldots, s_n$ respectively, and we call $\bar{t} = t_1, \ldots, t_n$, then the call strategy invocation $CS(\bar{t})$ is a strategy in $Strat_{\mathcal{R},Call_\mathcal{R}}$. If $\rho = \{\bar{x} \mapsto \bar{t}\}$ then $vars(CS(\bar{t})) = ran(\rho)$. If $\delta : \mathcal{X} \to \mathcal{T}_\Sigma(\mathcal{X} \setminus \hat{x})$, then we define $CS(\bar{t})\delta = CS(\bar{t}\delta)$. For every renaming γ such that $dom(\gamma) \subseteq vars(ST) \setminus \hat{x}$ and $ran(\gamma)$ is away from any known variable, there is a derivation rule
$$\frac{t \to v/ST(\gamma \cup \rho)}{t \to v/CS(\bar{t})}$$
in $\mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$.

- If `csd` $CS(\bar{x})$:= ST *if* $C \in Call_\mathcal{R}$, with $\bar{x} = x^1_{s_1}, \ldots, x^n_{s_n}$ and $C = \bigwedge_{j=1}^m (l_j = r_j) \wedge \phi$, call $V_{CS} = vars(ST) \cup vars(C)$, $\hat{x} \subseteq V_{CS}$, $t_1, \ldots, t_n$ are terms in $\mathcal{T}_\Sigma(\mathcal{X} \setminus V_{\mathcal{R},Call_\mathcal{R}})$, with sorts $s_1, \ldots, s_n$ respectively, call $\bar{t} = t_1, \ldots, t_n$, then the call strategy invocation $CS(\bar{t})$ is a strategy in $Strat_{\mathcal{R},Call_\mathcal{R}}$. If $\rho = \{\bar{x} \mapsto \bar{t}\}$ then $vars(CS(\bar{t})) = ran(\rho)$. If $\delta : \mathcal{X} \to \mathcal{T}_\Sigma(\mathcal{X} \setminus (ran(\rho) \cup V_{\mathcal{R},Call_\mathcal{R}}))$, then we define $CS(\bar{t})\delta = CS(\bar{t}\delta)$. For every renaming γ such that $dom(\gamma) \subseteq V_{CS} \setminus \hat{x}$ and $ran(\gamma)$ is away from any known variable, and each substitution $\delta : vars(C(\gamma \cup \rho)) \to \mathcal{T}_\Sigma$ such that $l_j(\gamma \cup \rho)\delta =_E r_j(\gamma \cup \rho)\delta$, for $1 \leq j \leq n$, and $E_0 \vDash \phi(\gamma \cup \rho)\delta$, there is a derivation rule
$$\frac{t \to v/ST(\gamma \cup \rho)\delta}{t \to v/CS(\bar{t})}$$
in $\mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$.

The meaning of γ in all three cases is that the names of the variables in ST that we could call free, with respect to CS, have no relevance. By using renaming, different instances of a call strategy will get different variable names in the narrowing calculus that we have developed.

Example 14. *The call strategy definition* $\texttt{sd toasts} := \texttt{top}(\texttt{toast1}[none]) \mid \texttt{top}(\texttt{toast2}[none])$ *allows us to rewrite the strategy in example 12 as* $\texttt{top}(\texttt{cook}[none]\{\texttt{toasts}, \texttt{toasts}\})$.

5.2.5 Tests

Tests are strategies that check a property on an equivalence class $[t]_E$ in $\mathcal{H}_{\Sigma/E}$. If the property holds then the test returns a set containing $[t]_E$ as its only element. Otherwise, the test returns the empty set.

$$\begin{array}{lcl} \langle \text{Test} \rangle & ::= & \texttt{match}\ \langle \mathcal{H}_\Sigma(\mathcal{X})\text{-term} \rangle\ \texttt{s.t.}\ \langle \text{Strat Condition} \rangle \\ \langle \text{Strat} \rangle & ::= & \langle \text{Test} \rangle \end{array}$$

For simplicity of notation, there will always be one quantifier-free formula $\phi \in QF(\mathcal{X}_0)$ as last element of the test condition, which will be the boolean term *true* if there are no built-in conditions to check.

For each equivalence class $[t]_E$ in $\mathcal{H}_{\Sigma/E}$, and each test strategy $TS = \texttt{match}\, u\, \texttt{s.t.}\ \bigwedge_{j=1}^{m}(l_j = r_j) \wedge \phi$, if there exists a substitution $\delta : vars(TS) \to \mathcal{T}_\Sigma$, where we define $vars(TS) = vars(u) \cup vars(\phi) \cup \bigcup_{j=1}^{m} vars((l_j, r_j))$, such that $t =_E u\delta$, $l_j\delta =_E r_j\delta$, for $1 \leq j \leq m$, and $E_0 \vDash \phi\delta$, then there is a rule

$$\frac{}{t \to t/\texttt{match}\, u\, \texttt{s.t.}\ \bigwedge_{j=1}^{m}(l_j = r_j) \wedge \phi}$$

in $\mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$. If $\delta : \mathcal{X} \to \mathcal{T}_\Sigma(\mathcal{X} \backslash vars(TS))$ then $TS\delta = \texttt{match}\, u\delta\, \texttt{s.t.}\ \bigwedge_{j=1}^{m}(l_j\delta = r_j\delta) \wedge \phi\delta$.

Example 15. *The set* $Call_\mathcal{R}$ *for the running example contains the definition*
$\texttt{sd test} := \texttt{match}\ N/B_\texttt{b}/Y; V_\texttt{t}W_\texttt{t}/OK\ \texttt{s.t.}\ Y < \texttt{ft}$.
This test will be used to verify that the system has not reached the fail time.

5.2.6 If-then-else

Strategies can be combined to be applied over execution paths in several ways. The first way is the if-then-else strategy where a subset of the test strategies, called *simple test*, is used. The term must match some pattern u. If the quantifier-free formula ϕ instantiated with the matching substitution holds, the strategy in the then clause is applied; if not, the strategy in the else clause is applied.

$$\begin{array}{lcl} \langle \text{Simple Test} \rangle & ::= & \texttt{match}\ \langle \mathcal{H}_\Sigma(\mathcal{X})\text{-term} \rangle\ \texttt{s.t.}\ \langle \text{quantifier-free formula} \rangle \\ \langle \text{Strat} \rangle & ::= & \langle \text{Simple Test} \rangle\ ?\ \langle \text{Strat} \rangle : \langle \text{Strat} \rangle \end{array}$$

For each pair of equivalence classes $[t]_E$ and $[v]_E$ in $\mathcal{H}_{\Sigma/E}$, each if-then-else strategy $IS = \texttt{match}\, u\, \texttt{s.t.}\ \phi\, ?\, ST_1 : ST_2$ and each substitution $\delta : vars(u) \cup vars(\phi) \to \mathcal{T}_\Sigma$ such that $t =_E u\delta$, if $E_0 \vDash \phi\delta$, then there is a rule

$$\frac{t \to v/ST_1\delta}{t \to v/\texttt{match}\, u\, \texttt{s.t.}\ \phi\, ?\, ST_1 : ST_2}$$

in $\mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$, and if $E_0 \vDash \neg\phi\delta$ then there is a rule

$$\frac{t \to v/ST_2\delta}{t \to v/\texttt{match}\, u\, \texttt{s.t.}\ \phi\, ?\, ST_1 : ST_2}$$

in $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$. We define $vars(IS) = vars(u) \cup vars(\phi) \cup vars(ST_1) \cup vars(ST_2)$.

$IS\delta = \texttt{match}\, u\delta\, \texttt{s.t.}\, \phi\delta \,?\, ST_1\delta \;:\; ST_2\delta$, for any substitution $\delta : \mathcal{X} \to \mathcal{T}_\Sigma(\mathcal{X} \backslash vars(IS))$.

The restriction to *SMT* conditions is needed to ensure the completeness of the narrowing calculus since, in general, a reachability condition cannot be proved false.

Example 16. *One alternative set* $Call_{\mathcal{R}}$ *for the running example contained the definition*

$\texttt{sd checkExtract} \;:=\; \texttt{match}\, N/B_{\texttt{b}}/Y; [\texttt{ct}, \texttt{ct}]V_{\texttt{t}}/OK \;\texttt{s.t.}\; true \,?\, \texttt{top}(\texttt{dish}[none]) \;:\; \texttt{idle}$

This if-then-else strategy was meant to force the extraction of a fully cooked toast to the dish, pruning the state space of the search for a solution.

5.2.7 Regular expressions

Another way of combining strategies is the use of regular expressions.

<Strat>	::=	<Strat> ; <Strat>	concatenation
<Strat>	::=	<Strat> | <Strat>	union
<Strat>	::=	<Strat> +	iteration (1 or more)
<Strat>	::=	<Strat> *	iteration (0 or more)

Of course, $ST* = \texttt{idle} \mid ST+$. Let ST and ST' be strategies, and let t, v and u be terms in $\mathcal{H}_\Sigma$ such that $ls(t) \equiv_{\le} ls(u) \equiv_{\le} ls(v)$. Then, we have rules

$$\frac{t \to u/ST_1 \quad u \to v/ST_2}{t \to v/ST_1\,;ST_2} \qquad \frac{t \to v/ST_1}{t \to v/ST_1 \mid ST_2} \qquad \frac{t \to v/ST_2}{t \to v/ST_1 \mid ST_2} \qquad \frac{t \to v/ST}{t \to v/ST+} \qquad \frac{t \to v/ST\,;ST+}{t \to v/ST+}$$

in $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$.

We define $vars(ST_1\,;ST_2) = vars(ST_1 \mid ST_2) = vars(ST_1) \cup vars(ST_2)$, and $vars(ST+) = vars(ST)$. The concatenation and union combinators are defined to be right associative, e.g., $ST_1;ST_2;ST_3 = ST_1;(ST_2;ST_3)$. The scope of this work is restricted to concatenated strategies that have no variables in common; this forces iterated strategies to be ground. Substitutions are applied to all the strategies in the regular expression.

We define $tokens(ST+) = tokens(ST)$, $tokens(ST_1\ op\ ST_2) = tokens(ST_1) \cup tokens(ST_2)$ if op is a binary combinator, and $tokens(ST) = ST$ otherwise.

Example 17. *The set* $Call_{\mathcal{R}}$ *for the running example contains the definition*

$\texttt{sd kitchCook} \;:=\; \texttt{top}(\texttt{kitchen}[none]) \;;\; \texttt{top}(\texttt{cook}[none]\{\texttt{toasts}, \texttt{toasts}\})$.

After applying the strategy $\texttt{top}(\texttt{kitchen}[none])$ *to a term with sort* `Kitchen`*, the strategy* $\texttt{top}(\texttt{cook}[none]\{\texttt{toasts}, \texttt{toasts}\})$ *will be applied to each term in the resulting set.*

5.2.8 Rewriting of subterms

The `matchrew` combinator allows the selection of a subterm to apply a rule and extends the scope of the substitution that validates a test strategy to subsequent steps of the execution path.

⟨TermStratList⟩	::=	⟨Variable⟩ `using` ⟨Strat⟩
	|	⟨TermStratList⟩ , ⟨TermStratList⟩
⟨Strat⟩	::=	`matchrew` ⟨$\mathcal{H}_\Sigma(\mathcal{X})$-term⟩ `s.t.` ⟨Strat Condition⟩ `by` ⟨TermStratList⟩

`Matchrew` strategies have the form $MS = \texttt{matchrew}\, u\, \texttt{s.t.}\, \bigwedge_{j=1}^{m}(l_j = r_j) \wedge \phi\ \texttt{by}\ x^1_{s_1} \texttt{using}\ ST_1, \ldots, x^n_{s_n}\, \texttt{using}\, ST_n$, where $\bar{x} = x^1_{s_1}, \ldots, x^n_{s_n}$ are the *match parameters* of *MS*, $\hat{x} \subset \mathcal{X}_1$, $|\hat{x}| = n$, $u = u[\bar{x}]_{\bar{p}}$, for proper $\bar{p}$, $\hat{l} \cup \hat{r} \subset \mathcal{H}_\Sigma(\mathcal{X})$, and, for $1 \le i \le n$, $x^i_{s_i}$ does not appear as a match parameter of another `matchrew` strategy in $\overline{ST}$ and for each $i \in \{1, \ldots, n\}$ such that $ST_i \neq \texttt{idle}$ there exists $j \in \{1, \ldots, m\}$ such that $l_j = x^i_{s_i}$ and $r_j \in \mathcal{H}_\Sigma(\mathcal{X}) \setminus \mathcal{X}$. We define $vars(MS) =$

$V_{u,\phi,\bar{l},\bar{r},\overline{ST}}$. We will also use the short-form MS = $\texttt{matchrew}\, u\, \texttt{s.t.}\, \bar{l} = \bar{r} \wedge \phi\ \texttt{by}\ \bar{x}\ \texttt{using}\ \overline{ST}$. If $\delta : \mathcal{X} \to \mathcal{T}_\Sigma(\mathcal{X}\backslash vars(MS))$, call $\delta' = \delta_{\backslash \hat{x}}$, then $MS\delta = \texttt{matchrew}\, u\delta'\, \texttt{s.t.}\, \bar{l}\delta' = \bar{r}\delta' \wedge \phi\delta'\ \texttt{by}\ \bar{x}\ \texttt{using}\ \overline{ST}\delta'$. For each n-tuple $(t_1, \ldots, t_n)$ of terms in $\mathcal{H}^n_\Sigma$ such that $ls(\bar{t}) \leq \bar{s}$, and each substitution δ such that $\delta_{vars(MS)} : \mathcal{X} \to \mathcal{T}_\Sigma(\mathcal{X}\backslash vars(MS))$, so $\delta_{vars(MS)}$ is idempotent, $u\delta \in \mathcal{T}_\Sigma$, $\{l_j\delta, r_j\delta\}^m_{j=1} \subset \mathcal{T}_\Sigma$, $\bar{l}\delta =_E \bar{r}\delta$, $\phi\delta \in \mathcal{T}_\Sigma$, and $E_0 \vDash \phi\delta$, so $ran(\delta_{vars(MS)}) \subseteq vars(\overline{ST}\delta)$, there is a derivation rule

$$\frac{x^1_{s_1}\delta \to t_1/ST_1\delta \cdots x^n_{s_n}\delta \to t_n/ST_n\delta}{u\delta \to u\delta[t_1, \ldots, t_n]_{p_1 \ldots p_n}/MS}$$

in $\mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$. For any structure Δ, we call $matchParam(\Delta)$ the set of all the match parameters that appear in Δ.

In narrowing, rewrite rules are intended to be applied, using unification, to non variable terms. The restriction that forces a variable $x^i_{s_i}$ to match with a non variable term of $\mathcal{H}_\Sigma(\mathcal{X})$, ensures that the narrowing calculus developed does not loose any solution, because this variable will be instantiated to a non variable term prior to trying to apply a rewrite rule to it.

Example 18. *The set $Call_\mathcal{R}$ for the running example contains the definition*
$\texttt{sd cook1 := matchrew}\ N/B_\texttt{b}/K_\texttt{k}/OK\ \texttt{s.t.}\ K_\texttt{k} = Y; R_\texttt{rt}V_\texttt{t}\ \texttt{by}\ K_\texttt{k}\ \texttt{using kitchCook}$.
The strategy `kitchCook` *will be applied to the* ***Kitchen*** $K_\texttt{k}$ *of a* ***State****, whenever there is a* ***RealToast*** *($R_\texttt{rt}$) in $K_\texttt{k}$, and $K_\texttt{k}$ will get instantiated to a non-variable term by the condition.*

Definition 31 (Subterms, holes, and replacement in a strategy)**.** *We extend the use of subterms and holes to strategies. If ST is a strategy, i is a positive integer, p is a position, and t is a term, then $ST|_{i.p}$ is the subterm that appears at position p in the term i of the tuple formed by all terms that appear in ST, taken from left to right, $ST[]_{i.p}$ consists in the replacement in $ST|_i$ of its subterm at position p with $[]$, and $ST[t]_{i.p}$ consists in the replacement in $ST|_i$ of its subterm at position p with t.*

Definition 32 (Equality modulo of strategies)**.** *Given two strategies ST and ST', we say that ST is equal modulo E to ST', and write $ST =_E ST'$ iff $ST = ST'[\bar{t}]_{\bar{p}}$, for proper $\bar{t}$ and $\bar{p}$, and for each position p in $\bar{p}$ $ST|_p =_E ST'|_p$ and $V_{ST|_p} = V_{ST'|_p}$.*

5.3 Interpretation of the semantics. Generalization of strategies

Lemma 5 (Interpretation of the semantics)**.** *Given a rewrite theory $\mathcal{R} = (\Sigma, E_0 \cup B, R)$, a set of call strategy definitions $Call_\mathcal{R}$, and terms $t, v \in \mathcal{H}_\Sigma$, for each c.p.t. T formed using the rules in $\mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$ with head $t \to v/ST$, so $[v]_E \in ST@[t]_E$, each renaming α such that $ran(\alpha) \cap (V_T \cup V_{\mathcal{R},Call_\mathcal{R}}) = \emptyset$, and each strategy $ST' =_E ST$ it holds that:*

1. *Main property: $t \to_{R/E} v$ and there exist closed proof trees for $[v]_E \in ST\alpha@[t]_E$ and $[v]_E \in ST'@[t]_E$ with the same depth and number of nodes as T.*
2. *If $ST = \texttt{idle}$ then $[t]_E = [v]_E$.*
3. *If $ST = c[\gamma]$ then $t \xrightarrow[c\gamma]{}{}^1_{R/E}\ v$.*
4. *If $ST = \texttt{top}(c[\gamma])$, then $t \xrightarrow[c\gamma,\epsilon]{}{}^1_{R/E}\ v$ (i.e., the rewrite happens at the top position of t).*
5. *If $ST = \texttt{match}\, u\, \texttt{s.t.}\, \bigwedge^m_{j=1}(l_j = r_j) \wedge \phi$ then $[t]_E = [v]_E$ and there exists a substitution σ such that $t =_E u\sigma$, $l_j\sigma =_E r_j\sigma$, for $1 \leq j \leq m$, and $E_0 \vDash \phi\sigma$.*
6. *If $ST = ST_1 ; ST_2$ then there exists a term $u \in \mathcal{H}_\Sigma$ such that $[u]_E \in ST_1@[t]_E$ and $[v]_E \in ST_2@[u]_E$.*

7. *If* $ST = ST_1+$ *then there exist* $i+1$ *terms* $u_0 = t, u_1, \ldots, u_{i-1}, u_i = v \in \mathcal{H}_\Sigma$*, with* $i > 0$*, such that* $[u_j]_E \in ST_1@[u_{j-1}]_E$*, for* $1 \leq j \leq i$*, where* i *is equal to one plus the number of times that a rule with the form* $\frac{w_1 \to w_2/ST_1\,;\,ST_1+}{w_1 \to w_2/ST_1+}$*, followed by the application of a rule with the form* $\frac{\overline{w_1 \to w'/ST_1} \quad \overline{w' \to w_2/ST_1+}}{w_1 \to w_2/ST_1\,;\,ST_1+}$*, is applied in the rightmost branch of the subtree before applying a rule with the form* $\frac{w_1 \to w_2/ST_1}{w_1 \to w_2/ST_1+}$.

8. *If* $ST = ST_1 \mid ST_2$ *then* $[v]_E \in ST_1@[t]_E$ *or* $[v]_E \in ST_2@[t]_E$.

9. *If* $ST = \texttt{match}\, u\, \texttt{s.t.}\, \phi\, ?\, ST_1 \;:\; ST_2$ *then there exists a substitution* δ *such that* $t =_E u\delta$ *and either* $E_0 \vDash \phi\delta$ *and* $[v]_E \in ST_1\delta@[t]_E$ *or* $E_0 \vDash \neg\phi\delta$ *and* $[v]_E \in ST_2\delta@[t]_E$.

10. *If* $ST = CS$*, where* $\texttt{sd}\ CS := ST_1 \in Call_\mathcal{R}$*, then: (i)* $[v]_E \in ST_1@[t]_E$*, and (ii)* $[v]_E \in ST_1\gamma@[t]_E$*, for every renaming* γ *such that* $dom(\gamma) \subseteq vars(ST_1) \setminus V_\mathcal{R}$ *and* $ran(\gamma) \cap V_{\mathcal{R},Call_\mathcal{R}} = \emptyset$.

11. *If* $ST = CS(\bar{t})$*, where* $\texttt{sd}\ CS(\bar{x}) := ST_1 \in Call_\mathcal{R}$*,* $\bar{x} = x^1_{s_1}, \ldots, x^n_{s_n}$*,* $\bar{t} = t_1, \ldots, t_n$*, and* $\rho = \{\bar{x} \mapsto \bar{t}\}$*, then: (i)* $[v]_E \in ST_1\rho@[t]_E$ *and (ii) if* γ *is a renaming such that* $dom(\gamma) \subseteq vars(ST_1)\setminus\hat{x}$ *and* $ran(\gamma)\cap(ran(\rho)\cup V_{\mathcal{R},Call_\mathcal{R}}) = \emptyset$ *(so* $\frac{t \to v/ST_1(\gamma\cup\rho)}{t \to v/CS(\bar{t})} \in \mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$*), then* $[v]_E \in ST_1(\gamma\cup\rho)@[t]_E$.

12. *If* $ST = CS(\bar{t})$*, where* $\texttt{csd}\ CS(\bar{x}) := ST_1$ *if* $C \in Call_\mathcal{R}$*, with* $\bar{x} = x^1_{s_1}, \ldots, x^n_{s_n}$ *and* $C = \bigwedge_{j=1}^m (l_j = r_j) \wedge \phi$*, call* $V_{CS} = vars(ST_1) \cup vars(C)$*,* $\hat{x} \subseteq V_{CS}$*,* $\bar{t} = t_1, \ldots, t_n$*, and* $\rho = \{\bar{x} \mapsto \bar{t}\}$*, then (i) there exists a substitution* $\delta_1 : vars(C\rho) \to \mathcal{T}_\Sigma$*, such that* $l_j\rho\delta_1 =_E r_j\rho\delta_1$*, for* $1 \leq j \leq n$*,* $E_0 \vDash \phi\rho\delta_1$ *(so* $\frac{t \to v/ST_1\rho\delta_1}{t \to v/CS(\bar{t})} \in \mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$*), and* $[v]_E \in ST_1\rho\delta_1@[t]_E$*, and (ii) for every renaming* γ *such that* $dom(\gamma) \subseteq V_{CS}\setminus\hat{x}$ *and* $ran(\gamma)\cap(ran(\rho)\cup V_{\mathcal{R},Call_\mathcal{R}}) = \emptyset$*, there exists a substitution* $\delta_2 : vars(C(\gamma\cup\rho)) \to \mathcal{T}_\Sigma$*, such that* $l_j(\gamma\cup\rho)\delta_2 =_E r_j(\gamma\cup\rho)\delta_2$*, for* $1 \leq j \leq n$*,* $E_0 \vDash \phi(\gamma\cup\rho)\delta_2$ *(so* $\frac{t \to v/ST_1(\gamma\cup\rho)\delta_2}{t \to v/CS(\bar{t})} \in \mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$*), and* $[v]_E \in ST_1(\gamma\cup\rho)\delta_2@[t]_E$.

13. *If* $ST = c[\gamma]\{ST_1, \ldots, ST_m\}$*, with* $c : l \to r$ *if* $\bigwedge_{j=1}^m l_j \to r_j \mid \psi$ *a rule in* R*, then there is a substitution* δ *such that* $[r_i\gamma\delta]_E \in ST_i\delta\ @\ [l_i\gamma\delta]_E$*, for* $1 \leq i \leq m$*, and* $t \xrightarrow[c,\gamma\delta]{}{}^1_{R/E}\ v$.

14. *If* $ST = \texttt{top}(c[\gamma]\{ST_1, \ldots, ST_m\})$*, with* $c : l \to r$ *if* $\bigwedge_{j=1}^m l_j \to r_j \mid \psi$ *a rule in* R *then there is a substitution* δ *such that* $[r_i\gamma\delta]_E \in ST_i\delta@[l_i\gamma\delta]_E$*, for* $1 \leq i \leq m$*, and* $t \xrightarrow[c,\epsilon,\gamma\delta]{}{}^1_{R/E}\ v$.

15. *If* $ST = \texttt{matchrew}\ u\ \texttt{s.t.}\ \bigwedge_{j=1}^m (l_j = r_j) \wedge \phi\ \texttt{by}\ x^1_{s_1}\ \texttt{using}\ ST_1, \ldots,\ x^n_{s_n}\ \texttt{using}\ ST_n$*, where* $u = u[x^1_{s_1}, \ldots, x^n_{s_n}]_{p_1\ldots p_n}$ *then there exist a substitution* δ*, where* $\delta_{V_{u,\phi,\bar{l},\bar{r}}}$ *is ground, and terms* $t_1, \ldots, t_n \in \mathcal{H}_\Sigma$ *such that* $t =_E u\delta$*,* $l_j\delta =_E r_j\delta$*, for* $1 \leq j \leq m$*,* $E_0 \vDash \phi\delta$*,* $[t_i]_E \in ST_i\delta\ @\ [x^i_{s_i}\delta]_E$*, for* $1 \leq i \leq n$*, and* $v =_E u\delta[t_1, \ldots, t_n]_{p_1\ldots p_n}$.

Proof. The proof is done by induction on the depth of the c.p.t for $t \to v/ST$. □

Lemma 6 (Generalization of strategies)**.** *Given a rewrite theory* $\mathcal{R} = (\Sigma, E_0 \cup B, R)$*, a set of call strategy definitions* $Call_\mathcal{R}$*, terms* $t, v \in \mathcal{H}_\Sigma$*, a strategy* $ST \in Strat_{\mathcal{R},Call_\mathcal{R}}$*, and a substitution* σ *such that* $dom(\sigma) \cap V_R = \emptyset$ *and* $ran(\sigma) \cap (V_R \cup V_{ST}) = \emptyset$*, if* $[v]_E \in ST\sigma@[t]_E$ *can be proved with a c.p.t.* T *then* $[v]_E \in ST@[t]_E$ *and a c.p.t.* T' *with head* $t \to v/ST$ *and the same depth as* T *can be constructed.*

Proof. The proof is done by structural induction on the depth of T. □

6 Reachability problems

In this section we present the concept of reachability problem, together with its solutions and the properties that a solution to one of these problems has. From now on, we will consider as valid those rewrite theories $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ whose axioms B are any combination of **associativity**, **commutativity**, and **identity** (*ACU rewrite theories*).

Then, the only rules that will be added to the closure under B-extensions of R will have the form $l : f(x_s, f(t_1, t_2)) \to f(x_s, t_3)$ for each rule $l : f(t_1, t_2) \to t_3 \in R$ such that f has the associative property (it could also be $l : f(f(t_1, t_2), x_s) \to f(t_3, x_s)$, we choose the other form). The commutative property has no non-variable subterms, and for the identity property, $f(x_s, 0) = x_s$, the non-variable subterm 0 only matches rules of the form $l : 0 \to t$ yielding a rule $l : f(x_s, 0) \to f(x_s, t)$, which is subsumed by the original rule $l : 0 \to t$ with the substitution $\{x_s \mapsto 0\}$.

Definition 33 (Reachability problem)**.** *Given a rewrite theory $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ and a set of call strategy definitions $Call_{\mathcal{R}}$, a* reachability problem *is an expression P with the form $\bigwedge_{i=1}^{n} u_i \to v_i/ST_i \mid \phi \mid V, \nu$, where u_i and v_i are terms in $\mathcal{H}_\Sigma(\mathcal{X})$, ST_i is a strategy in $Strat_{\mathcal{R},Call_{\mathcal{R}}}$, $\phi \in QF(\mathcal{X}_0)$, V is the finite set of* parameters *of the problem, i.e., variables of $\mathcal{X}$ that have to be given a ground value, and ν is a substitution such that $dom(\nu) \subseteq V$ and $ran(\nu)$ consists only of new variables, not seen before, that may hold the initial values, either constants or patterns, of some of these parameters. The formula ϕ is the* reachability formula *of P. We define $vars(P) = vars(\bar{u}, \bar{v}, \phi)$. The set V allows the declaration of variables in $V_{\mathcal{R},Call_{\mathcal{R}}}$ or $V_{\overline{ST}}$, as parameters of the problem. V must always verify:*

1. *$vars(P) \subseteq V$, $vars(B) \cap V = \emptyset$, and $V_{\mathcal{R}} \cap V_{Call_{\mathcal{R}}} \subseteq V$, i.e, $V_{\mathcal{R}}$ and $V_{Call_{\mathcal{R}}}$ have no variables in common, with the exception of the parameters of the problem,*

2. *concatenated strategies may have in common only variables from V, since they will be given a ground value; this is also mandatory for strategies from different open goals; also, only variables from V may appear in iterated strategies and call strategy invocations, since they may become concatenated ones, and*

3. *V cannot contain:*
   - *any variable in $dom(\gamma)$ for any strategy $c[\gamma]$ that may appear in $Call_{\mathcal{R}}$ or ST_i, $1 \leq i \leq n$,*
   - *any variable in $\hat{x}$ for any call strategy definition* `sd` *$C(\bar{x})$ or* `csd` *$C(\bar{x})$ that may appear in $Call_{\mathcal{R}}$, or*
   - *any variable in $matchParam(\overline{ST}) \cup matchParam(Call_{\mathcal{R}})$.*

Definition 34 (Instances)**.** *Given a rewrite theory $\mathcal{R} = (\Sigma, E_0 \cup B, R)$, a set of call strategy declarations $Call_{\mathcal{R}}$, and a substitution σ such that $vars(B) \cap (dom(\sigma) \cup ran(\sigma)) = \emptyset$, the* instance *$\mathcal{R}^\sigma$ of $\mathcal{R}$ is the rewrite theory that results from the simultaneous replacement of every instance in R of any variable $x \in dom(\sigma)$ with $x\sigma$, $Call^\sigma_{\mathcal{R}}$ is the set of call strategy declarations that results from the simultaneous replacement of every instance in $Call_{\mathcal{R}}$ of any variable $x \in dom(\sigma)$ with $x\sigma$, and $Strat^\sigma_{\mathcal{R},Call_{\mathcal{R}}}$ is their set of associated strategies. For every strategy ST in $Strat_{\mathcal{R},Call_{\mathcal{R}}}$ we denote by ST^σ its corresponding strategy in $Strat^\sigma_{\mathcal{R},Call_{\mathcal{R}}}$. We denote by $\mathcal{D}^\sigma_{\mathcal{R},Call_{\mathcal{R}}}$ the associated set of derivation rules. If γ is a substitution, $dom(\gamma) \cap (dom(\sigma) \cup ran(\sigma)) = \emptyset$, and $ST = ST_1\gamma$ then $ST^\sigma = ST_1^\sigma(\gamma \cdot \sigma)$. If $t \in \mathcal{T}_\Sigma(\mathcal{X})$, then $t^\sigma = t\sigma$. If $\phi \in QF(\mathcal{X}_0)$, then $\phi^\sigma = \phi\sigma$. For any structure S formed with terms, formulas and strategies, the instance S^σ of S will consist in the instantiation with σ of each one of its elements.*

Although the label, say c, of an instantiated rule remains the same, we will use superscripts, say c^σ, when it is needed to distinguish which instance of the rule we are referring to.

Proposition 9 (Equality of $(R^\sigma)_B$ and $(R_B)^\sigma$)**.** *For any ACU rewrite theory* $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ *and any substitution* σ *such that* $vars(B) \cap (dom(\sigma) \cup ran(\sigma)) = \emptyset$ *it holds that* $(R^\sigma)_B = (R_B)^\sigma$*.*

Proof. We prove $(c^\sigma)_B = (c_B)^\sigma$ for every rule $c \in R$. If $c : l \to r \textit{ if } C \in R$ then, by definition, $l \in \mathcal{H}_\Sigma(\mathcal{X}) \setminus \mathcal{X}$, so l has the form $f(\bar{l})$, for proper f and $\bar{l}$.

- If f is binary associative then c has the form $c : f(l_1, l_2) \to r \textit{ if } C \in R$, and $c : f(x_s, f(l_1, l_2)) \to r \textit{ if } C \in c_B$, so $c : f(x_s, f(l_1\sigma, l_2\sigma)) \to r\sigma \textit{ if } C\sigma \in (c_B)^\sigma$ since $x_s\sigma = x_s$. Then, $c : f(l_1\sigma, l_2\sigma) \to r\sigma \textit{ if } C\sigma \in R^\sigma$, so also $c : f(x_s, f(l_1\sigma, l_2\sigma)) \to r\sigma \textit{ if } C\sigma \in (c^\sigma)_B$, and $(c^\sigma)_B = (c_B)^\sigma$.

- Else, $c_B = \{c\}$, and $(c_B)^\sigma = \{c^\sigma\}$. Now, c^σ has the form $c : f(\bar{l}\sigma) \to r\sigma \textit{ if } C\sigma$ where f is not binary associative, so also $(c^\sigma)_B = \{c^\sigma\}$, hence $(c^\sigma)_B = (c_B)^\sigma$.

□

We will write R^σ_B to refer to either $(R^\sigma)_B$ or $(R_B)^\sigma$, indistinctly.

Definition 35 (Solution of a reachability problem)**.** *Given a rewrite theory* $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ *and a set of call strategy definitions* $Call_\mathcal{R}$*, a solution of the reachability problem* $P = \bigwedge_{i=1}^n u_i \to v_i/ST_i \mid \phi \mid V, \nu$ *is a substitution* $\sigma : V \to \mathcal{T}_\Sigma$ *such that* $\sigma = \nu \cdot \sigma'$ *for some substitution* σ'*,* $E_0 \vDash \phi\sigma$*, and* $[v_i\sigma]_E \in ST_i^\sigma @[u_i\sigma]_E$ *(hence* $u_i\sigma \to_{R^\sigma/E} v_i\sigma$*), for* $1 \le i \le n$*.*

Given a rewrite theory $\mathcal{R} = (\Sigma, E_0 \cup B, R)$, a set of call strategy definitions $Call_\mathcal{R}$, and the reachability problems $P = \bigwedge_{i=1}^n u_i \to v_i/ST_i \mid \phi \mid V, \nu$ and $P' = \bigwedge_{i=1}^n u_i \to v_i/ST_i \,;\, \texttt{idle} \mid \phi \mid V, \nu$, both problems yield the same solutions. For any solution σ of P, $E_0 \vDash \phi\sigma$ and $[v_i\sigma]_E$ in $ST_i^\sigma @[u_i\sigma]_E$, for $1 \le i \le n$, so there are closed proof trees

$$\frac{F_i}{u_i\sigma \to v_i\sigma/ST_i^\sigma},$$

where $1 \le i \le n$, formed with the rules in $\mathcal{D}^\sigma_{\mathcal{R}, Call_\mathcal{R}}$. Then, also

$$\frac{\dfrac{F_i}{u_i\sigma \to v_i\sigma/ST_i^\sigma} \quad \dfrac{}{v_i\sigma \to v_i\sigma/\texttt{idle}}}{u_i\sigma \to v_i\sigma/ST_i^\sigma \,;\, \texttt{idle}},$$

where $1 \le i \le n$, are closed proof trees, so σ is a solution of P', and vice versa.

Given a reachability problem $\bigwedge_{i=1}^n u_i \to v_i/ST_i \mid \phi \mid V, \nu$, we will solve the equivalent problem $\bigwedge_{i=1}^n u_i \to v_i/ST_i \,;\, \texttt{idle} \mid \phi \mid V, \nu$, since it will allow us to use a smaller set of narrowing rules, by not having to distinguish between those strategies that are a concatenation of strategies, to process one strategy after the other, and those that are not.

7 Strategies in reachability by conditional narrowing modulo SMT and axioms

In this section, the narrowing calculus for reachability with strategies is introduced and its soundness and weak completeness are stated, as well as its completeness for *topmost* rewrite theories.

7.1 Reachability goals and calculus

Some definitions and the calculus for reachability with strategies by conditional narrowing modulo SMT and axioms are presented now.

Definition 36 (Instance of a set of variables). *Given a set of variables V and a substitution ν, we call $V^{\nu} = (V \setminus dom(\nu)) \cup ran(\nu_V)$.*

Definition 37 (Reachability goal). *Given a rewrite theory $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ and a set of call strategy definitions $Call_{\mathcal{R}}$, a* reachability goal *G is an expression with the form*

1. $(\bigwedge_{i=1}^{n} u'_i \to v'_i/ST_i \mid \phi')^{\nu}\varrho_{\nu} \mid V, \nu$, *or*
2. $(u'_1|_p \to^1 x_k, u'_1[x_k]_p \to v'_1/ST_1 \wedge \bigwedge_{i=2}^{n} u'_i \to v'_i/ST_i \mid \phi')^{\nu}\varrho_{\nu} \mid V, \nu$,

where ν and ϱ_{ν} are substitutions, $dom(\nu) \subseteq V$, $dom(\varrho_{\nu}) \cap (V \cup V^{\nu}) = \emptyset$, $V \subset \mathcal{X}$ is finite, call $(\bar{u}, \bar{v}, \phi) = (\bar{u}', \bar{v}', \phi')^{\nu}\varrho_{\nu}$, $n \geq 1$, u'_i and v'_i are terms in $\mathcal{H}_{\Sigma}(\mathcal{X})$, $ST_i \in Strat_{\mathcal{R},Call_{\mathcal{R}}}$, for $1 \leq i \leq n$, and $\phi \in QF(\mathcal{X}_0)$; also, in the second case, $p \in pos(u_1)$, $k = [ls(u_1|_p)]$, the kind of the least sort of $u_1|_p$, $x_k \notin V_{\bar{u}',\bar{v}',\phi',\overline{ST}} \cup V \cup ran(\nu) \cup dom(\varrho_{\nu}) \cup ran(\varrho_{\nu})$, and ST_1 has the form $RA\,;ST$, with RA a rule application.

In the first case, each one of the elements in the conjunctions is an open goal, for which we define $V_{u \to v/ST} = V_{u,v}$, and $V_G = V_{\bar{u},\bar{v},\phi} \cup V^{\nu}$; in the second case, we say that x_k is the connecting variable *of the goal and we define $V_G = \{x_k\} \cup V_{\bar{u},\bar{v},\phi} \cup V^{\nu}$. We will write 'goal' as a synonym of reachability goal.*

Reachability goals with the second form, where we always can recover u_1 form $u_1|_p$ and $u_1[]_p$, can be generated by the calculus rules in Figures 3 - 5 from a reachability goal with the first form when the first open goal has the form $u_1 \to v_1/RA; ST$, with RA a rule application strategy. This second form prevents the repeated application in a derivation of rule transitivity, that maintains the problem in the second form, forcing the application to the first open goal of the rule application rule, that reverts the problem to the first form.

The substitution ϱ_{ν} will be used in our calculus to hold instantiations or renamings, that will be generated by the calculus rules, of the variables not in V.

Definition 38 (Instance of a goal). *If G is a goal of the form $(\bigwedge_{i=1}^{n} S_i \mid \phi)^{\nu}\varrho_{\nu} \mid V, \nu$ and σ is a substitution such that $dom(\sigma) \cap V^{\nu} \neq \emptyset$, then we define the* instance *$G\sigma$ of G as $G\sigma = (\bigwedge_{i=1}^{n} S_i \mid \phi)^{\mu}\varrho_{\mu} \mid V, \mu$, where $\mu = (\nu\sigma)_V$ and $\varrho_{\mu} = (\varrho_{\nu}\sigma)_{V_G \setminus V}$.*

Definition 39 (Instance of a conjunction of open goals). *If G is a goal of the form $(\bigwedge_{i=1}^{n} S_i \mid \phi)^{\nu}\varrho_{\nu} \mid V, \nu$, let $SG = (\bigwedge_{i=1}^{n} S_i)^{\nu}\varrho_{\nu}$, and σ is a substitution such that $dom(\sigma) \cap V^{\nu} \neq \emptyset$, then we define the* instance *$SG\sigma$ of SG as $SG\sigma = (\bigwedge_{i=1}^{n} S_i)^{\mu}\varrho_{\mu}$, where $\mu = (\nu\sigma)_V$ and $\varrho_{\mu} = (\varrho_{\nu}\sigma)_{V_{SG} \setminus V}$.*

When $dom(\sigma) \cap V^{\nu} = \emptyset$, σ is directly applied to every term and formula in G and SG, respectively, thus avoiding circularity in these definitions.

Definition 40 (Admissible goals). *From now on, we will only consider in our work two types of goals:*

(a) those goals coming from a reachability problem $\bigwedge_{i=1}^{n} u_i \to v_i/ST_i \mid \phi \mid V, \nu$, which is transformed into the goal $\bigwedge_{i=1}^{n} u_i\nu \to v_i\nu/ST_i^{\nu}; \texttt{idle} \mid \phi\nu \mid V, \nu$, with $\varrho_{\nu} = none$, and

(b) those goals generated by repeatedly applying the calculus rules in Figures 3 - 5 to one goal of type (a).

The notation $G \leadsto_{[r],\sigma} G'$, will be used in the calculus to indicate that rule $[r]$ of the calculus has been applied with substitution σ to G, yielding G'. We call this application a *narrowing step*. If σ is the identity substitution it can be omitted. The rule $[r]$ can also be omitted in the expression. The superscripts $\leadsto^n$, with $n > 0$, $\leadsto^+$, and $\leadsto^*$ will be used with their standard meanings, maybe with no rule in the subscript ($\leadsto$ and $\leadsto^1$ are equivalent).

Proposition 10 (Invariants of the goals). *Given a rewrite theory $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ and a set of call strategy definitions $Call_{\mathcal{R}}$, and an admissible goal G with the form*

- $\bigwedge_{i=1}^{n} u_i \to v_i/ST_i^{\nu}\varrho_\nu \mid \phi \mid V, \nu$, *or*
- $u_1|_p \to^1 x_k, u_1[x_k]_p \to v_1/ST_1^{\nu}\varrho_\nu \wedge \bigwedge_{i=2}^{n} u_i \to v_i/ST_i^{\nu}\varrho_\nu \mid \phi \mid V, \nu$,

if G_0 is a goal of type (a), with substitution ν_0 (ϱ_{ν_0} = none by definition), and $G_0 \leadsto_\theta^* G$ then the following invariants hold:

1. $vars(B) \cap V = \emptyset$ *and* $V_{\mathcal{R}} \cap V_{Call_{\mathcal{R}}} \subseteq V$,
2. $V \cap ran(\nu) = \emptyset$ *and* $\nu = (\nu_0\theta)_V$, *hence* $dom(\nu) \subseteq V$, *so* $dom(\nu)$ *satisfies the restrictions given for* V *in Definition 33.2,*
3. $\varrho_\nu = \theta_{\setminus V}$, *hence* $dom(\varrho_\nu) \cap V = \emptyset$ *and* ϱ_ν *is idempotent,*
4. $ran(\theta) \cap (V \cup V_{\mathcal{R},Call_{\mathcal{R}}} \cup vars(\overline{ST})) = \emptyset$ *and* $ran(\varrho_\nu) \cap V = \emptyset$,
5. $dom(\varrho_\nu) \cap ran(\nu) = \emptyset$,
6. $dom(\varrho_\nu) \cap V^\nu = \emptyset$,
7. $V_{\mathcal{R}^\nu} \cap V_{Call_{\mathcal{R}^\nu}} \subseteq V^\nu$,
8. *if* $t \in \mathcal{T}_\Sigma(\mathcal{X})$ *then* $t^\nu \varrho_\nu = t(\nu \uplus \varrho_\nu)$,
9. u_i, v_i, $1 \leq i \leq n$, *and each term in* $\hat{\phi}$ *have the form* $t^\nu \varrho_\nu$,
10. $vars(\bar{u}, \bar{v}, \phi) \cap dom(\nu) = \emptyset$, *and*
11. G *has also the form* $G_1^\nu \varrho'_\nu$, *where* $\varrho'_\nu = \theta_{V_{G_1} \setminus V}$, *so* $dom(\varrho'_\nu) \subseteq V_{G_1} \setminus V$.

Proof. By induction on the number of applied calculus rules from Figures 3 and 4. □

We extend the definition of solution of a reachability problem to goals.

Definition 41 (Solution of a goal). *Given a rewrite theory $\mathcal{R} = (\Sigma, E_0 \cup B, R)$, a set of call strategy definitions $Call_{\mathcal{R}}$ for $\mathcal{R}$, and a goal G, a substitution $\sigma : vars(G) \to \mathcal{T}_\Sigma$, where $\nu' = (\nu\sigma)_V$ and $\varrho_{\nu'} = (\varrho_\nu\sigma)_{\setminus V}$, is a* solution *of G iff:*

1. *if* $G = \bigwedge_{i=1}^{n} u_i \to v_i/ST_i^{\nu}\varrho_\nu \mid \phi \mid V, \nu$ *then* $E_0 \vDash \phi\sigma$ *and* $[v_i\sigma]_E \in ST_i^{\nu'}\varrho_{\nu'}@[u_i\sigma]_E$ *(hence* $u_i\sigma \to_{R^{\nu'}/E} v_i\sigma$*), for* $1 \leq i \leq n$, *and*
2. *if* $G = u_1|_p \to^1 x_k, u_1[x_k]_p \to v_1/ST_1^{\nu}\varrho_\nu \wedge \bigwedge_{i=2}^{n} u_i \to v_i/ST_i^{\nu}\varrho_\nu \mid \phi \mid V, \nu$, *where* $ST_1 = RA; ST$, *then* $E_0 \vDash \phi\sigma$, $[x_k\sigma]_E \in RA^{\nu'}\varrho_{\nu'} @ [u_1\sigma|_p]_E$, $[v_1\sigma]_E \in ST^{\nu'}\varrho_{\nu'}@[u_1[x_k]_p\sigma]_E$, *and* $[v_i\sigma]_E \in ST_i^{\nu'}\varrho_{\nu'}@[u_i\sigma]_E$, *for* $2 \leq i \leq n$.

In the second case, as $[x_k\sigma]_E \in RA^{\nu'}\varrho_{\nu'} @ [u_1|_p\sigma]_E$ implies $[u_1[x_k]_p\sigma]_E \in RA^{\nu'}\varrho_{\nu'} @ [u_1\sigma]_E$, and $[v_1\sigma]_E \in ST^{\nu'}\varrho_{\nu'}@[u_1[x_k]_p\sigma]_E$ then $[v_1\sigma_{\setminus\{x_k\}}]_E \in ST_1^{\nu'}(\varrho_{\nu'})_{\setminus\{x_k\}}@[u_1\sigma_{\setminus\{x_k\}}]_E$, i.e., $\sigma_{\setminus\{x_k\}}$ is a solution of $\bigwedge_{i=1}^{n} u_i \to v_i/ST_i^{\nu}\varrho_\nu \mid \phi \mid V, \nu$.

We call $nil \mid \phi \mid V, \nu$, where ϕ is satisfiable and $\nu : \mathcal{X} \to \mathcal{T}_\Sigma(\mathcal{X})$ such that $dom(\nu) \subseteq V$, an *empty goal.* Given $\mathcal{R}_B = (\Sigma, E_0 \cup B, R_B)$, a closed under B-extensions associated rewrite theory of $\mathcal{R} = (\Sigma, E_0 \cup B, R)$, both with built-in subtheory (Σ_0, E_0), a reachability problem $P = \bigwedge_{i=1}^{n} u_i \to v_i/ST_i \mid \phi \mid V, \nu$ is solved by applying the calculus rules in Figures 3 and 4, starting with $G = \bigwedge_{i=1}^{n} u_i\nu \to v_i\nu/(ST_i^{\nu}\ ;\ \texttt{idle}) \mid \phi\nu \mid V, \nu$ in a top-down manner, until an empty goal is obtained, where $(\bigwedge_{i=1}^{n} u_i \to v_i/ST_i^{\nu}\varrho_\nu)\sigma = \bigwedge_{i=1}^{n} u_i\sigma \to v_i\sigma/ST_i^{(\nu\sigma)_V}(\varrho_\nu\sigma)_{\setminus V}$.

- $[d1]$ idle

$$\frac{u \to v/\texttt{idle}\ (\wedge\,\Delta) \mid \phi \mid V, \nu}{(\Delta\sigma) \mid \psi \mid V, (\nu\sigma)_V}$$

where $abstract_{\Sigma_1}((u, v)) = \langle \lambda(\bar{x}, \bar{y}).(u^\circ, v^\circ); (\theta_u^\circ, \theta_v^\circ); (\phi_u^\circ, \phi_v^\circ)\rangle$, σ in $CSU_B(u^\circ = v^\circ)$, $vars(\psi) \subseteq vars((\phi \wedge \phi_u^\circ \wedge \phi_v^\circ)\sigma)$, $E_0 \vdash \psi \Leftrightarrow (\phi \wedge \phi_u^\circ \wedge \phi_v^\circ)\sigma$, and ψ is satisfiable

- $[d2]$ idle

$$\frac{u \to v/\texttt{idle}\,; ST\ (\wedge\,\Delta) \mid \phi \mid V, \nu}{u \to v/ST\ (\wedge\,\Delta) \mid \phi \mid V, \nu}$$

- $[o1]$ or

$$\frac{u \to v/(ST_1 \mid ST_2)\,; ST\ (\wedge\,\Delta) \mid \phi \mid V, \nu}{u \to v/ST_1\,; ST\ (\wedge\,\Delta) \mid \phi \mid V, \nu}$$

- $[o2]$ or

$$\frac{u \to v/(ST_1 \mid ST_2)\,; ST\ (\wedge\,\Delta) \mid \phi \mid V, \nu}{u \to v/ST_2\,; ST\ (\wedge\,\Delta) \mid \phi \mid V, \nu}$$

- $[p1]$ plus

$$\frac{u \to v/ST_1+\,; ST\ (\wedge\,\Delta) \mid \phi \mid V, \nu}{u \to v/ST_1\,; ST\ (\wedge\,\Delta) \mid \phi \mid V, \nu}$$

- $[p2]$ plus

$$\frac{u \to v/ST_1+\,; ST\ (\wedge\,\Delta) \mid \phi \mid V, \nu}{u \to v/ST_1\,; ST_1+\,; ST\ (\wedge\,\Delta) \mid \phi \mid V, \nu}$$

- $[s1]$ star

$$\frac{u \to v/ST_1*\,; ST\ (\wedge\,\Delta) \mid \phi \mid V, \nu}{u \to v/ST\ (\wedge\,\Delta) \mid \phi \mid V, \nu}$$

- $[s2]$ star

$$\frac{u \to v/ST_1*\,; ST\ (\wedge\,\Delta) \mid \phi \mid V, \nu}{u \to v/ST_1+\,; ST\ (\wedge\,\Delta) \mid \phi \mid V, \nu}$$

- $[i1]$ if then else

$$\frac{u \to v/\texttt{match}\ t\ \texttt{s.t.}\ \phi'\ ?\ ST_1 : ST_2\,; ST\ (\wedge\,\Delta) \mid \phi \mid V, \nu}{(u \to v\ /ST_1\,; ST\ (\wedge\,\Delta))\sigma \mid \psi \mid V, (\nu\sigma)_V}$$

where $abstract_{\Sigma_1}((u, t)) = \langle \lambda(\bar{x}, \bar{y}).(u^\circ, t^\circ); (\theta_u^\circ, \theta_t^\circ); (\phi_u^\circ, \phi_t^\circ)\rangle$, σ in $CSU_B(u^\circ = t^\circ)$, $vars(\psi) \subseteq vars((\phi \wedge \phi' \wedge \phi_u^\circ \wedge \phi_t^\circ)\sigma)$, $E_0 \vdash \psi \Leftrightarrow (\phi \wedge \phi' \wedge \phi_u^\circ \wedge \phi_t^\circ)\sigma$, and ψ is satisfiable

- $[i2]$ if then else

$$\frac{u \to v/\texttt{match}\ t\ \texttt{s.t.}\ \phi'\ ?\ ST_1 : ST_2\,; ST\ (\wedge\,\Delta) \mid \phi \mid V, \nu}{(u \to v\ /ST_2\,; ST\ (\wedge\,\Delta))\sigma \mid \psi \mid V, (\nu\sigma)_V}$$

where $abstract_{\Sigma_1}((u, t)) = \langle \lambda(\bar{x}, \bar{y}).(u^\circ, t^\circ); (\theta_u^\circ, \theta_t^\circ); (\phi_u^\circ, \phi_t^\circ)\rangle$, σ in $CSU_B(u^\circ = t^\circ)$, $vars(\psi) \subseteq vars((\phi \wedge \neg\phi' \wedge \phi_u^\circ \wedge \phi_t^\circ)\sigma)$, $E_0 \vdash \psi \Leftrightarrow (\phi \wedge \neg\phi' \wedge \phi_u^\circ \wedge \phi_t^\circ)\sigma$, and ψ is satisfiable

Figure 3: Inference rules for reachability with strategies modulo SMT plus axioms I

- $[t]$ transitivity

$$\frac{u \to v/RA\,;ST\ (\wedge\,\Delta) \mid \phi \mid V,\nu}{u \to^1 x_k, x_k \to v/RA\,;ST\ (\wedge\,\Delta) \mid \phi \mid V,\nu}$$

where RA is a rule application, $u \in \mathcal{H}_\Sigma(\mathcal{X}) \setminus \mathcal{X}$, $k = [ls(u)]$, and x_k fresh variable

- $[c]$ congruence

$$\frac{u|_p \to^1 x_k, u[x_k]_p \to v/RA\,;ST\ (\wedge\,\Delta) \mid \phi \mid V,\nu}{u_i \to^1 y_{k'}, u[y_{k'}]_{p.i} \to v/RA\,;ST\ (\wedge\,\Delta) \mid \phi \mid V,\nu}$$

where RA is a rule application, $u|_p = f(u_1,\ldots,u_n)$, $u_i \in \mathcal{H}_\Sigma(\mathcal{X}) \setminus \mathcal{X}$,

$k' = [ls(u_i)]$, $y_{k'}$ fresh variable, and $\sigma_1 = \{x_k \mapsto u|_p[y_{k'}]_i\}$

- $[r]$ rule application

$$\frac{u|_p \to^1 x_k, u[x_k]_p \to v/c[\gamma]\{ST_1,\ldots,ST_n\}\,;ST\ (\wedge\,\Delta) \mid \phi \mid V,\nu}{(\bigwedge_{i=1}^n (l_i\gamma \to r_i\gamma/ST_i;\texttt{idle}) \wedge u[r\gamma]_p \to v\ /ST\ (\wedge\,\Delta))\sigma \mid \psi \mid V,(\nu\sigma)_V}$$

where $c : l \to r\ \mathit{if}\ \bigwedge_{i=1}^n (l_i \to r_i) \mid \phi'$ fresh version, except for $\mathit{dom}(\gamma) \cup V^\nu$, of a rule c in R^ν,

$\mathit{abstract}_{\Sigma_1}((u|_p, l\gamma)) = \langle \lambda(\bar{u},\bar{y}).(u^\circ, l^\circ); (\sigma_u^\circ, \sigma^\circ); (\phi_u^\circ, \phi_l^\circ)\rangle$, σ' in $\mathit{CSU}_B(u^\circ = l^\circ)$,

$\sigma = \sigma' \cup \{x_k \mapsto r\gamma\sigma'\}$, $\mathit{vars}(\psi) \subseteq \mathit{vars}((\phi \wedge \phi_u^\circ \wedge \phi_l^\circ \wedge (\phi'\gamma))\sigma)$,

$E_0 \vdash \psi \Leftrightarrow (\phi \wedge \phi_u^\circ \wedge \phi_l^\circ \wedge (\phi'\gamma))\sigma$, and ψ is satisfiable

- $[tp]$ top

$$\frac{u \to v/\texttt{top}(c[\gamma]\{ST_1,\ldots,ST_n\})\,;ST\ (\wedge\,\Delta) \mid \phi \mid V,\nu}{(\bigwedge_{i=1}^n (l_i\gamma \to r_i\gamma/ST_i;\texttt{idle}) \wedge r\gamma \to v\ /ST\ (\wedge\,\Delta))\sigma \mid \psi \mid V,(\nu\sigma)_V}$$

where $c : l \to r\ \mathit{if}\ \bigwedge_{i=1}^n (l_i \to r_i) \mid \phi'$ fresh version, except for $\mathit{dom}(\gamma) \cup V^\nu$, of a rule c in R^ν,

$\mathit{abstract}_{\Sigma_1}((u, l\gamma)) = \langle \lambda(\bar{u},\bar{y}).(u^\circ, l^\circ); (\sigma_u^\circ, \sigma^\circ); (\phi_u^\circ, \phi_l^\circ)\rangle$, σ in $\mathit{CSU}_B(u^\circ = l^\circ)$,

$\mathit{vars}(\psi) \subseteq \mathit{vars}((\phi \wedge \phi_u^\circ \wedge \phi_l^\circ \wedge (\phi'\gamma))\sigma)$, $E_0 \vdash \psi \Leftrightarrow (\phi \wedge \phi_u^\circ \wedge \phi_l^\circ \wedge (\phi'\gamma))\sigma$, and ψ is satisfiable

Figure 4: Inference rules for reachability with strategies modulo SMT plus axioms II

- $[m]$ match

$$\frac{u \to v/\texttt{match}\ t\ \texttt{s.t.}\ \bigwedge_{j=1}^{m}(l_j = r_j) \wedge \phi' \,;ST\ (\wedge\ \Delta) \mid \phi \mid V, \nu}{(\bigwedge_{j=1}^{m}(l_j \to r_j/\texttt{idle}) \wedge u \to v\ /ST\ (\wedge\ \Delta))\sigma \mid \psi \mid V, (\nu\sigma)_V}$$

where $\mathit{abstract}_{\Sigma_1}((u,t)) = \langle \lambda(\bar{x},\bar{y}).(u^\circ, t^\circ); (\theta_u^\circ, \theta_t^\circ); (\phi_u^\circ, \phi_t^\circ)\rangle$, σ in $\mathit{CSU}_B(u^\circ = t^\circ)$,

$\mathit{vars}(\psi) \subseteq \mathit{vars}((\phi \wedge \phi' \wedge \phi_u^\circ \wedge \phi_t^\circ)\sigma)$, $E_0 \vdash \psi \Leftrightarrow (\phi \wedge \phi' \wedge \phi_u^\circ \wedge \phi_t^\circ)\sigma$, and ψ is satisfiable

- $[w]$ matchrew

$$\frac{u \to v/\texttt{matchrew}\ t[\bar{z}]_{\bar{p}}\ \texttt{s.t.}\ \bigwedge_{j=1}^{m}(l_j = r_j) \wedge \phi'\ \texttt{by}\ z_1\ \texttt{using}\ ST_1, \ldots, z_n\ \texttt{using}\ ST_n\,;ST\ (\wedge\ \Delta) \mid \phi \mid V, \nu}{(\bigwedge_{j=1}^{m}(l_j\gamma \to r_j\gamma/\texttt{idle}) \wedge \bigwedge_{i=1}^{n}(x_i \to y_i/ST_i\gamma;\texttt{idle}) \wedge t[\bar{y}]_{\bar{p}} \to v/\ ST\ (\wedge\ \Delta))\sigma \mid \psi \mid V, (\nu\sigma)_V}$$

where $\bar{z} = z_1, \ldots, z_n$, $\bar{x}$ and $\bar{y}$ fresh versions of $\bar{z}$, γ renaming from $\bar{z}$ to $\bar{x}$,

$\mathit{abstract}_{\Sigma_1}((u, t[\bar{x}]_{\bar{p}})) = \langle \lambda(\bar{w},\bar{w}').(u^\circ, t^\circ); (\theta_u^\circ, \theta_t^\circ); (\phi_u^\circ, \phi_t^\circ)\rangle$, σ in $\mathit{CSU}_B(u^\circ = t^\circ)$,

$\mathit{vars}(\psi) \subseteq \mathit{vars}((\phi \wedge \phi' \wedge \phi_u^\circ \wedge \phi_t^\circ)\sigma)$, $E_0 \vdash \psi \Leftrightarrow (\phi \wedge \phi' \wedge \phi_u^\circ \wedge \phi_t^\circ)\sigma$, and ψ is satisfiable

- $[c1]$ call strategy

$$\frac{u \to v/CS\,;ST\ (\wedge\ \Delta) \mid \phi \mid V, \nu}{u \to v/ST_2\,;ST\ (\wedge\ \Delta) \mid \phi \mid V, \nu} \qquad \frac{u \to v/CS(\bar{t})\,;ST\ (\wedge\ \Delta) \mid \phi \mid V, \nu}{u \to v/ST_2\gamma\,;ST\ (\wedge\ \Delta) \mid \phi \mid V, \nu}$$

where $\texttt{sd}\ CS := ST_1$, or $\texttt{sd}\ CS(\bar{x}) := ST_1$ in $\mathit{Call}_{\mathcal{R}}^{\nu}$, $\gamma = \{\bar{x} \mapsto \bar{t}\}$,

and ST_2 fresh version of ST_1, except for $\mathit{dom}(\gamma) \cup V^\nu$

- $[c2]$ call strategy

$$\frac{u \to v/CS(\bar{t})\,;ST\ (\wedge\ \Delta) \mid \phi \mid V, \nu}{\bigwedge_{j=1}^{m}(l_j\gamma \to r_j\gamma/\texttt{idle}) \wedge u \to v/ST_2\gamma\,;ST\ (\wedge\ \Delta) \mid \psi \mid V, \nu}$$

where $\texttt{csd}\ CS(\bar{x}) := ST_1\ \mathit{if}\ C$ in $\mathit{Call}_{\mathcal{R}}^{\nu}$, $\gamma = \{\bar{x} \mapsto \bar{t}\}$,

$ST_2\ \mathit{if}\ \bigwedge_{j=1}^{m}(l_j = r_j) \wedge \phi'$ fresh version of $ST_1\ \mathit{if}\ C$, except for $\mathit{dom}(\gamma) \cup V^\nu$,

$\mathit{vars}(\psi) \subseteq \mathit{vars}(\phi'\gamma \wedge \phi)$, $E_0 \vdash \psi \Leftrightarrow \phi'\gamma \wedge \phi$, and ψ is satisfiable

Figure 5: Inference rules for reachability with strategies modulo SMT plus axioms III

We briefly explain rule $[w]$ (matchrew): we rename the matching parameters from $\bar{z}$ to the fresh variables $\bar{x}$ with γ. Once abstracted u and $t[\bar{x}]_{\bar{p}}$ to u° and t° and B-unified u° and t° with σ, we search for a unifier of $\bar{l}\gamma\sigma$ and $\bar{r}\gamma\sigma$, say α, using the `idle` strategy. Once found, the open goals $(\bar{x}\sigma \to \bar{y}/\overline{ST}\gamma\sigma)\alpha$, where $\bar{y}$ is fresh, will find a substitution β that makes $[y_i\beta]_E$ an element of $ST_i\gamma\sigma\alpha\beta @ [x_i\sigma\alpha\beta]_E$, for $1 \leq i \leq n$, and go on trying to find solutions for the open goal $(t[\bar{y}]_{\bar{p}} \to v/ST)\sigma\alpha\beta$.

Definition 42 (Narrowing path and computed answer). *Given $\mathcal{R}_B = (\Sigma, E_0 \cup B, R_B)$, an associated rewrite theory of $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ closed under B-extensions, and a goal G with set of parameters V and substitution ν_0, if there is a* narrowing path $G \rightsquigarrow_{\sigma_1} G_1 \rightsquigarrow_{\sigma_2} \cdots \rightsquigarrow_{\sigma_{n-1}} G_{n-1} \rightsquigarrow_{\sigma_n}$ *nil* $\mid \psi \mid V, \nu$, *using the calculus rules in Figures 3 and 4, hence ψ is satisfiable, then we write $G \rightsquigarrow^n_\sigma$ nil $\mid \psi \mid V, \nu$, where $\sigma = \sigma_1 \cdots \sigma_n$, and we call $\nu \mid \psi$ a* computed answer *for G.*

If $\nu_0 = none$ then ν is the restriction of σ to V by construction. In this case, as the unifiers σ_i, $1 \leq i \leq n$, returned by CSU_B are idempotent and away from all the variables that have previously appeared in the computation, so $ran(\sigma_i) \cap \bigcup_{j=1}^{i-1} ran(\sigma_j) = \emptyset$, then ν is also idempotent.

Although several rules allow for simplification in the reachability formula obtained, e.g., we can replace $X - Y + Z > 0 \wedge X = Y$ with $Z > 0$, it is always possible to obtain the same computed answer without using simplifications.

Proposition 11 (Canonical narrowing path). *Given $\mathcal{R}_B = (\Sigma, E_0 \cup B, R_B)$, an associated rewrite theory of $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ closed under B-extensions, and a narrowing path from a goal G (with set of parameters V), $G = \Delta_0 \mid \psi_0 \mid V, none \rightsquigarrow_{\sigma_1} \Delta_1 \mid \psi_1 \mid V, \nu_1 \rightsquigarrow_{\sigma_2} \cdots \Delta_{m-1} \mid \psi_{m-1} \mid V, \nu_{m-1} \rightsquigarrow_{\sigma_m} nil \mid \psi_m \mid V, \nu_m$, there exists another narrowing path $G = \Delta_0 \mid \psi_0 \mid V, none \rightsquigarrow_{\sigma_1} \Delta_1 \mid \chi_1 \mid V, \nu_1 \rightsquigarrow_{\sigma_2} \cdots \Delta_{m-1} \mid \chi_{m-1} \mid V, \nu_{m-1} \rightsquigarrow_{\sigma_m} nil \mid \chi_m \mid V, \nu_m$, where the same inference rule, with the same substitution, is applied at each step in both paths, there is no simplification of the reachability formula on the second path, and $E_0 \vdash \psi_i \Leftrightarrow \chi_i$, for $1 \leq i \leq m$.*

Proof. As the applied rule at each step i only depends on Δ_{i-1} which is the same on both paths, as long as ψ_i and χ_i are satisfiable, all that it has to be proved is $E_0 \vdash \psi_i \Leftrightarrow \chi_i$. Then as ψ_i is satisfiable so is χ_i.

By the definition of the proposition, $\chi_0 = \psi_0$, so $E_0 \vdash \psi_0 \Leftrightarrow \chi_0$. The check for $E_0 \vdash \psi_{i-1} \Leftrightarrow \chi_{i-1}$ implies $E_0 \vdash \psi_i \Leftrightarrow \chi_i$, for $1 \leq i \leq m$, is trivial since there are only two type of inference rules in the calculus:

- those rules that do not modify the formula, so $\psi_i = \psi_{i-1}$, $\chi_i = \chi_{i-1}$, and $E_0 \vdash \psi_{i-1} \Leftrightarrow \chi_{i-1}$ implies $E_0 \vdash \psi_i \Leftrightarrow \chi_i$, and
- those rules where $\chi_i = (\chi_{i-1} \wedge \chi'_{i-1})\theta$, for suitable χ'_{i-1} and θ, and $E_0 \vdash \psi_i \Leftrightarrow (\chi_{i-1} \wedge \chi'_{i-1})\theta$, i.e., $E_0 \vdash \psi_i \Leftrightarrow \chi_i$.

□

The aim of this work is to solve reachability problems; it must be born in mind that a goal with the second form comes from a reachability problem. Now it is proved that the calculus rules are a sound method for solving goals. A distinction is made depending on the form of the goal. For goals of the second form it is necessary to be very careful with the connecting variable of the goal, since this variable does not appear in the original reachability problem.

7.2 Soundness and weak completeness of the calculus

The soundness and weak completeness, i.e., completeness with respect to R/E-normalized solutions, of the calculus for reachability problems are now proved.

Theorem 2 (Soundness of the Calculus for Reachability Goals)**.** *Given an associated rewrite theory $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ closed under B-extensions and a reachability goal G, if $\nu \mid \psi$ is a computed answer for G then for each substitution $\rho : V^\nu \to \mathcal{T}_\Sigma$ such that $\psi\rho$ is satisfiable, $\nu \cdot \rho$ is a solution for G.*

Proof. By structural induction over the length of the corresponding canonical narrowing path and the first inference rule applied. □

The following lemma will be used in the proof of the weak completeness of the calculus.

Lemma 7 (Narrowing of equational conditions)**.** *Given an associated rewrite theory $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ closed under B-extensions, and a goal $G = \bigwedge_{j=1}^{m}(l_j \to r_j/\mathtt{idle}) \wedge \Delta^\mu \varrho_\mu \mid \psi \mid V, \mu$, if α is a ground substitution such that $V_G \subseteq dom(\alpha)$, $E_0 \vDash \psi\alpha$, and $\bar{l}\alpha =_E \bar{r}\alpha$, then there exist a ground substitution α°, substitutions $\beta_1, \ldots, \beta_m$ from CSUs, let $\beta_i^k = \beta_i\beta_{i+1}\cdots\beta_k$, and abstractions $abstract_{\Sigma_1}((l_j\beta_1^{j-1}, r_j\beta_1^{j-1})) = \langle\lambda(\bar{x}_j, \bar{y}_j).(l_j^\circ, r_j^\circ); (\theta_{l_j}^\circ, \theta_{r_j}^\circ); (\phi_{l_j}^\circ, \phi_{r_j}^\circ)\rangle$, for $1 \le j \le m$, where $\beta_1^0 = none$, let $\beta = \beta_1^m$, such that $dom(\alpha^\circ) = dom(\alpha) \cup V_{\hat{x},\hat{y}}$, $\alpha =_{E_0} \alpha^\circ_{dom(\alpha)}$, $\bar{l}^\circ\alpha^\circ =_E \bar{r}^\circ\alpha^\circ$, $\alpha^\circ \ll_E \beta_{dom(\alpha^\circ)}$, $G \rightsquigarrow^m_{[d1]} \Delta^\nu\varrho_\nu \mid \psi\beta \wedge \bigwedge_{j=1}^{m}(\phi_{l_j}^\circ \wedge \phi_{r_j}^\circ)\beta_j^m \mid V, \nu$, and for every pair of substitutions ρ and γ such that $ran(\rho)$ is away from all known variables, $\alpha^\circ \ll_E (\beta\rho)_{dom(\alpha^\circ)}$, and $\alpha^\circ =_E (\beta\rho)_{dom(\alpha^\circ)} \cdot \gamma$, it holds that $E_0 \vDash (\psi\beta \wedge \bigwedge_{j=1}^{m}(\phi_{l_j}^\circ \wedge \phi_{r_j}^\circ)\beta_j^m)\rho\gamma$ and $\Delta^\mu\varrho_\mu\alpha =_E \Delta^\mu\varrho_\mu\beta\rho\gamma$.*

Theorem 3 (Weak Completeness of the Calculus for Reachability Goals)**.** *Given an associated rewrite theory $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ closed under B-extensions and a reachability problem $P = \bigwedge_{i=1}^{n} u_i \to v_i/ST_i \mid \phi \mid V, \mu$, where μ is R/E-normalized, if $\sigma : V \to \mathcal{T}_\Sigma$ is a R/E-normalized solution for P then there exist a formula $\psi \in QF(\mathcal{X}_0)$ and two substitutions, say λ and ρ, such that $\bigwedge_{i=1}^{n} u_i\mu \to v_i\mu/ST_i^\mu; \mathtt{idle} \mid \phi\mu \mid V, \mu \rightsquigarrow^+_\lambda nil \mid \psi \mid V, \nu$, $\sigma =_E \nu{\cdot}\rho$, and $\psi\rho$ is satisfiable, where $\nu = (\mu\lambda)_V$.*

Proof. The proof is by induction over the sum of the number of nodes in each closed proof tree. □

Remark 2. *In the previous theorem, by Definition 35 there exists a substitution σ' such that $\sigma = \mu \cdot \sigma'$. As σ is R/E-normalized then, by Proposition 5, μ has to be R/E-normalized too. Also, as σ is R/E-normalized and the substitution η obtained after each narrowing step is always a generalization of σ then, by Proposition 6, η is R/E-normalized too.*

7.3 Completeness of the calculus, for topmost rewrite theories

In the proof of weak completeness of the calculus for reachability, the only places where the hypothesis of σ being R/E-normalized is used are in the initial substitution μ and in the induction case, *(ii)*, where it limits the positions where rewriting can happen at some proper subterm of $u_1\sigma$, an instance of the first term in the reachability problem P (u_1). It is immediate then to prove the *completeness of the calculus for topmost rewrite theories*, those rewrite theories $\mathcal{R} = (\Sigma, E, R)$ such that for some top sort `state`, no operator in Σ has `state` as argument sort, each rule $l \to r$ *if* $\bigwedge_{i=1}^{n} l_i \to r_i \mid \phi$ in R satisfies $l, r \in \mathcal{T}_\Sigma(\mathcal{X})_{\mathtt{state}}$ and $l_i, r_i \in \mathcal{T}_\Sigma(\mathcal{X})_{\mathtt{state}}$, for $1 \le i \le n$, since rewriting always happens at position ϵ of $u_1\sigma$, so the hypothesis of σ being R/E-normalized is not needed for this type of rewrite theories in the proof of completeness, when no variable in V has sort `state`, so μ is R/E-normalized.

8 Example

Three applications of the calculus using the running example are shown, recall the abbreviations: `i` − `Integer`, `p` − `Pan`, `rt` − `RealToast`, `t` − `Toast`, `k` − `Kitchen`, `b` − `Bin`, `s` − `System`, $\mathtt{ct_i}$ − `cookTime`, and $\mathtt{ft_i}$ − `failTime`. We will omit the use of the subindex `i` in all variables for readability. In both cases we take $\texttt{ct} = 20$. In the first case, from an initial system with an empty toaster, an empty dish, and at most one toast in the bin, we want to reach in no more than 60 seconds the same final system as in the previous case. In the second case, we want to know if there is value for `ft` lower than 61 seconds that allows us to get from an initial system where there are three toasts in the bag and the remaining elements are empty to a final system where there are three toasts in the dish and all the remaining elements are empty.

We choose $Call_{\mathcal{R}}$ to consist of the following call strategy definitions:

- $\texttt{sd test} := \texttt{match}\ N/B_{\texttt{b}}/Y; V_{\texttt{t}}W_{\texttt{t}}/OK\ \texttt{s.t.}\ Y < \texttt{ft}$
- $\texttt{sd cook1} := \texttt{matchrew}\ N/B_{\texttt{b}}/K_{\texttt{k}}/OK\ \texttt{s.t.}\ K_{\texttt{k}} = Y; R_{\texttt{rt}}V_{\texttt{t}}\ \texttt{by}\ K_{\texttt{k}}\ \texttt{using kitchCook}$
- $\texttt{sd kitchCook} := \texttt{top}(\texttt{kitchen}[none])\ ;\ \texttt{top}(\texttt{cook}[none]\{\texttt{toasts}, \texttt{toasts}\})$
- $\texttt{sd toasts} := \texttt{top}(\texttt{toast1}[none])\ |\ \texttt{top}(\texttt{toast2}[none])$
- $\texttt{sd noCook} := \texttt{top}(\texttt{bin}[none])\ |\ \texttt{top}(\texttt{pan}[none])\ |\ \texttt{top}(\texttt{dish}[none])$
- $\texttt{sd loop} := (\texttt{noCook}\ |\ (\texttt{cook1}\ ;\ \texttt{test}\ ;\ \texttt{noCook}))+$
- $\texttt{sd solve1} := \texttt{top}(\texttt{bag}[none])\ ;\ \texttt{top}(\texttt{bag}[none])\ ;\ \texttt{top}(\texttt{bag}[none])\ ;\ \texttt{loop}.$
- $\texttt{sd solve2} := \texttt{top}(\texttt{bag}[none])\ ;\ \texttt{top}(\texttt{bag}[none])\ ;\ (\texttt{top}(\texttt{bag}[none])\ |\ \texttt{idle})\ ;\ \texttt{loop}.$

Our reachability problems are:

$P_1 = N\ /\ T_{\texttt{t}}\ /\ 0\ ;\ \texttt{zt zt}\ /\ 0\ \rightarrow\ 0\ /\ \texttt{zt}\ /\ Y\ ;\ \texttt{zt zt}\ /\ 3\ /\ \texttt{solve2}\ \mid\ N > 0 \wedge N < 3\ \mid \{\texttt{ct}, \texttt{ft}, N, T_{\texttt{t}}, Y\}, \{\texttt{ct} \mapsto 20, \texttt{ft} \mapsto 61\}$, and

$P_2 = 3\ /\ \texttt{zt}/\ 0\ ;\ \texttt{zt zt}\ /\ 0\ \rightarrow\ 0\ /\ \texttt{zt}\ /\ Y\ ;\ \texttt{zt zt}\ /\ 3\ /\ \texttt{solve1}\ \mid\ \texttt{ft} < 61\ \mid \{\texttt{ct}, \texttt{ft}, Y\}, \{\texttt{ct} \mapsto 20\}$.

The most important feature of $Call_{\mathcal{R}}$ is the invocation of the call strategy `test` after each invocation of `cook1`. This renders the search state space of both problems finite, since there is a limit in both cases in the value of `ft` that gets checked against the timer, which initially has value 0, through the invocation of `test`.

Further pruning of the search tree is achieved through several facts: (i) all rule applications are used inside `top` strategies, preventing rule `congruence` of the narrowing calculus to be applied, (ii) in the call strategy definition `cook1`, where a rule must be applied in a subterm of the state, the `matchrew` strategy selects the precise subterm where to apply a `top` strategy in a much efficient way that the blind search of rule applications, and (iii) the use of the call strategy `noCook` after `test` prevents consecutive calls to `cook1` since rule `toast2` always well-toasts one side, so it cannot be invoked in the next strategy call. The definition of `noCook` could be further optimized but it is left as is for the sake of simplicity.

In P_1, as we can infer from the problem that, initially, there must be either two or three toasts in the bag, we impose the application of the rule `bag` twice, followed by the nondeterministic strategy $\texttt{top}(\texttt{bag}[none])\ |\ \texttt{idle}$, before applying any other rule, also preventing its application later, pruning the search tree. In the initial state we use the variable $T_{\texttt{t}}$ to represent the bin. This use is valid because `Toast` is a subsort of `Bin`, and it also covers both initial cases: the one without toasts in the bin and the one with one toast in the bin, since both `EmptyToast` and `RealToast` are subsorts of `Toast`.

Among the answers returned by the prototype we have:

a - $\texttt{ct} \mapsto 20, \texttt{ft} \mapsto 61, N \mapsto 3,\ Y \mapsto 60,\ T_{\texttt{t}} \mapsto \texttt{zt}$,

b - $\texttt{ct} \mapsto 20, \texttt{ft} \mapsto 61, N \mapsto 2,\ Y \mapsto 60,\ T_{\texttt{t}} \mapsto [0, 0]$,

c - $\texttt{ct} \mapsto 20, \texttt{ft} \mapsto 61, N \mapsto 2,\ Y \mapsto 40,\ T_{\texttt{t}} \mapsto [20, 20]$, and d - $\mathtt{ct} \mapsto 20, \mathtt{ft} \mapsto 61, N \mapsto 2,\ Y \mapsto 40 + U + V,\ T_{\mathtt{t}} \mapsto [C, D]$ *such that*

$C + U = 20 \wedge D + V = 20 \wedge U + V \leq 20 \wedge U > 0 \wedge V > 0,$

stating that we need 60 seconds when (a) 3 toasts are in the bag or (b) 2 toasts are in the bag and one fresh toast is in the bin. The required amount of time can be smaller: (c) 40 seconds if the toast in the bin is well-cooked or, if it is not, (d) 40 seconds plus the remaining toasting time for the toast in the bin, as long as this remaining time is not above 20 seconds.

In P_2, as we know that there are three toasts in the bag, we impose the application of the rule `bag` three times before applying any other rule, also preventing its application later, pruning the search tree. This problem has only one initial state, but what we are trying to find is a value for the parameter `ft` that fits the restrictions of the problem. The search for a solution ends, since our search state space is finite thanks to the call strategy `test`, without finding a solution.

For the third example, if we take P_2 and we allow `ft` to be below 62 seconds instead of 61, then the prototype returns the answer $Y \mapsto\ 60$ *such that* $\mathtt{ft} < 62 \wedge \mathtt{ft} > 60$, i.e., we can cook three toasts in 60 seconds when $\mathtt{ft} = 61$, fulfilling all the restrictions of the problem.

9 Conclusions and related work

In our previous work [AMPP17], we extended the admissible conditions in [RMM17] by: (i) allowing for reachability subgoals in the rewrite rules and (ii) removing all restrictions regarding the variables that appear in the rewrite rules. A narrowing calculus for conditional narrowing modulo $E_0 \cup B$ when E_0 is a subset of the theories handled by SMT solvers, B are the axioms not related to the algebraic data types handled by the SMT solvers, and the conditions in the rules in the rewrite theory are either rewrite conditions or quantifier-free SMT formulas, was presented, and the soundness and weak completeness of the calculus, as well as the completeness of the calculus for topmost rewrite theories was proved.

The current work extends the previous one by adding two novel features: (1) the use of strategies, to drive the search and reduce the state space, and (2) the support for parameters both in the rewrite theories and in the strategies, that allows for the resolution of some reachability problems that could not be specified in the previous calculi that we had developed. A calculus for conditional narrowing modulo $E_0 \cup B$ with strategies and parameters has been presented, and the soundness and weak completeness of the calculus have been proved. To the best of our knowledge, a similar calculus did not previously exist in the literature.

The strategy language that we have proved suitable for our narrowing calculus in this work is a subset of the Maude strategy language [MOMV04, EMOMV07, RMPV18]. This strategy language and a connection with SMT solvers have been incorporated into the latest version of the Maude language [DEE+20], which is being used to develop the prototype for the calculus in this work.

A classic reference in equational conditional narrowing modulo is the work of Bockmayr [Boc93]. The topic is addressed here for Church-Rosser equational conditional term rewriting systems without axioms. The intimate relationship between rewriting and reachability problems was shown by Hullot [Hul80], where he proved that any normalized solution to a reachability problem could be lifted to a narrowing derivation that computed a more general solution.

Narrowing modulo order-sorted unconditional equational logics is covered by Meseguer and Thati [MT07], being currently used for cryptographic protocol analysis.

The idea of constraint solving by narrowing in combined algebraic domains was presented by Kirchner and Ringeissen [KR94], where the supported theories had unconstrained equalities and the rewrite rules had constraints from an algebraic built-in structure, but they did not allow for reachability problems.

Escobar, Sasse, and Meseguer [ESM12] have developed the concepts of variant and folding variant narrowing, a narrowing strategy for order-sorted unconditional rewrite theories that terminates on those theories having the *finite variant property*, but it has no counterpart for conditional rewrite theories and it does not allow the use of constraint solvers or strategies.

Foundations for order-sorted conditional rewriting have been published by Meseguer [Mes17]. Cholewa, Escobar, and Meseguer [CEM15] have defined a new hierarchical method, called layered constraint narrowing, to solve narrowing problems in order-sorted conditional equational theories, an approach similar to ours, and given new theoretical results on that matter, including the definition of constrained variants for order-sorted conditional rewrite theories, but with no specific support for SMT solvers.

In [Mes20], Meseguer studies reachability in Generalized Rewrite Theories, that include constructors and variants, using equational theories beyond our setup of $E_0 \cup B$ (that only asks for strict B-coherence), but with no rewrite conditions in the rules. Frozenness is used as a type of strategy.

In previous work [AMPP14, AMPP15], the relationship between verifiable and computable answers for reachability problems in rewrite theories with an underlying membership equational logic has been studied, presenting two correct and weakly complete narrowing calculi, the second being a refinement of the first one. In this second calculus only normalized terms, in a similar way to the reduction phase of Fribourg in the language SLOG [Fri85], were considered in order to find an answer to a reachability problem. The rewriting language Maude [CDE$^+$07], which allows the use of reflection, was used as a framework to develop the prototype for the calculus.

Order-sorted conditional rewriting with constraint solvers has been addressed by Rocha et al. [RMM17], where the only admitted conditions in the rules are quantifier-free SMT formulas, and the only non-ground terms admitted in the reachability problems are those whose variables have sorts belonging to the SMT sorts supported.

Future work will focus in broadening the applicability of the calculus. One line of work will involve the development of a narrowing calculus for $E_0 \cup (E_1 \cup B)$ unification with strategies, where E_1 is a non-SMT equational theory; another line of work will study the extension of the strategies and reachability problems supported by the calculus.

References

[AMPP14] Luis Aguirre, Narciso Martí-Oliet, Miguel Palomino, and Isabel Pita. Conditional narrowing modulo in rewriting logic and Maude. In Escobar [Esc14], pages 80–96.

[AMPP15] Luis Aguirre, Narciso Martí-Oliet, Miguel Palomino, and Isabel Pita. Sentence-normalized conditional narrowing modulo in rewriting logic and maude. In Narciso Martí-Oliet, Peter Csaba Ölveczky, and Carolyn L. Talcott, editors, *Logic, Rewriting, and Concurrency - Essays dedicated to José Meseguer on the Occasion of His 65th Birthday*, volume 9200 of *Lecture Notes in Computer Science*, pages 48–71. Springer, 2015.

[AMPP17] Luis Aguirre, Narciso Martí-Oliet, Miguel Palomino, and Isabel Pita. Conditional narrowing modulo SMT and axioms. In Wim Vanhoof and Brigitte Pientka, editors, *Proceedings of the 19th International Symposium on Principles and Practice of Declarative Programming, Namur, Belgium, October 09 - 11, 2017*, pages 17–28. ACM, 2017.

[BM06] Roberto Bruni and José Meseguer. Semantic foundations for generalized rewrite theories. *Theoretical Computer Science*, 360(1-3):386–414, 2006.

[BM12] Kyungmin Bae and José Meseguer. Model checking LTLR formulas under localized fairness. In Francisco Durán, editor, *Rewriting Logic and Its Applications - 9th International Workshop, WRLA 2012, Held as a Satellite Event of ETAPS,* *Tallinn, Estonia, March 24-25, 2012, Revised Selected Papers*, volume 7571 of *Lecture Notes in Computer Science*, pages 99–117. Springer, 2012.

[BM14] Kyungmin Bae and José Meseguer. Infinite-state model checking of LTLR formulas using narrowing. In Escobar [Esc14], pages 113–129.

[Boc93] Alexander Bockmayr. Conditional narrowing modulo a set of equations. *Applicable Algebra in Engineering, Communication and Computing*, 4:147–168, 1993.

[CDE+07] Manuel Clavel, Francisco Durán, Steven Eker, Patrick Lincoln, Narciso Martí-Oliet, José Meseguer, and Carolyn Talcott. *All About Maude - A High-Performance Logical Framework: How to Specify, Program, and Verify Systems in Rewriting Logic*, volume 4350 of *Lecture Notes in Computer Science*. Springer, 2007.

[CEM15] Andrew Cholewa, Santiago Escobar, and José Meseguer. Constrained narrowing for conditional equational theories modulo axioms. *Sci. Comput. Program.*, 112:24–57, 2015.

[DEE+20] Francisco Durán, Steven Eker, Santiago Escobar, Narciso Martí-Oliet, José Meseguer, Rubén Rubio, and Carolyn L. Talcott. Programming and symbolic computation in Maude. *J. Log. Algebr. Meth. Program.*, 110, 2020.

[DLM+08] Francisco Durán, Salvador Lucas, Claude Marché, José Meseguer, and Xavier Urbain. Proving operational termination of membership equational programs. *Higher-Order and Symbolic Computation*, 21(1-2):59–88, 2008.

[DM12] Francisco Durán and José Meseguer. On the Church-Rosser and coherence properties of conditional order-sorted rewrite theories. *Journal of Logic and Algebraic Programming*, 81(7-8):816–850, 2012.

[dMB08] Leonardo de Moura and Nikolaj Bjørner. Z3: an efficient SMT solver. In C. R. Ramakrishnan and Jakob Rehof, editors, *Tools and Algorithms for the Construction and Analysis of Systems, 14th International Conference, TACAS 2008, Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2008, Budapest, Hungary, March 29-April 6, 2008. Proceedings*, volume 4963 of *Lecture Notes in Computer Science*, pages 337–340. Springer, 2008.

[EMM09] Santiago Escobar, Catherine Meadows, and José Meseguer. Maude-NPA: Cryptographic protocol analysis modulo equational properties. In Alessandro Aldini, Gilles Barthe, and Roberto Gorrieri, editors, *Foundations of Security Analysis and Design V*, volume 5705 of *Lecture Notes in Computer Science*, pages 1–50. Springer, 2009.

[EMOMV07] Steven Eker, Narciso Martí-Oliet, José Meseguer, and Alberto Verdejo. Deduction, strategies, and rewriting. In Myla Archer, Thierry Boy de la Tour, and César Muñoz, editors, *Proceedings of the 6th International Workshop on Strategies in Automated Deduction, STRATEGIES 2006, Seattle, WA, USA, August 16, 2006*, volume 174(11) of *Electronic Notes in Theoretical Computer Science*, pages 3–25. Elsevier, 2007.

[Esc14] Santiago Escobar, editor. *Rewriting Logic and Its Applications - 10th International Workshop, WRLA 2014, Held as a Satellite Event of ETAPS, Grenoble, France, April 5-6, 2014, Revised Selected Papers*, volume 8663 of *Lecture Notes in Computer Science*. Springer, 2014.

[ESM12] Santiago Escobar, Ralf Sasse, and José Meseguer. Folding variant narrowing and optimal variant termination. *Journal of Logic and Algebraic Programming*, 81(7-8):898–928, 2012.

[Fay79] M. Fay. First-order unification in an equational theory. In *Proc. 4th Workshop on Automated Deduction*, pages 161–167, Austin, TX, USA, 1979. Academic Press.

[Fri85] Laurent Fribourg. SLOG: A logic programming language interpreter based on clausal superposition and rewriting. In *Proceedings of the 1985 Symposium on Logic Programming, Boston, Massachusetts, USA, July 15-18, 1985*, pages 172–184. IEEE-CS, 1985.

[GK01] Jürgen Giesl and Deepak Kapur. Dependency pairs for equational rewriting. In Aart Middeldorp, editor, *Rewriting Techniques and Applications, 12th International Conference, RTA 2001, Utrecht, The Netherlands, May 22-24, 2001, Proceedings*, volume 2051 of *Lecture Notes in Computer Science*, pages 93–108. Springer, 2001.

[GM86] Elio Giovannetti and Corrado Moiso. A completeness result for E-unification algorithms based on conditional narrowing. In Mauro Boscarol, Luigia Carlucci Aiello, and Giorgio Levi, editors, *Foundations of Logic and Functional Programming, Workshop, Trento, Italy, December 15-19, 1986, Proceedings*, volume 306 of *Lecture Notes in Computer Science*, pages 157–167. Springer, 1986.

[Ham00] Mohamed Hamada. Strong completeness of a narrowing calculus for conditional rewrite systems with extra variables. *Electronic Notes in Theoretical Computer Science*, 31:89–103, 2000.

[Hul80] Jean-Marie Hullot. Canonical forms and unification. In Wolfgang Bibel and Robert A. Kowalski, editors, *5th Conference on Automated Deduction, Les Arcs, France, July 8-11, 1980, Proceedings*, volume 87 of *Lecture Notes in Computer Science*, pages 318–334. Springer, 1980.

[KR94] Hélène Kirchner and Christophe Ringeissen. Constraint solving by narrowing in combined algebraic domains. In Pascal Van Hentenryck, editor, *Logic Programming, Proceedings of the Eleventh International Conference on Logic Programming, Santa Marherita Ligure, Italy, June 13-18, 1994*, pages 617–631. MIT Press, 1994.

[LMM05] Salvador Lucas, Claude Marché, and José Meseguer. Operational termination of conditional term rewriting systems. *Information Processing Letters*, 95(4):446–453, 2005.

[Mes90] José Meseguer. Rewriting as a unified model of concurrency. In J.C.M. Baeten and J.W. Klop, editors, *CONCUR '90 Theories of Concurrency: Unification and Extension*, volume 458 of *Lecture Notes in Computer Science*, pages 384–400. Springer, 1990.

[Mes97] José Meseguer. Membership algebra as a logical framework for equational specification. In Francesco Parisi-Presicce, editor, *Recent Trends in Algebraic Development Techniques, 12th International Workshop, WADT'97, Tarquinia, Italy, June 1997, Selected Papers*, volume 1376 of *Lecture Notes in Computer Science*, pages 18–61. Springer, 1997.

[Mes12] José Meseguer. Twenty years of rewriting logic. *Journal of Logic and Algebraic Programming*, 81(7-8):721–781, 2012.

[Mes17] José Meseguer. Strict coherence of conditional rewriting modulo axioms. *Theor. Comput. Sci.*, 672(C):1–35, April 2017.

[Mes20] José Meseguer. Generalized rewrite theories, coherence completion, and symbolic methods. *J. Log. Algebraic Methods Program.*, 110, 2020.

[MH94] Aart Middeldorp and Erik Hamoen. Completeness results for basic narrowing. *Applicable Algebra in Engineering, Communication and Computing*, 5:213–253, 1994.

[MM02] Narciso Martí-Oliet and José Meseguer. Rewriting logic: roadmap and bibliography. *Theoretical Computer Science*, 285(2):121–154, 2002.

[MOMV04] Narciso Martí-Oliet, José Meseguer, and Alberto Verdejo. Towards a strategy language for Maude. In Narciso Martí-Oliet, editor, *Proceedings of the Fifth International Workshop on Rewriting Logic and its Applications, WRLA 2004, Barcelona, Spain, March 27-April 4, 2004*, volume 117 of *Electronic Notes in Theoretical Computer Science*, pages 417–441. Elsevier, 2004.

[MT07] José Meseguer and Prasanna Thati. Symbolic reachability analysis using narrowing and its application to verification of cryptographic protocols. *Higher-Order and Symbolic Computation*, 20(1-2):123–160, 2007.

[Plo72] Gordon Plotkin. Building in equational theories. *Machine Intelligence 7*, pages 73–90, 1972.

[RMM17] Camilo Rocha, José Meseguer, and César A. Muñoz. Rewriting modulo SMT and open system analysis. *J. Log. Algebr. Meth. Program.*, 86(1):269–297, 2017.

[RMPV18] Rubén Rubio, Narciso Martí-Oliet, Isabel Pita, and Alberto Verdejo. Parameterized strategies specification in Maude. In José Luiz Fiadeiro and Ionut Tutu, editors, *Recent Trends in Algebraic Development Techniques - 24th IFIP WG 1.3 International Workshop, WADT 2018, Egham, UK, July 2-5, 2018, Revised Selected Papers*, volume 11563 of *Lecture Notes in Computer Science*, pages 27–44. Springer, 2018.

A Appendix

This appendix holds the rest of the proofs of this work.

Lemma 5. Given a rewrite theory $\mathcal{R} = (\Sigma, E_0 \cup B, R)$, a set of call strategy definitions $Call_{\mathcal{R}}$, and terms $t, v \in \mathcal{H}_\Sigma$, for each c.p.t. T formed using the rules in $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$ with head $t \to v/ST$, so $[v]_E \in ST@[t]_E$, each renaming α such that $ran(\alpha) \cap (V_T \cup V_{\mathcal{R},Call_{\mathcal{R}}}) = \emptyset$, and each strategy $ST' =_E ST$ it holds that:

1. Main property: $t \to_{R/E} v$ and there exist closed proof trees for $[v]_E \in ST\alpha@[t]_E$ and $[v]_E \in ST'@[t]_E$ with the same depth and number of nodes as T.
2. If $ST = \texttt{idle}$ then $[t]_E = [v]_E$.
3. If $ST = c[\gamma]$ then $t \underset{c\gamma}{\longrightarrow}{}^{1}_{R/E} v$.
4. If $ST = \texttt{top}(c[\gamma])$, then $t \underset{c\gamma,\epsilon}{\longrightarrow}{}^{1}_{R/E} v$ (i.e., the rewrite happens at the top position of t).
5. If $ST = \texttt{match}\, u\, \texttt{s.t.}\, \bigwedge_{j=1}^{m}(l_j = r_j) \wedge \phi$ then $[t]_E = [v]_E$ and there exists a substitution σ such that $t =_E u\sigma$, $l_j\sigma =_E r_j\sigma$, for $1 \leq j \leq m$, and $E_0 \vDash \phi\sigma$.
6. If $ST = ST_1 \,;\, ST_2$ then there exists a term $u \in \mathcal{H}_\Sigma$ such that $[u]_E \in ST_1@[t]_E$ and $[v]_E \in ST_2@[u]_E$.
7. If $ST = ST_1+$ then there exist $i+1$ terms $u_0 = t, u_1, \ldots, u_{i-1}, u_i = v \in \mathcal{H}_\Sigma$, with $i > 0$, such that $[u_j]_E \in ST_1@[u_{j-1}]_E$, for $1 \leq j \leq i$, where i is equal to one plus the number of times that a rule with the form $\frac{w_1 \to w_2/ST_1 \,;\, ST_1+}{w_1 \to w_2/ST_1+}$, followed by the application of a rule with the form $\frac{\overline{w_1 \to w'/ST_1} \quad \overline{w' \to w_2/ST_1+}}{w_1 \to w_2/ST_1 \,;\, ST_1+}$, is applied in the rightmost branch of the subtree before applying a rule with the form $\frac{w_1 \to w_2/ST_1}{w_1 \to w_2/ST_1+}$.
8. If $ST = ST_1 \mid ST_2$ then $[v]_E \in ST_1@[t]_E$ or $[v]_E \in ST_2@[t]_E$.
9. If $ST = \texttt{match}\, u\, \texttt{s.t.}\, \phi \,?\, ST_1 \,:\, ST_2$ then there exists a substitution δ such that $t =_E u\delta$ and either $E_0 \vDash \phi\delta$ and $[v]_E \in ST_1\delta@[t]_E$ or $E_0 \vDash \neg\phi\delta$ and $[v]_E \in ST_2\delta@[t]_E$.
10. If $ST = CS$, where $\texttt{sd}\, CS := ST_1 \in Call_{\mathcal{R}}$, then: (i) $[v]_E \in ST_1@[t]_E$, and (ii) $[v]_E \in ST_1\gamma@[t]_E$, for every renaming γ such that $dom(\gamma) \subseteq vars(ST_1) \setminus V_{\mathcal{R}}$ and $ran(\gamma) \cap V_{\mathcal{R},Call_{\mathcal{R}}} = \emptyset$.
11. If $ST = CS(\bar{t})$, where $\texttt{sd}\, CS(\bar{x}) := ST_1 \in Call_{\mathcal{R}}$, $\bar{x} = x^1_{s_1}, \ldots, x^n_{s_n}$, $\bar{t} = t_1, \ldots, t_n$, and $\rho = \{\bar{x} \mapsto \bar{t}\}$, then: (i) $[v]_E \in ST_1\rho@[t]_E$ and (ii) if γ is a renaming such that $dom(\gamma) \subseteq vars(ST_1) \setminus \hat{x}$ and $ran(\gamma) \cap (ran(\rho) \cup V_{\mathcal{R},Call_{\mathcal{R}}}) = \emptyset$ (so $\frac{t \to v/ST_1(\gamma \cup \rho)}{t \to v/CS(\bar{t})} \in \mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$), then $[v]_E \in ST_1(\gamma \cup \rho)@[t]_E$.
12. If $ST = CS(\bar{t})$, where $\texttt{csd}\, CS(\bar{x}) := ST_1$ *if* $C \in Call_{\mathcal{R}}$, with $\bar{x} = x^1_{s_1}, \ldots, x^n_{s_n}$ and $C = \bigwedge_{j=1}^{m}(l_j = r_j) \wedge \phi$, call $V_{CS} = vars(ST_1) \cup vars(C)$, $\hat{x} \subseteq V_{CS}$, $\bar{t} = t_1, \ldots, t_n$, and $\rho = \{\bar{x} \mapsto \bar{t}\}$, then (i) there exists a substitution $\delta_1 : vars(C\rho) \to \mathcal{T}_\Sigma$, such that $l_j\rho\delta_1 =_E r_j\rho\delta_1$, for $1 \leq j \leq n$, $E_0 \vDash \phi\rho\delta_1$ (so $\frac{t \to v/ST_1\rho\delta_1}{t \to v/CS(\bar{t})} \in \mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$), and $[v]_E \in ST_1\rho\delta_1@[t]_E$, and (ii) for every renaming γ such that $dom(\gamma) \subseteq V_{CS} \setminus \hat{x}$ and $ran(\gamma) \cap (ran(\rho) \cup V_{\mathcal{R},Call_{\mathcal{R}}}) = \emptyset$, there exists a substitution $\delta_2 : vars(C(\gamma \cup \rho)) \to \mathcal{T}_\Sigma$, such that $l_j(\gamma \cup \rho)\delta_2 =_E r_j(\gamma \cup \rho)\delta_2$, for $1 \leq j \leq n$, $E_0 \vDash \phi(\gamma \cup \rho)\delta_2$ (so $\frac{t \to v/ST_1(\gamma \cup \rho)\delta_2}{t \to v/CS(\bar{t})} \in \mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$), and $[v]_E \in ST_1(\gamma \cup \rho)\delta_2@[t]_E$.

13. If $ST = c[\gamma]\{ST_1, \ldots, ST_m\}$, with $c : l \to r$ *if* $\bigwedge_{j=1}^m l_j \to r_j \mid \psi$ a rule in R, then there is a substitution δ such that $[r_i\gamma\delta]_E \in ST_i\delta \; @ \; [l_i\gamma\delta]_E$, for $1 \leq i \leq m$, and $t \xrightarrow[c,\gamma\delta]{1}_{R/E} v$.

14. If $ST = \texttt{top}(c[\gamma]\{ST_1, \ldots, ST_m\})$, with $c : l \to r$ *if* $\bigwedge_{j=1}^m l_j \to r_j \mid \psi$ a rule in R then there is a substitution δ such that $[r_i\gamma\delta]_E \in ST_i\delta @ [l_i\gamma\delta]_E$, for $1 \leq i \leq m$, and $t \xrightarrow[c,\epsilon,\gamma\delta]{1}_{R/E} v$.

15. If $ST = \texttt{matchrew}\ u\ \texttt{s.t.}\ \bigwedge_{j=1}^m (l_j = r_j) \wedge \phi\ \texttt{by}\ x_{s_1}^1\ \texttt{using}\ ST_1, \ldots,\ x_{s_n}^n\ \texttt{using}\ ST_n$, where $u = u[x_{s_1}^1, \ldots, x_{s_n}^n]_{p_1 \ldots p_n}$ then there exist a substitution δ, where $\delta_{V_{u,\phi,\bar{l},\bar{r}}}$ is ground, and terms $t_1, \ldots, t_n \in \mathcal{H}_\Sigma$ such that $t =_E u\delta$, $l_j\delta =_E r_j\delta$, for $1 \leq j \leq m$, $E_0 \vDash \phi\delta$, $[t_i]_E \in ST_i\delta \; @ \; [x_{s_i}^i\delta]_E$, for $1 \leq i \leq n$, and $v =_E u\delta[t_1, \ldots, t_n]_{p_1 \ldots p_n}$.

Proof. The proof for the first property is done by induction on the depth of the c.p.t. T for $t \to v/ST$. The rest of the properties are proved when the related strategy is treated in the proof for the first property. As $ran(\alpha) \cap vars(ST) = \emptyset$ then $vars(ST) \cap dom(\alpha^{-1}) = \emptyset$, so $ST\alpha\alpha^{-1} = ST$.

- There are five strategies in the base case: `fail`, `idle`, $c[\gamma]$, $\texttt{top}(c[\gamma])$, and the `match` test. The depth and number of nodes of all the closed proof trees is one in this case.

  1. As there are no derivation rules for `fail`, there is nothing to prove in this case.
  2. If $[v]_E \in \texttt{idle}@[t]_E = \{[t]_E\}$ then, as shown in example 10, $[v]_E = [t]_E$ (**property 2**), so $v =_E t$ and, by definition, $t \to_{R/E} v$. As $\texttt{idle}\,\alpha = \texttt{idle}$ then also $[v]_E \in \texttt{idle}\,\alpha@[t]_E$ using the original c.p.t. T. As only $\texttt{idle} =_E \texttt{idle}$, there is nothing to prove about the strategies that are equal modulo E to `idle`.
  3. If $[v]_E \in c[\gamma]@[t]_E$, with $c : l \to r$ *if* ϕ, then $\overline{t \to v/c[\gamma]}$ must come from a derivation rule $\overline{t' \to v'/c[\gamma]}$ in $\mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$, where $t' \xrightarrow[c,p,\gamma\delta]{1}_R v'$ for proper p and δ such that $t =_E t' = t'[l\gamma\delta]_p$, $v =_E v' = t'[r\gamma\delta]_p$, and $E_0 \vDash \phi\gamma\delta$, so $t \xrightarrow[c,p,\gamma\delta]{1}_{R/E} v$ (**property 3**).

     $c[\gamma]\alpha = c[(\gamma\alpha)_{dom(\gamma)}]$, call $\beta = (\gamma\alpha)_{dom(\gamma)}$ and let $\delta' = \alpha^{-1}\delta$. As $ran(\alpha) \cap (V_T \cup V_{\mathcal{R},Call_\mathcal{R}}) = \emptyset$ then $c\beta\delta' = c(\gamma\alpha)_{dom(\gamma)}\alpha^{-1}\delta = c\gamma\delta$, so also $t \xrightarrow[c,\beta\delta']{1}_{R/E} v$, and there is a derivation rule $\overline{t' \to v'/c[\beta]} \in \mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$, so $\overline{t \to v/c[\gamma]\alpha}$ is a c.p.t. for $[v]_E \in c[\gamma]\alpha@[t]_E$ because $t =_E t'$, $v =_E v'$, and $c[\gamma]\alpha = c[\beta]$.
     As $ST = c[\gamma] =_E ST'$, then $ST' = c[\gamma']$ where $\gamma =_E \gamma'$, so $(l, r, \phi)\gamma =_E (l, r, \phi)\gamma'$, with $V_{l\gamma} = V_{l\gamma'}$ and $V_{r\gamma} = V_{r\gamma'}$, hence $E_0 \vDash \phi\gamma'\delta$, $t =_E t'[l\gamma\delta]_p =_E t'[l\gamma'\delta]_p$ and $v =_E t'[r\gamma\delta]_p =_E t'[r\gamma'\delta]_p$, ground terms, and $t'[l\gamma'\delta]_p \xrightarrow[c,p,\gamma'\delta]{1}_R t'[r\gamma'\delta]_p$. Then, there is a derivation rule $\overline{t'[l\gamma'\delta]_p \to t'[r\gamma'\delta]_p/c[\gamma']}$ in $\mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$, so $\overline{t \to v/c[\gamma']}$ is a c.p.t. for $[v]_E \in c[\gamma']@[t]_E$.
  4. If $[v]_E \in \texttt{top}(c[\gamma])@[t]_E$, where $c : l \to r$ is a rule in R, then $T = \overline{t \to v/\texttt{top}(c[\gamma])}$ must come from a derivation rule $\overline{l\gamma\delta \to r\gamma\delta/\texttt{top}(c[\gamma])} \in \mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$, meaning that $l\gamma\delta \xrightarrow[c,\epsilon,\gamma\delta]{1}_R r\gamma\delta$, such that $l\gamma\delta =_E t$ and $r\gamma\delta =_E v$, so $t \xrightarrow[c,\epsilon,\gamma\delta]{1}_{R/E} v$ (**property 4**). Call $\beta = (\gamma\alpha)_{dom(\gamma)}$. As in the previous case, $\texttt{top}(c[\gamma])\alpha = \texttt{top}(c[\gamma]\alpha) = \texttt{top}(c[\beta])$. If we take $\delta' = \alpha^{-1}\delta$, then $c\beta\delta' = c\gamma\delta$ so also $l\gamma\delta \xrightarrow[c,\epsilon,\beta\delta']{1}_R r\gamma\delta$ and $\overline{l\gamma\delta \to r\gamma\delta/\texttt{top}(c[\beta])} \in \mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$, so $\overline{t \to v/\texttt{top}(c[\gamma])\alpha}$ is a c.p.t. for $[v]_E \in \texttt{top}(c[\gamma])\alpha@[t]_E$, because $l\gamma\delta =_E t$, $r\gamma\delta =_E v$, and $\texttt{top}(c[\gamma])\alpha = \texttt{top}(c[\beta])$.
     As $ST = \texttt{top}(c[\gamma]) =_E ST'$ then $ST' = \texttt{top}(c[\gamma'])$ where $\gamma =_E \gamma'$, so $(l, r, \phi)\gamma =_E (l, r, \phi)\gamma'$, with $V_{l\gamma} = V_{l\gamma'}$ and $V_{r\gamma} = V_{r\gamma'}$, hence $E_0 \vDash \phi\gamma'\delta$, $t =_E l\gamma\delta =_E l\gamma'\delta$ and $v =_E r\gamma\delta =_E r\gamma'\delta$, ground terms, and $l\gamma'\delta \xrightarrow[c,\epsilon,\gamma'\delta]{1}_R r\gamma'\delta$. Then, there is a

derivation rule $\frac{}{t'[l\gamma'\delta]_p \to t'[r\gamma'\delta]_p/\mathtt{top}(c[\gamma'])}$ in $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$, so $\frac{}{t \to v/\mathtt{top}(c[\gamma'])}$ is a c.p.t. for $[v]_E \in \mathtt{top}(c[\gamma'])@[t]_E$.

5. If $ST = \mathtt{match}\, u\, \mathtt{s.t.}\, \bigwedge_{j=1}^{m}(l_j = r_j) \wedge \phi$ and $[v]_E \in ST@[t]_E$, then $T = \frac{}{t \to v/ST}$ must come from a rule $\frac{}{w \to w/ST}$ in $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$ such that $t =_E w$ and $v =_E w$, so $t =_E v$ (i.e. $[t]_E = [v]_E$), and there exists a substitution σ such that $w =_E u\sigma$, so $t =_E u\sigma$, $l_j\sigma =_E r_j\sigma$, for $1 \leq j \leq m$, and $E_0 \vDash \phi\sigma$ (**property 5**). As $t =_E v$ then, by definition, $t \to_{R/E} v$.

   As $ST\alpha = \mathtt{match}\, u\alpha\, \mathtt{s.t.}\, \bigwedge_{j=1}^{m}(l_j\alpha = r_j\alpha) \wedge \phi\alpha$, if we take $\sigma' = \alpha^{-1}\sigma$ then, trivially, $w =_E u\alpha\sigma'$, so $t =_E u\alpha\sigma'$, $l_j\alpha\sigma' =_E r_j\alpha\sigma'$, for $1 \leq j \leq m$, and $E_0 \vDash \phi\alpha\sigma'$, so there is a rule $\frac{}{w \to w/ST\alpha} \in \mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$, hence $\frac{}{t \to v/ST\alpha}$ is a c.p.t. for $[v]_E \in ST\alpha@[t]_E$.

   As $ST =_E ST'$, then $ST' = \mathtt{match}\, u'\, \mathtt{s.t.}\, \bigwedge_{j=1}^{m}(l'_j = r'_j) \wedge \phi'$ where $(u, \bar{l}, \bar{r}, \phi) =_E (u', \bar{l}', \bar{r}', \phi')$, with $V_{u,\bar{l},\bar{r},\phi} = V_{u',\bar{l}',\bar{r}',\phi'}$, so $V_{(u',\bar{l}',\bar{r}',\phi')\sigma} = \emptyset$, hence $\frac{}{w \to w/ST'}$ is a derivation rule in $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$, since $w =_E u\sigma =_E u'\sigma$, $\bar{l}'\sigma =_E \bar{l}\sigma =_E \bar{r}\sigma =_E \bar{r}'\sigma$, and $E_0 \vDash \phi\sigma$ and $\phi =_E \phi'$ implies $E_0 \vDash \phi'\sigma$. Then, as $t =_E w$ and $v =_E w$, $\frac{}{t \to v/ST'}$ is a c.p.t. for $[v]_E \in ST'@[t]_E$.

- Inductive step:

6. $ST = ST_1 \,;\, ST_2$.

   If $[v]_E \in ST@[t]_E$ then $T = \frac{T_1 \quad T_2}{t \to v/ST_1\,;\,ST_2}$ comes from a rule $\frac{t \to u/ST_1 \quad u \to v/ST_2}{t \to v/ST_1\,;\,ST_2}$, where T_1 and T_2 are closed proof trees with head $t \to u/ST_1$ and $u \to v/ST_2$, respectively, so $[u]_E \in ST_1@[t]_E$ and $[v]_E \in ST_2@[u]_E$ (**property 6**). As these closed proof trees are of a smaller depth then, by I.H. and property 1, $t \to_{R/E} u$ and $u \to_{R/E} v$, so $t \to_{R/E} v$.

   As $ST\alpha = ST_1\alpha ; ST_2\alpha$, we can apply the I.H. to T_1 and T_2, so there are closed proof trees T'_1 and T'_2 with head $t \to u/ST_1\alpha$ and $u \to v/ST_2\alpha$, respectively. As there is a rule $\frac{t \to u/ST_1\alpha \quad u \to v/ST_2\alpha}{t \to v/ST\alpha} \in \mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$ then $\frac{T'_1 \quad T'_2}{t \to v/ST\alpha}$ is a c.p.t. for $[v]_E \in ST\alpha@[t]_E$.

   As $ST =_E ST'$, then $ST' = ST'_1 \,;\, ST'_2$ where $ST_1 =_E ST'_1$ and $ST_2 =_E ST'_2$. As T_1 and T_2 are of a smaller depth than T then, by I.H., there are closed proof trees T'_1 and T'_2 for $[u]_E \in ST'_1@[t]_E$ and $[v]_E \in ST'_2@[u]_E$, with the same depth and number of nodes as T_1 and T_2, respectively, and $\frac{T'_1 \quad T'_2}{t \to v/ST'}$ is a c.p.t. for $[v]_E \in ST'@[t]_E$ with the same depth and number of nodes as T.

7. $ST = ST_1+$.

   T must be either of the form $\frac{T_1}{t \to v/ST_1+}$ or $\frac{T_2}{t \to v/ST_1+}$, where T_1 has head $t \to v/ST_1$ or T_2 has head $t \to v/ST_1 \,;\, ST_1+$.

   In the first case, $i = 1$ because no rule with the form $\frac{w_1 \to w_2/ST_1\,;ST_1+}{w_1 \to w_2/ST_1+}$ has been applied, and there are 2 terms, u_0 (we take t) and u_1 (we take v), in $\mathcal{H}_\Sigma$ such that $u_0 = t$, $u_1 = v$, and $[u_1]_E \in ST_1@[u_0]_E$, because we have a c.p.t. for $t \to v/ST_1$.

   In the second case, we can apply I.H. to the c.p.t. for $u_1 \to v/ST_1+$ so there are i terms $w_0 = u_1, \ldots, w_{i-2}, w_{i-1} = v$ such that $[w_j]_E \in ST_1@[w_{j-1}]_E$, for $1 \leq j \leq i-1$. As there is a c.p.t. for $t \to u_1/ST_1$ in the left branch, then also $[u_1]_E \in ST_1@[t]_E$. Taking $u_0 = t$ and $u_{j+1} = w_j$ for $1 \leq j \leq i-1$ we get $u_0 = t$, $u_i = w_{i-1} = v$, and $[u_j]_E \in ST_1@[u_{j-1}]_E$, for $1 \leq j \leq i$ (**property 7**).

   In either case we also have a c.p.t. of a smaller depth whose head has the form $t \to v/\ldots$ so, by I.H., $t \to_{R/E} v$. Also by I.H., we have either a c.p.t. T'_1 with head $= t \to v/ST_1\alpha$ or T'_2 with head $t \to v/ST_1\alpha \,;\, ST_1\alpha+$ with depth equal, whichever the case, to $depth(T) - 1$. As $ST_1 + \alpha = ST_1\alpha+$ and there are rules $\frac{t \to v/ST_1\alpha}{t \to v/ST_1\alpha+}$

and $\frac{t \to v/ST_1\alpha\,;\,ST_1\alpha+}{t \to v/ST_1\alpha+}$ in $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$ then either $\frac{T_1'}{t \to v/ST_1\alpha+}$ or $\frac{T_2'}{t \to v/ST_1\alpha+}$ is a c.p.t. for $[v]_E \in ST\alpha @ [t]_E$ with the same depth and number of nodes as T.
As $ST =_E ST'$, then $ST' = ST_1'+$ where $ST_1 =_E ST_1'$. As T_j, where j in $\{1,2\}$, has smaller depth than T then, by I.H., there is a c.p.t. T' for $[v]_E \in ST_1' @ [t]_E$ or $[v]_E \in ST_1'\,;\,ST_1 + @[t]_E$ with the same depth and number of nodes as T_j, and $\frac{T'}{t \to v/ST'}$ is a c.p.t. for $[v]_E \in ST' @ [t]_E$ with the same depth and number of nodes as T.

8. $ST = ST_1 \mid ST_2$.
T must be either of the form $\frac{T_1}{t \to v/ST_1 \mid ST_2}$ or $\frac{T_2}{t \to v/ST_1 \mid ST_2}$, where T_1 has head $t \to v/ST_1$ or T_2 has head $t \to v/ST_2$, so either $[v]_E \in ST_1 @ [t]_E$ or $[v]_E \in ST_2 @ [t]_E$ must hold (**property 8**) and, by I.H., $t \to_{R/E} v$. Also by I.H. there is a c.p.t T_1', with head $t \to v/ST_1\alpha$, or T_2', with head $t \to v/ST_2\alpha$ with depth equal, whichever the case, to $depth(T) - 1$.
As $ST\alpha = ST_1\alpha \mid ST_2\alpha$ and there are rules $\frac{t \to v/ST_1\alpha}{t \to v/ST_1\alpha \mid ST_2\alpha}$ and $\frac{t \to v/ST_2\alpha}{t \to v/ST_1\alpha \mid ST_2\alpha}$ in $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$, then either $\frac{T_1'}{t \to v/ST_1\alpha \mid ST_2\alpha}$ or $\frac{T_2'}{t \to v/ST_1\alpha \mid ST_2\alpha}$ is a c.p.t. for $[v]_E \in ST\alpha @ [t]_E$ with the same depth and number of nodes as T.
As $ST =_E ST'$, then $ST' = ST_1' \mid ST_2'$ where $ST_1 =_E ST_1'$ and $ST_2 =_E ST_2'$. As T_j, where j in $\{1,2\}$, has smaller depth than T then, by I.H., there is a c.p.t. T' for $[v]_E \in ST_1' @ [t]_E$ or $[v]_E \in ST_2' @ [t]_E$, with the same depth and number of nodes as T_j, and $\frac{T'}{t \to v/ST'}$ is a c.p.t. for $[v]_E \in ST' @ [t]_E$ with the same depth and number of nodes as T.

9. $ST = \texttt{match}\, u\, \texttt{s.t.}\, \phi\, ?\, ST_1 \,:\, ST_2$.
By the definition of the derivation rules for the if-then-else strategy, T must be of the form $\frac{T_1}{t \to v/ST}$ or $\frac{T_2}{t \to v/ST}$, where T_1 has head $t \to v/ST_1\delta$ or T_2 has head $t \to v/ST_2\delta$, coming from the application of a rule with the form $\frac{t' \to v'/ST_1\delta}{t' \to v'/ST}$ or $\frac{t' \to v'/ST_2\delta}{t' \to v'/ST}$, with $t =_E t' =_E u\delta$ and $v =_E v'$. In the first case, by definition of the rule, $E_0 \vDash \phi\delta$ and, as T_1 is a c.p.t. for $t \to v/ST_1\delta$, $[v]_E \in ST_1\delta @ [t]_E$; in the second case, also by definition of the rule, $E_0 \vDash \neg\phi\delta$ and, as T_2 is a c.p.t. for $t \to v/ST_2\delta$, $[v]_E \in ST_2\delta @ [t]_E$ (**property 9**). In either case, as T_1 and T_2 are closed proof trees of a smaller depth whose head has the form $t \to v/\ldots$ then, by I.H., $t \to_{R/E} v$.
$ST\alpha = \texttt{match}\, u\alpha\, \texttt{s.t.}\, \phi\alpha\, ?\, ST_1\alpha \,:\, ST_2\alpha$. If we take $\delta' = \alpha^{-1}\delta$ then $\alpha\delta' = \delta$, so $u\alpha\delta' = u\delta$, $\phi\alpha\delta' = \phi\delta$, $ST_1\alpha\delta' = ST_1\delta$, and $ST_2\alpha\delta' = ST_2\delta$.
   - If $E_0 \vDash \phi\alpha\delta'$ (so $E_0 \vDash \phi\delta$) then T_1 exists and there is a rule $\frac{t' \to v'/ST_1\alpha\delta'}{t' \to v'/ST\alpha}$ (i.e., $\frac{t' \to v'/ST_1\delta}{t' \to v'/ST\alpha}$) in $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$, so $\frac{T_1}{t \to v/ST\alpha}$ is a c.p.t. for $[v]_E \in ST\alpha @ [t]_E$ with the same depth and number of nodes as T.
   - Else, T_2 exists and there is a rule $\frac{t' \to v'/ST_2\alpha\delta'}{t' \to v'/ST\alpha}$ (i.e., $\frac{t' \to v'/ST_2\delta}{t' \to v'/ST\alpha}$ in $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$), so $\frac{T_2}{t \to v/ST\alpha}$ is a c.p.t. for $[v]_E \in ST\alpha @ [t]_E$ with the same depth and number of nodes as T.

As $ST =_E ST'$, then $ST' = \texttt{match}\, u'\, \texttt{s.t.}\, \phi'\, ?\, ST_1' \,:\, ST_2'$ where $u =_E u'$, $\phi =_E \phi'$, $ST_1 =_E ST_1'$, $ST_2 =_E ST_2'$, $V_u = V_{u'}$, $V_\phi = V_{\phi'}$, $V_{ST_1} = V_{ST_1'}$, and $V_{ST_2} = V_{ST_2'}$. We prove the case where $E_0 \vDash \phi\delta$, the case where $E_0 \vDash \neg\phi\delta$ is proved in exactly the same way. As $\phi =_E \phi'$ and $V_\phi = V_{\phi'}$ then $E_0 \vDash \phi'\delta$, ground formula. Also, as $u =_E u'$ and $V_u = V_{u'}$, then $t =_E t' =_E u\delta =_E u'\delta$, so there is a derivation rule $\frac{t' \to v'/ST_1'\delta}{t' \to v'/ST'}$. As $ST_1 =_E ST_1'$ then $ST_1\delta =_E ST_1'\delta$ so, by I.H. since $t =_E t'$, $v =_E v'$, and T_1 has smaller depth than T, there is a c.p.t. $T_1' = \frac{T'}{t \to v/ST_1'\delta}$ for $[v]_E \in ST_1'\delta @ [t]_E$, with the same depth and number of nodes as T_1, and $\frac{T_1'}{t \to v/ST'}$ is a c.p.t. for $[v]_E \in ST' @ [t]_E$ with the same depth and number of nodes as T.

10. $ST = CS$, where `sd` $CS := ST_1$, and γ renaming such that $dom(\gamma) \subseteq vars(ST_1) \setminus V_{\mathcal{R}}$ and $ran(\gamma) \cap V_{\mathcal{R},Call_{\mathcal{R}}} = \emptyset$.
T must be of the form $\frac{T_1}{t \to v/CS}$, where T_1 has head $t \to v/ST_1\beta$, so $t \to_{R/E} v$, by I.H., for some renaming β such that $ran(\beta) \cap V_{\mathcal{R},Call_{\mathcal{R}}} = \emptyset$ (hence $dom(\beta^{-1}) \cap V_{\mathcal{R}} = \emptyset$). Also by I.H., if we take β^{-1}, as $dom(\beta^{-1}) \cap V_{\mathcal{R}} = \emptyset$ then there is a c.p.t. T_1' with head $t \to v/ST_1$ and the same depth and number of nodes as T_1, so $[v]_E \in ST_1@[t]_E$ (i), and if we take $\gamma' = \beta^{-1}\gamma$, as also $dom(\gamma) \cap V_{\mathcal{R}} = \emptyset$, there must be a c.p.t. with head $t \to v/ST_1\beta\gamma'$ (i.e., $t \to v/ST_1\gamma$), with the same depth and number of nodes as T_1, so $[v]_E \in ST_1\gamma@[t]_E$ (ii) (**property 10**).
As $dom(\alpha) \subseteq vars(CS) = \emptyset$ then $\alpha =$ `none`, so $ST\alpha = CS$ and T is also a c.p.t. for $[v]_E \in ST\alpha@[t]_E$.
As $ST' =_E ST$, then $ST' = CS = ST$, and T is also a c.p.t. for $[v]_E \in ST'@[t]_E$.
11. $ST = CS(\bar{t})$, where $\bar{t} = t_1, \ldots, t_n$, `sd` $CS(\bar{x}) := ST_1 \in Call_{\mathcal{R}}$, $\bar{x} = x^1_{s_1}, \ldots, x^n_{s_n}$, $\hat{x} \subseteq V_{CS}$, $\rho = \{x^1_{s_1} \mapsto t_1, \ldots, x^n_{s_n} \mapsto t_n\}$, with $ran(\rho) \subset \mathcal{X} \setminus V_{\mathcal{R},Call_{\mathcal{R}}}$ by the definition of call strategy, and γ is a renaming such that $dom(\gamma) \subseteq vars(ST_1) \setminus \hat{x}$ and $ran(\gamma) \cap (ran(\rho) \cup V_{\mathcal{R},Call_{\mathcal{R}}}) = \emptyset$.
T must be of the form $\frac{T_1}{t \to v/CS(\bar{t})}$, where T_1 has head $t \to v/ST_1(\beta \cup \rho)$ (so, by I.H., $t \to_{R/E} v$) for some renaming β such that $dom(\beta) \subseteq vars(ST_1) \setminus (\hat{x} \cup V_{\mathcal{R}})$ and $ran(\beta) \cap (ran(\rho) \cup V_{\mathcal{R},Call_{\mathcal{R}}}) = \emptyset$, hence $(\beta \cup \rho)\beta^{-1} = \rho$. Then, by I.H., there must exist a c.p.t T_2 with head $t \to v/ST_1\rho$ and the same depth and number of nodes as T_1 so $[v]_E \in ST_1\rho@[t]_E$ (i).
As $dom(\gamma) \subseteq vars(ST_1) \setminus \hat{x} \subseteq V_{\mathcal{R},Call_{\mathcal{R}}}$, $dom(\rho) = \hat{x}$, and $ran(\rho) \subset \mathcal{X} \setminus V_{\mathcal{R},Call_{\mathcal{R}}}$, so $ran(\rho) \cap dom(\gamma) = \emptyset$, then $ST_1(\gamma \cup \rho) = ST_1\rho\gamma$, with $dom(\gamma) \subseteq vars(ST_1\rho)$. Then $dom(\gamma) \subseteq vars(ST_1\rho)$. As T_2 has head $t \to v/ST_1\rho$ and the same depth and number of nodes as T_1, $dom(\gamma) \subseteq vars(ST_1\rho \setminus V_{\mathcal{R}})$, and $ran(\gamma) \cap (vars(ST_1\rho) \cup V_{\mathcal{R},Call_{\mathcal{R}}}) = \emptyset$ then, by I.H., there must exist a c.p.t. T_3 with head $t \to v/ST_1\rho\gamma$ (i.e., $t \to v/ST_1(\gamma \cup \rho)$), so $[v]_E \in ST_1(\gamma \cup \rho)@[t]_E$ (ii) (**property 11**).
As $dom(\alpha) \subseteq vars(ST) \setminus \hat{x} = vars(CS(\bar{t})) \setminus \hat{x} = ran(\rho)$, because $\hat{x} \notin vars(CS(\bar{t}))$ and $ran(\rho) \cap V_{\mathcal{R},Call_{\mathcal{R}}} = \emptyset$, then $ST\alpha = CS(\bar{t}\alpha)$ and as $ran(\rho) \subset \mathcal{X} \setminus V_{\mathcal{R},Call_{\mathcal{R}}} \subset \mathcal{X} \setminus vars(ST_1)$, so $dom(\alpha) \cap vars(ST_1) = \emptyset$, then $ST_1(\rho\alpha) = (ST_1\rho)\alpha$ and there is a derivation rule $\frac{t \to v/(ST_1\rho)\alpha}{t \to v/CS(\bar{t}\alpha)}$ in $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$. Now, as T_2 has head $t \to v/ST_1\rho$ and depth one less than the depth of T, $dom(\alpha) \subseteq ran(\rho) \subseteq vars(ST_1\rho)$ and $vars(ST_1) \subseteq V_{\mathcal{R},Call_{\mathcal{R}}}$, so $ran(\alpha) \cap (vars(ST_1\rho) \cup V_{\mathcal{R},Call_{\mathcal{R}}}) \subseteq ran(\alpha) \cap (ran(\rho) \cup V_{\mathcal{R},Call_{\mathcal{R}}}) = ran(\alpha) \cap (vars(ST) \cup V_{\mathcal{R},Call_{\mathcal{R}}}) = \emptyset$ then, by I.H., there is a c.p.t. T_4 with head $t \to v/(ST_1\rho)\alpha$ and the same depth and number of nodes as T_1, so $\frac{T_4}{t \to v/ST\alpha}$ is a c.p.t. for $[v]_E \in ST\alpha@[t]_E$ with the same depth and number of nodes as T.
As $ST' =_E ST$, then $ST' = CS(\bar{t}')$, where $\bar{t} =_E \bar{t}'$. Let $\rho' = \bar{x} \mapsto \bar{t}'$, so $\rho' =_E \rho$. As $T = \frac{T_1}{t \to v/CS(\bar{t})}$, where T_1 has head $t \to v/ST_1(\beta \cup \rho)$, then there is a derivation rule $\frac{t \to v/ST_1(\beta \cup \rho)}{t \to v/CS(\bar{t})}$, so there is also a derivation rule $\frac{t \to v/ST_1(\beta \cup \rho')}{t \to v/CS(\bar{t}')}$. As $ST_1(\beta \cup \rho) =_E ST_1(\beta \cup \rho')$ then, by I.H., there is a c.p.t. T_1' for $[v]_E \in ST_1(\beta \cup \rho')@[t]_E$ with the same depth and number of nodes as T_1, so $\frac{T_1'}{t \to v/CS(\bar{t}')}$ is a c.p.t. for $[v]_E \in CS(\bar{t}')@[t]_E$ with the same depth and number of nodes as T.
12. $ST = CS(\bar{t})$, where $\bar{t} = t_1, \ldots, t_n$, `csd` $CS(\bar{x}) := ST_1$ *if* $C \in Call_{\mathcal{R}}$, with $\bar{x} = x^1_{s_1}, \ldots, x^n_{s_n}$, $\hat{x} \subseteq V_{CS}$, and $C = \bigwedge_{j=1}^{m}(l_j = r_j) \wedge \phi$, call $V_C = vars(C)$, $V_{CS} = vars(ST_1) \cup V_C$, and $\rho = \{x^1_{s_1} \mapsto t_1, \ldots, x^n_{s_n} \mapsto t_n\}$, with $ran(\rho) \cap V_{\mathcal{R},Call_{\mathcal{R}}} = \emptyset$, and γ is a renaming such that $dom(\gamma) \subseteq V_{CS} \setminus \hat{x} = V_{CS} \setminus dom(\rho)$ and $ran(\gamma) \cap (ran(\rho) \cup V_{\mathcal{R},Call_{\mathcal{R}}}) = \emptyset$, so $C(\gamma \cup \rho)(\gamma_{vars(C)})^{-1} = C\rho$.
T must be of the form $\frac{T_1}{t \to v/CS(\bar{t})}$, where T_1 has head $t \to v/ST_1(\beta \cup \rho)\delta$ (so, by I.H., $t \to_{R/E} v$) for some renaming β such that $dom(\beta) \subseteq V_{CS} \setminus \hat{x} = V_{CS} \setminus dom(\rho)$, so

$dom(\beta) \cap dom(\rho) = \emptyset$ and $ran(\beta) \cap (ran(\rho) \cup V_{\mathcal{R},Call_{\mathcal{R}}}) = \emptyset$, so $ran(\beta) \cap (ran(\rho) \cup dom(\rho)) = \emptyset$ hence $\rho\beta = \beta \cup \rho$, and some substitution $\delta : vars(C(\beta \cup \rho)) \to \mathcal{T}_\Sigma$ such that $\bar{l}(\beta \cup \rho)\delta =_E \bar{r}(\beta \cup \rho)\delta$ and $E_0 \vDash \phi(\beta \cup \rho)\delta$.

Call $\delta_1 = \beta\delta$. As $\rho\beta = \beta \cup \rho$ then $\delta_1 : vars(C\rho) \to \mathcal{T}_\Sigma$ is a substitution such that $l_j\rho\delta_1 =_E r_j\rho\delta_1$, for $1 \leq j \leq n$, $E_0 \vDash \phi\rho\delta_1$. Also as $\rho\beta = \beta \cup \rho$, so $(\beta \cup \rho)\delta = \rho\beta\delta = \rho\delta_1$, T_1 is a c.p.t with head $t \to v/ST_1\rho\delta_1$ so, by definition, $[v]_E \in ST_1\rho\delta_1@[t]_E$ (i).

As $C(\gamma \cup \rho)(\gamma_{V_C})^{-1} = C\rho$ then $C(\gamma \cup \rho)(\gamma_{V_C})^{-1}\delta_1 = C\rho\delta_1$, call $\delta_2 = (\gamma_{V_C})^{-1}\delta_1$, hence $\delta_2 : vars(C(\gamma \cup \rho)) \to \mathcal{T}_\Sigma$ is a substitution such that $l_j(\gamma \cup \rho)\delta_2 =_E r_j(\gamma \cup \rho)\delta_2$, for $1 \leq j \leq n$, and $E_0 \vDash \phi(\gamma \cup \rho)\delta_2$. As $dom(\delta_1) = vars(C\rho)$ then $ST_1(\gamma \cup \rho)\delta_2 = ST_1(\gamma \cup \rho)(\gamma_{V_C})^{-1}\delta_1 = ST_1(\gamma_{V_C} \cup \gamma_{\setminus V_C} \cup \rho)(\gamma_{V_C})^{-1}\delta_1 = ST_1(\gamma_{\setminus V_C} \cup \rho)\delta_1 = ST_1(\gamma_{\setminus V_C} \cup \rho\delta_1)$, because as $ran(\gamma) \cap (ran(\rho) \cup V_{\mathcal{R},Call_{\mathcal{R}}}) = \emptyset$ and $vars(ST_1) \subseteq V_{\mathcal{R},Call_{\mathcal{R}}}$ then after $\gamma_{\setminus V_C}$ instantiates ST_1 in $ST_1(\gamma_{\setminus V_C} \cup \rho)$, δ_1 does not instantiate any renamed variable in $ran(\gamma_{\setminus V_C})$. Now, as δ_1 ground implies $ran(\rho\delta_1) \subseteq ran(\rho)$, $ran(\rho) \cap V_{\mathcal{R},Call_{\mathcal{R}}} = \emptyset$, and $dom(\gamma_{\setminus V_C}) \subseteq vars(ST_1) \subseteq V_{\mathcal{R},Call_{\mathcal{R}}}$, then $ST_1(\gamma_{\setminus V_C} \cup \rho\delta_1) = ST_1\rho\delta_1\gamma_{\setminus V_C}$, i.e., $ST_1(\gamma \cup \rho)\delta_2 = ST_1\rho\delta_1\gamma_{\setminus V_C}$.

In order to use I.H. we need to prove $ran(\gamma_{\setminus V_C}) \cap (vars(ST_1\rho\delta_1) \cup V_{\mathcal{R},Call_{\mathcal{R}}}) = \emptyset$ and $dom(\gamma_{\setminus V_C}) \subseteq vars(ST_1\rho\delta_1)$.

- By definition, $ran(\gamma) \cap (ran(\rho) \cup V_{\mathcal{R},Call_{\mathcal{R}}}) = \emptyset$. As $ran(\rho\delta_1) \subseteq ran(\rho)$ then also $ran(\gamma) \cap (ran(\rho\delta_1) \cup V_{\mathcal{R},Call_{\mathcal{R}}}) = \emptyset$, so $ran(\gamma_{\setminus V_C}) \cap (vars(ST_1\rho\delta_1) \cup V_{\mathcal{R},Call_{\mathcal{R}}}) = \emptyset$ because $vars(ST_1) \subseteq V_{\mathcal{R},Call_{\mathcal{R}}}$.
- As $dom(\gamma) \subseteq V_{CS} \setminus dom(\rho)$ and $V_{CS} = vars(ST_1) \cup V_C$ then $dom(\gamma_{\setminus V_C}) \subseteq vars(ST_1) \setminus (dom(\rho) \cup V_C)$ so $dom(\gamma_{\setminus V_C}) \subseteq vars(ST_1\rho) \setminus V_C$. Now, as $ran(\rho) \cap V_{\mathcal{R},Call_{\mathcal{R}}} = \emptyset$, so $ran(\rho) \cap vars(ST_1) = \emptyset$, and $dom(\gamma_{\setminus V_C}) \subseteq vars(ST_1) \setminus V_C$, then $dom(\gamma_{\setminus V_C}) \subseteq vars(ST_1\rho) \setminus (V_C \cup ran(\rho))$ so, as $dom(\delta_1) = vars(C\rho) \subseteq V_C \cup ran(\rho)$, then $dom(\gamma_{\setminus V_C}) \subseteq vars(ST_1\rho\delta_1)$.

Then, by I.H., there is a c.p.t. for $[v]_E \in ST_1\rho\delta_1\gamma_{\setminus V_C}@[t]_E$ hence, as $ST_1(\gamma \cup \rho)\delta_2 = ST_1\rho\delta_1\gamma_{\setminus V_C}$, also $[v]_E \in ST_1(\gamma \cup \rho)\delta_2@[t]_E$ (ii) (**property 12**).

As $dom(\alpha) \subseteq vars(ST) \setminus \hat{x} = vars(CS(\bar{t})) \setminus \hat{x} = ran(\rho)$, because $\hat{x} \notin vars(CS(\bar{t}))$ and $ran(\rho) \cap V_{\mathcal{R},Call_{\mathcal{R}}} = \emptyset$, then $ST\alpha = CS(\bar{t}\alpha)$. Also, as $ran(\alpha) \cap (vars(ST) \cup V_{\mathcal{R},Call_{\mathcal{R}}}) = \emptyset$, then $ran(\alpha) \cap (ran(\rho) \cup dom(\rho)) = \emptyset$ and $dom(\alpha^{-1}) \cap (vars(ST) \cup V_{\mathcal{R},Call_{\mathcal{R}}}) = \emptyset$ so, as $V_{CS} \subseteq V_{\mathcal{R},Call_{\mathcal{R}}}$, $C\rho\alpha\alpha^{-1} = C\rho$ and $ST_1\rho\alpha\alpha^{-1} = ST_1\rho$, hence $C\rho\alpha\alpha^{-1}\delta_1 = C\rho\delta_1$ and $ST_1\rho\alpha\alpha^{-1}\delta_1 = ST_1\rho\delta_1$, call $\delta_3 = \alpha^{-1}\delta_1$, so $\delta_3 : vars(C\rho\alpha) \to \mathcal{T}_\Sigma$ is a substitution such that $l_j\rho\alpha\delta_3 =_E r_j\rho\alpha\delta_3$, for $1 \leq j \leq n$ and $E_0 \vDash \phi\rho\alpha\delta_3$ and there is a derivation rule $\frac{t \to v/ST_1\rho\alpha\delta_3}{t \to v/CS(\bar{t}\alpha)} \in \mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$. Then, as $ST_1\rho\alpha\delta_3 = ST_1\rho\delta_1$ implies $t \to v/ST_1\rho\alpha\delta_3 = t \to v/ST_1\rho\delta_1$ and T_1 has head $t \to v/ST_1\rho\delta_1$, $\frac{T_1}{t \to v/CS(\bar{t}\alpha)}$ is a c.p.t. for $[v]_E \in ST\alpha@[t]_E$ with the same depth and number of nodes as T.

As $ST' =_E ST$, then $ST' = CS(\bar{t}')$, where $\bar{t} =_E \bar{t}'$. Let $\rho' = \bar{x} \mapsto \bar{t}'$, so $\rho' =_E \rho$. As $T = \frac{T_1}{t \to v/CS(\bar{t})}$, where T_1 has head $t \to v/ST_1(\beta \cup \rho)$, then there is a derivation rule $\frac{t \to v/ST_1(\beta \cup \rho)}{t \to v/CS(\bar{t})}$. As $\rho =_E \rho'$, then $\bar{l}(\beta \cup \rho')\delta =_E \bar{l}(\beta \cup \rho)\delta =_E \bar{r}(\beta \cup \rho)\delta =_E \bar{r}(\beta \cup \rho')\delta$ and $E_0 \vDash \phi(\beta \cup \rho')\delta$, so there is also a derivation rule $\frac{t \to v/ST_1(\beta \cup \rho')}{t \to v/CS(\bar{t}')}$. As $ST_1(\beta \cup \rho) =_E ST_1(\beta \cup \rho')$ then, by I.H., there is a c.p.t. T_1' for $[v]_E \in ST_1(\beta \cup \rho')@[t]_E$ with the same depth and number of nodes as T_1, so $\frac{T_1'}{t \to v/CS(\bar{t}')}$ is a c.p.t. for $[v]_E \in CS(\bar{t}')@[t]_E$ with the same depth and number of nodes as T.

13. $ST = c[\gamma]\{\overline{ST}\}$, with $c : l \to r$ *if* $\bigwedge_{j=1}^{m} l_j \to r_j \mid \psi$ a rule in R, $\overline{ST} = ST_1, \ldots, ST_m$, and $dom(\gamma) \cap vars(\overline{ST}) = \emptyset$.

    T must be of the form $\frac{T_1 \cdots T_m}{t \to v/c[\gamma]\{\overline{ST}\}}$, where T_i, $1 \leq i \leq m$, are closed proof trees with head $l_i\gamma\delta \to r_i\gamma\delta/ST_i\delta$ (so, by I.H., $l_i\gamma\delta \to_{R/E} r_i\gamma\delta$ and $[r_j\gamma\delta]_E \in ST_j\delta@[l_j\gamma\delta]_E$), because there is a derivation rule $\frac{l_1\gamma\delta \to r_1\gamma\delta/ST_1\delta \cdots l_m\gamma\delta \to r_m\gamma\delta/ST_m\delta}{u \to u[r\gamma\delta]_p/c[\gamma]\{\overline{ST}\}} \in \mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$, where

$u \in \mathcal{H}_\Sigma$, $p \in pos(u)$, $\delta : vars(c\gamma) \to \mathcal{T}_\Sigma$, $u = u[l\gamma\delta]_p =_E t$, $u[r\gamma\delta]_p =_E v$, and $E_0 \vDash \psi\gamma\delta$ so, by definition as also $l_i\gamma\delta \to_{R/E} r_i\gamma\delta$, $1 \leq i \leq m$, $t \xrightarrow[c,u,p,\gamma\delta]{}{}^1_{R/E} v$ (**property 13**).

Call $\gamma' = (\gamma\alpha)_{dom(\gamma)}$ so $ST\alpha = c[\gamma']\{\overline{ST}\alpha\}$.

If we take $\delta' = \alpha^{-1}\delta$, as $dom(\alpha^{-1}) = ran(\alpha)$, $ran(\alpha) \cap (V_T \cup V_{\mathcal{R},Call_\mathcal{R}}) = \emptyset$, $\delta : vars(c\gamma) \to \mathcal{T}_\Sigma$, then $c\gamma'\delta' = c(\gamma\alpha)_{dom(\gamma)}\alpha^{-1}\delta = c\gamma\delta$, so $\delta' : vars(c\gamma') \to \mathcal{T}_\Sigma$ with $E_0 \vDash \psi\gamma'\delta'$, $u|_p = l\gamma'\delta'$, and $\overline{ST}\alpha\delta' = \overline{ST}\delta$.

Then, $\frac{l_1\gamma'\delta' \to r_1\gamma'\delta'/ST_1\alpha\delta' \cdots l_m\gamma'\delta' \to r_m\gamma'\delta'/ST_m\alpha\delta'}{u \to u[r\gamma'\delta']_p/ST\alpha}$, i.e., $\frac{l_1\gamma\delta \to r_1\gamma\delta/ST_1\delta \cdots l_m\gamma\delta \to r_m\gamma\delta/ST_m\delta}{u \to u[r\gamma\delta]_p/ST\alpha}$ is a derivation rule in $\mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$, so $\frac{T_1 \cdots T_m}{t \to v/ST\alpha}$ is a c.p.t. for $[v]_E \in ST\alpha@[t]_E$ with the same depth and number of nodes as T.

As $ST = c[\gamma]\{\overline{ST}\} =_E ST'$, then $ST' = c[\gamma']\{\overline{ST}'\}$ where $\overline{ST} =_E \overline{ST}'$ and $\gamma =_E \gamma'$, so $(l, r, \psi, \bar{l}, \bar{r})\gamma =_E (l, r, \psi, \bar{l}', \bar{r}')\gamma'$, with $V_{l\gamma} = V_{l\gamma'}$, $V_{r\gamma} = V_{r\gamma'}$, $V_{\bar{l}\gamma} = V_{\bar{l}\gamma'}$ and $V_{\bar{r}\gamma} = V_{\bar{r}\gamma'}$, hence $E_0 \vDash \psi\gamma'\delta$, $t =_E t'[l\gamma\delta]_p =_E t'[l\gamma'\delta]_p$ and $v =_E t'[r\gamma\delta]_p =_E t'[r\gamma'\delta]_p$, ground terms and formula. Then, $\frac{l_1\gamma'\delta \to r_1\gamma'\delta/ST'_1\delta \cdots l_m\gamma'\delta \to r_m\gamma'\delta/ST'_m\delta}{u \to u[r\gamma'\delta]_p/c[\gamma']\{\overline{ST}'\}}$ is a derivation rule in $\mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$. Again, by I.H., since $\overline{ST}\delta =_E \overline{ST}'\delta$ and $(\bar{l}, \bar{r})\gamma\delta =_E (\bar{l}, \bar{r})\gamma'\delta$, there exist a c.p.t. T'_j with the same depth and number of nodes as T_j for $[r_j\gamma'\delta]_E \in ST'_j\delta@[l_j\gamma'\delta]_E$, for $1 \leq j \leq m$, so $\frac{T'_1 \cdots T'_m}{t \to v/c[\gamma']\{\overline{ST}'\}}$ is a c.p.t. for $[v]_E \in c[\gamma']\{\overline{ST}'\}@[t]_E$.

14. $ST = \texttt{top}(c[\gamma]\{\overline{ST}\})$, with $c : l \to r \textit{ if } \bigwedge_{j=1}^m l_j \to r_j \mid \psi$ a rule in R, $\overline{ST} = ST_1, \ldots, ST_m$, and $dom(\gamma) \cap vars(\overline{ST}) = \emptyset$.

    T must be of the form $\frac{T_1 \cdots T_m}{t \to v/c[\gamma]\{\overline{ST}\}}$, where T_i, $1 \leq i \leq m$, are closed proof trees with head $l_i\gamma\delta \to r_i\gamma\delta/ST_i\delta$ (so, by I.H., $l_i\gamma\delta \to_{R/E} r_i\gamma\delta$ and $[r_j\gamma\delta]_E \in ST_j\delta@[l_j\gamma\delta]_E$), because there is a derivation rule $\frac{l_1\gamma\delta \to r_1\gamma\delta/ST_1\delta \cdots l_m\gamma\delta \to r_m\gamma\delta/ST_m\delta}{l\gamma\delta \to r\gamma\delta/\texttt{top}(c[\gamma]\{ST_1,\ldots,ST_m\})} \in \mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$, where $\delta : vars(c\gamma) \to \mathcal{T}_\Sigma$, $l\gamma\delta =_E t$, $r\gamma\delta =_E v$, and $E_0 \vDash \psi\gamma\delta$.

    As $l_i\gamma\delta \to_{R/E} r_i\gamma\delta$, $1 \leq i \leq m$, $t =_E l\gamma\delta$, $v =_E r\gamma\delta$, and $E_0 \vDash \psi\gamma\delta$ then, by definition, $t \xrightarrow[c,u,\epsilon,\gamma\delta]{}{}^1_{R/E} v$ (**property 14**).

    The proofs for the existence of a c.p.t. for $[v]_E \in ST\alpha@[t]_E$ and $[v]_E \in ST'@[t]_E$ with the same depth and number of nodes as T are the same proofs shown in the previous subcase, particularized for the position $p = \epsilon$, so $u = l\gamma\delta$ and $u[r\gamma\delta]_p = r\gamma\delta$.

15. $ST = \texttt{matchrew}\ u\ \texttt{s.t.}\ C\ \texttt{by}\ x^1_{s_1}\ \texttt{using}\ ST_1, \ldots, x^n_{s_n}\ \texttt{using}\ ST_n$, call $\bar{x} = x^1_{s_1}, \ldots, x^n_{s_n}$, where $C = \bigwedge_{j=1}^m (l_j = r_j) \wedge \phi$, $u = u[x^1_{s_1}, \ldots, x^n_{s_n}]_{p_1 \ldots p_n}$, and $\hat{x} = \{\bar{x}\}$.

    T must be of the form $\frac{T_1 \cdots T_n}{t \to v/ST}$, where each T_i is a c.p.t. with head $x^i_{s_i}\delta \to t_i/ST_i\delta$, $1 \leq i \leq n$, by application of a rule $\frac{x^1_{s_1}\delta \to t_1/ST_1\delta \cdots x^n_{s_n}\delta \to t_n/ST_n\delta}{u\delta \to u\delta[\bar{t}]_{\bar{p}}/ST} \in \mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$, so $V_{(u,\bar{l},\bar{r},\phi)\delta} = \emptyset$ and $V_{\overline{ST}\delta} \subseteq V_T$, where $\delta_{V_{ST}} : \mathcal{X} \to \mathcal{T}_\Sigma(\mathcal{X} \backslash V_{ST})$, $ran(\delta_{V_{ST}}) \subseteq V_{\overline{ST}\delta}$, $t =_E u\delta$, $v =_E u\delta[\bar{t}]_{\bar{p}}$, $\bar{l}\delta =_E \bar{r}\delta$, and $E_0 \vDash \phi\delta$ so, by I.H., $[t_j]_E \in ST_j\delta@[x^j_{s_j}\delta]_E$, for $1 \leq j \leq n$ (**property 15**). Also by I.H., $x^j_{s_j}\delta \to_{R/E} t_j$, for $1 \leq j \leq n$. Then, by congruence of rewriting, $t =_E u\delta[x^1_{s_1}\delta, \ldots, x^n_{s_n}\delta]_{p_1 \ldots p_n} \to_{R/E} u\delta[\bar{t}]_{\bar{p}} =_E v$ (i.e., $t \to_{R/E} v$). Call $\alpha' = \alpha_{\backslash \bar{x}}$. Then $ST\alpha$ has the form $\texttt{matchrew}\ u\alpha'\ \texttt{s.t.}\ C\alpha'\ \texttt{by}\ x^1_{s_1}\ \texttt{using}\ ST_1\alpha', \ldots, x^n_{s_n}\ \texttt{using}\ ST_n\alpha'$, i.e., $ST\alpha = ST\alpha'$, with $ran(\alpha) \cap (V_T \cup V_{\mathcal{R},Call_\mathcal{R}}) = \emptyset$. Call $\delta' = (\alpha')^{-1}\delta$. As $ran(\alpha) \cap V_T = \emptyset$, $ran(\delta_{V_{ST}}) \subseteq V_{\overline{ST}\delta} \subseteq V_T$, and $ran(\delta_{V_{ST}}) \cap V_{ST} = \emptyset$, then $ran(\alpha) \cap ran(\delta_{V_{ST}}) = \emptyset$, hence $ran(\alpha') \cap ran(\delta_{V_{ST}}) = \emptyset$. As also $V_{ST} \cap ran(\delta_{V_{ST}}) = \emptyset$ and $V_{ST\alpha'} \subseteq V_{ST} \cup ran(\alpha')$ then, for each $x \in V_{ST}$, $x\alpha'\delta' = x\delta$ and:

    - if $x \in dom(\delta)$ then $V_{x\delta} \subseteq ran(\delta_{V_{ST}})$, so $V_{x\delta} \cap V_{ST\alpha'} = \emptyset$, i.e., $V_{x\alpha'\delta'} \cap V_{ST\alpha'} = \emptyset$, and
    - if $x \notin dom(\delta)$ then $x\delta = x$ and:

* if $x \in dom(\alpha')$ then, as $ran(\alpha) \cap V_T = \emptyset$, hence also $ran(\alpha') \cap V_T = \emptyset$, and $x \in V_{ST} \subseteq V_T$, then $x \notin V_{ST\alpha'}$, i.e., $\emptyset = V_{x\delta} \cap V_{ST\alpha'} = V_{x\alpha'\delta'} \cap V_{ST\alpha'}$;
* if $x \notin dom(\alpha')$ then $x \in V_{ST\alpha'} \setminus ran(\alpha') = V_{ST\alpha'} \setminus dom((\alpha')^{-1})$, so $x\delta' = x(\alpha')^{-1}\delta = x\delta = x$, i.e., $x \notin dom(\delta'_{V_{ST\alpha'}})$.

Then $\delta'_{V_{ST\alpha'}} : \mathcal{X} \to \mathcal{T}_\Sigma(\mathcal{X} \backslash vars(ST\alpha'))$ and $ST\alpha\delta' = ST\alpha'\delta' = ST\delta$, hence $t =_E u\alpha'\delta' = u\delta \in \mathcal{T}_\Sigma$, $\bar{l}\alpha'\delta' = \bar{l}\delta =_E \bar{r}\delta = \bar{r}\alpha'\delta'$, so $\{l_j\alpha'\delta', r_j\alpha'\delta'\}_{j=1}^m \subset \mathcal{T}_\Sigma$, $\phi\alpha'\delta' = \phi\delta \in \mathcal{T}_\Sigma$, and $E_0 \vDash \phi\alpha'\delta'$, hence there is a derivation rule $\frac{x^1_{s_1}\alpha'\delta' \to t_1/ST_1\alpha'\delta' \cdots x^n_{s_n}\alpha'\delta' \to t_n/ST_n\alpha'\delta'}{u\alpha'\delta' \to u\alpha'\delta'[\bar{t}]_{\bar{p}}/ST\alpha'}$ in $\mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$. As $u\alpha'\delta' = u\delta$, $ST\alpha = ST\alpha'$, $\overline{ST}\alpha'\delta' = \overline{ST}\delta$, and $\bar{x}\alpha'\delta' = \bar{x}\delta$, because $\bar{x} \subseteq mp(ST)$, this is the same as $\frac{x^1_{s_1}\delta \to t_1/ST_1\delta \cdots x^n_{s_n}\delta \to t_n/ST_n\delta}{u\delta \to u\delta[\bar{t}]_{\bar{p}}/ST\alpha} \in \mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$. Then, as $t =_E u\delta$ and $v =_E u\delta[\bar{t}]_{\bar{p}}$, $\frac{T_1 \cdots T_n}{t \to v/ST\alpha}$ is a c.p.t. for $[v]_E \in ST\alpha @ [t]_E$.

As $ST =_E ST'$, then $ST' = \texttt{matchrew}\, u'\ \texttt{s.t.}\ C'\ \texttt{by}\ \bar{x}\ \texttt{using}\ \overline{ST}'$ where $\overline{ST} =_E \overline{ST}'$ $C =_E C' = \bigwedge_{j=1}^m (l'_j = r'_j) \wedge \phi'$, so $(\phi, \bar{l}, \bar{r}) =_E (\phi', \bar{l}', \bar{r}')$, with $V_u = V_{u'} = \hat{x}$, $V_\phi = V_{\phi'}$, $V_{\bar{l}\gamma} = V_{\bar{l}\gamma'}$ and $V_{\bar{r}\gamma} = V_{\bar{r}\gamma'}$, so $t =_E u\delta =_E u'\delta$, $v =_E u\delta[\bar{t}]_{\bar{p}} =_E u\delta'[\bar{t}]_{\bar{p}}$, $\bar{l}\delta =_E \bar{r}\delta$, and $E_0 \vDash \phi'\delta$, ground terms and formula.

Then, there is a derivation rule $\frac{x^1_{s_1}\delta \to t_1/ST'_1\delta \cdots x^n_{s_n}\delta \to t_n/ST'_n\delta}{u'\delta \to u\delta[\bar{t}]_{\bar{p}}/ST'} \in \mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$ in $\mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$. Again, by I.H., since $\overline{ST}\delta =_E \overline{ST}'\delta$, there exist a c.p.t. T'_j with the same depth and number of nodes as T_j, for $[t_j]_E \in ST'_j\delta @ [x^j_{s_j}\delta]_E$, for $1 \le j \le n$, so $\frac{T'_1 \cdots T'_n}{t \to v/ST'}$ is a c.p.t. for $[v]_E \in \overline{ST}' @ [t]_E$ with the same depth and number of nodes as T.

□

Lemma 6. Given a rewrite theory $\mathcal{R} = (\Sigma, E_0 \cup B, R)$, a set of call strategy definitions $Call_\mathcal{R}$, terms $t, v \in \mathcal{H}_\Sigma$, a strategy $ST \in Strat_{\mathcal{R},Call_\mathcal{R}}$, and a substitution σ such that $dom(\sigma) \cap V_R = \emptyset$ and $ran(\sigma) \cap (V_R \cup V_{ST}) = \emptyset$, if $[v]_E \in ST\sigma @ [t]_E$ can be proved with a c.p.t. T then $[v]_E \in ST @ [t]_E$ and a c.p.t. T' with head $t \to v/ST$ and the same depth and number of nodes as T can be constructed.

Proof. The proof is done by structural induction on the depth of T.

* There are five strategies in the base case: `fail`, `idle`, $c[\gamma]$, $\texttt{top}(c[\gamma])$, and the `match` test. The depth of all the closed proof trees is one in this case.
  - As there are no derivation rules for `fail`, there is nothing to prove in this case.
  - If $ST = \texttt{idle}$ then $ST\sigma = ST$ and $T' = T$.
  - If $ST = c[\gamma]$ then $ST\sigma = c[(\gamma\sigma)_{dom(\gamma)}]$. As $dom(\sigma) \cap V_R = \emptyset$ then $c(\gamma\sigma)_{dom(\gamma)} = c\gamma\sigma_{ran(\gamma)}$. $T = \frac{}{t \to v/ST\sigma}$ because c has the form $c : l \to r\ if\ \phi$, and there exist $u \in \mathcal{H}_\Sigma$, $p \in pos(u)$, and $\delta : V_{c\gamma\sigma_{ran(\gamma)}} \to \mathcal{T}_\Sigma$ such that $u \xrightarrow[c\gamma\sigma_{ran(\gamma)},p,\delta]{}{}^1_R\ w$, i.e., $u = u[l\gamma\sigma_{ran(\gamma)}\delta]_p$ and $E_0 \vDash \phi\gamma\sigma_{ran(\gamma)}\delta$, so there is a derivation rule $\frac{}{u \to w/ST\sigma}$ in $\mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$, $t =_E u$, and $w = u[r\gamma\sigma_{ran(\gamma)}\delta]_p =_E v$. Then, also $u \xrightarrow[c\gamma,p,\sigma_{ran(\gamma)}\delta]{}{}^1_R\ w$, because as, by definition, $dom(\gamma) \subseteq vars(c)$ then $\sigma_{ran(\gamma)}\delta : V_{c\gamma} \to \mathcal{T}_\Sigma$, so there is a derivation rule $\frac{}{u \to w/ST}$ in $\mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$ and $T' = \frac{}{t \to v/ST}$.
  - if $ST = \texttt{top}(c[\gamma])$ then $ST\sigma = \texttt{top}(c[\gamma\sigma_{ran(\gamma)}])$. As $dom(\sigma) \cap V_R = \emptyset$ then $c(\gamma\sigma)_{dom(\gamma)} = c\gamma\sigma_{ran(\gamma)}$. $T = \frac{}{t \to v/ST\sigma}$ because c has the form $c : l \to r\ if\ \phi$, there exists $\delta : V_{c\gamma\sigma_{ran(\gamma)}} \to \mathcal{T}_\Sigma$ such that $E_0 \vDash \phi\gamma\sigma_{ran(\gamma)}\delta$, so $\frac{}{l\gamma\sigma_{ran(\gamma)} \to r\gamma\sigma_{ran(\gamma)}/ST\sigma}$ is a derivation rule in $\mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$, $t =_E l\gamma\sigma_{ran(\gamma)}\delta$, and $r\gamma\sigma_{ran(\gamma)}\delta =_E v$. Again, by definition, $dom(\gamma) \subseteq vars(c)$ so $\sigma_{ran(\gamma)}\delta : V_{c\gamma} \to \mathcal{T}_\Sigma$ and, as $E_0 \vDash \phi\gamma\sigma_{ran(\gamma)}\delta$, there is a derivation rule $\frac{}{l\gamma\sigma_{ran(\gamma)} \to r\gamma\sigma_{ran(\gamma)}/ST}$ in $\mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$, so $T' = \frac{}{t \to v/ST}$.

– if $ST = \texttt{match}\, u \,\texttt{s.t.}\ \bigwedge_{j=1}^{m}(l_j = r_j) \wedge \phi$ then there exists a substitution δ such that $t =_E u\sigma\delta$, $l_j\sigma\delta =_E r_j\sigma\delta$, for $1 \leq j \leq m$, and $E_0 \vDash \phi\sigma\delta$, so there are derivation rules $\frac{}{w \to w/ST\sigma}$ and $\frac{}{w \to w/ST}$ in $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$, where $w =_E u\sigma\delta$, and $T = \frac{}{t \to v/ST\sigma}$ because $t =_E w =_E v$, so also $T' = \frac{}{t \to v/ST}$.

- Inductive step:

  – $ST = ST_1\,;\,ST_2$ and T has the form $\frac{\frac{T_1}{t \to w/ST_1\sigma} \quad \frac{T_2}{w \to v/ST_2\sigma}}{t \to v/ST\sigma}$. By I.H there are closed proof trees with the forms $\frac{T_1'}{t \to w/ST_1}$ and $\frac{T_2'}{w \to v/ST_2}$ where T_1' and T_2' have the same depth and number of nodes as T_1 and T_2, respectively, so $T' = \frac{\frac{T_1'}{t \to w/ST_1} \quad \frac{T_2'}{t \to w/ST_2}}{t \to v/ST}$ is a c.p.t. with the same depth and number of nodes as T.

  – $ST = ST_1{+}$ and T must be either of the form $\frac{\frac{T_1}{t \to v/ST_1\sigma}}{t \to v/ST\sigma}$ or $\frac{\frac{T_2}{t \to v/ST_1\sigma\,;\,ST_1\sigma+}}{t \to v/ST\sigma}$. As $ST_1\sigma\,;\,ST_1\sigma{+} = (ST_1\,;\,ST_1{+})\sigma$ then, by I.H., there is either a c.p.t. with the form $\frac{T_1'}{t \to v/ST_1}$ or $\frac{T_2'}{t \to v/ST_1\,;\,ST_1+}$, hence either $T' = \frac{\frac{T_1'}{t \to v/ST_1}}{t \to v/ST}$ or $T' = \frac{\frac{T_2'}{t \to v/ST_1\,;\,ST_1+}}{t \to v/ST}$.

  – $ST = ST_1|ST_2$ and T must be either of the form $\frac{\frac{T_1}{t \to v/ST_1\sigma}}{t \to v/ST\sigma}$ or $\frac{\frac{T_2}{t \to v/ST_2\sigma}}{t \to v/ST\sigma}$. Then, by I.H., there is either a c.p.t. with the form $\frac{T_1'}{t \to v/ST_1}$ or $\frac{T_2'}{t \to v/ST_2}$, hence either $T' = \frac{\frac{T_1'}{t \to v/ST_1}}{t \to v/ST}$ or $T' = \frac{\frac{T_2'}{t \to v/ST_2}}{t \to v/ST}$.

  – $ST = \texttt{match}\, u \,\texttt{s.t.}\ \phi\ ?\ ST_1\ :\ ST_2$ and T must be either of the form $\frac{\frac{T_1}{t \to v/ST_1\sigma\delta}}{t \to v/ST\sigma}$ or $\frac{\frac{T_2}{t \to v/ST_2\sigma\delta}}{t \to v/ST\sigma}$ where $\delta : V_{u\sigma,\phi\sigma} \to \mathcal{T}_\Sigma$, $t =_E u\sigma\delta$, and either $E_0 \vDash \phi\sigma\delta$ or $E_0 \vDash \neg\phi\sigma\delta$, respectively.
  Let $\alpha = \sigma_{V_{u,\phi}}$, so $dom(\delta) = V_{u,\phi} \setminus dom(\alpha)$, and $\beta = \sigma_{\setminus V_{u,\phi}}$, so $dom(\delta) \cap dom(\beta) = \emptyset$. Then $\sigma = \alpha \uplus \beta$, $(u\sigma\delta, \phi\sigma\delta) = (u\alpha\delta, \phi\alpha\delta)$, so $E_0 \vDash \phi\sigma\delta$ iif $E_0 \vDash \phi\alpha\delta$, and $\alpha\delta : V_{u,\phi} \to \mathcal{T}_\Sigma$, so there is a derivation rule of the form $\frac{t \to v/ST_1\alpha\delta}{t \to v/ST}$ or $\frac{t \to v/ST_2\alpha\delta}{t \to v/ST}$ in $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$. Consider the open goal $t \to v/(ST_i\alpha\delta)\beta$, where $i = 1$ if $E_0 \vDash \phi\alpha\delta$ and $i = 2$ if $E_0 \vDash \neg\phi\alpha\delta$. As δ is ground and $dom(\delta) \cap dom(\beta) = \emptyset$ then $\alpha\delta\beta = \alpha\beta\delta = \sigma\delta$ and $\frac{T_i}{t \to v/(ST_i\alpha\delta)\beta}$ is a c.p.t. so, by I.H., there is a c.p.t. with the form $\frac{T_i'}{t \to v/ST_i\alpha\delta}$, where T_i' has the same depth and number of nodes as T_i, and $T' = \frac{\frac{T_i'}{t \to v/ST_i\alpha\delta}}{t \to v/ST}$.

  – $ST = CS$, where $\texttt{sd}\ CS\ :=\ ST_1 \in Call_{\mathcal{R}}$ and T has the form $\frac{\frac{T_1}{t \to v/ST_1\gamma}}{t \to v/ST\sigma}$, for some renaming γ, because $ST\sigma = CS\sigma = CS = ST$, so $T' = \frac{\frac{T_1}{t \to v/ST_1\gamma}}{t \to v/ST}$.

  – $ST = CS(\bar{t})$, where $\texttt{sd}\ CS(\bar{x})\ :=\ ST_1 \in Call_{\mathcal{R}}$, $\bar{x} = x^1_{s_1}, \ldots, x^n_{s_n}$, $\bar{t} = t_1, \ldots, t_n$, and $\rho = \{\bar{x} \mapsto \bar{t}\}$, call $\rho' = \{\bar{x} \mapsto \bar{t}\sigma\}$, and T has the form $\frac{\frac{T_1}{t \to v/ST_1(\gamma \cup \rho')}}{t \to v/ST\sigma}$, because $ST_1\sigma = CS(\bar{t})\sigma = CS(\bar{t}\sigma)$, and for some renaming γ such that $dom(\gamma) \subseteq V_{ST_1} \setminus \hat{x}$ and $ran(\gamma)$ is away from any known variable, so $V_{ST_1} = \bar{x} \cup ran(\gamma)$. As we also have $dom(\rho') = dom(\rho) = \hat{x}$, then $ST_1(\gamma \cup \rho') = ST_1\gamma\rho' = ST_1\gamma\rho\sigma$ and also $ST_1(\gamma \cup \rho) = ST_1\gamma\rho$. As $\frac{T_1}{t \to v/ST_1\gamma\rho\sigma}$ is a c.p.t. then, by I.H., there is a c.p.t. $\frac{T_1'}{t \to v/ST_1\gamma\rho}$, and $T' = \frac{\frac{T_1'}{t \to v/ST_1(\gamma \cup \rho)}}{t \to v/ST}$.

  – $ST = CS(\bar{t})$, where $\texttt{csd}\ CS(\bar{x})\ :=\ ST_1\ if\ C \in Call_{\mathcal{R}}$, with $\bar{x} = x^1_{s_1}, \ldots, x^n_{s_n}$ and $C = \bigwedge_{j=1}^{m}(l_j = r_j) \wedge \phi$, $\hat{x} \subseteq V_{CS}$, $\bar{t} = t_1, \ldots, t_n$, and $\rho = \{\bar{x} \mapsto \bar{t}\}$, call $\rho' = \{\bar{x} \mapsto \bar{t}\sigma\}$,

and T has the form $\frac{\frac{T_1}{t\to v/ST_1(\gamma\cup\rho')\delta}}{t\to v/ST\sigma}$, because $ST_1\sigma = CS(\bar{t})\sigma = CS(\bar{t}\sigma)$, and for some renaming γ such that $dom(\gamma) \subseteq V_{ST_1}\setminus\hat{x}$ and $ran(\gamma)$ is away from any known variable, so $V_{ST_1} = \bar{x} \cup ran(\gamma)$, and there is a substitution $\delta : vars(CS(\gamma\cup\rho')) \to \mathcal{T}_\Sigma$ such that $l_j(\gamma\cup\rho')\delta =_E r_j(\gamma\cup\rho')\delta$, for $1 \le j \le n$, and $E_0 \vDash \phi(\gamma\cup\rho')\delta$.
Let $\delta' = \delta_{ran(\gamma)} \cup (\sigma\delta_{\setminus ran(\gamma)})$. As δ is ground and $ran(\gamma)$ is away from all known variables, then $(\gamma\cup\rho)\delta' = (\gamma\cup\rho)\delta_{ran(\gamma)} \cup (\sigma\delta_{\setminus ran(\gamma)}) = (\gamma\delta_{ran(\gamma)}) \cup (\rho\sigma\delta_{\setminus ran(\gamma)})) = (\gamma\delta_{ran(\gamma)}) \cup (\rho'\delta_{\setminus ran(\gamma)})) = (\gamma\cup\rho')\delta$, so $\delta' : vars(C(\gamma\cup\rho)) \to \mathcal{T}_\Sigma$ verifies $l_j(\gamma\cup\rho)\delta' =_E r_j(\gamma\cup\rho)\delta'$, for $1 \le j \le n$, and $E_0 \vDash \phi(\gamma\cup\rho)\delta'$, and there is a derivation rule $\frac{t\to v/ST_1(\gamma\cup\rho)\delta'}{t\to v/ST}$ in $\mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$. Since $(\gamma\cup\rho)\delta' = (\gamma\cup\rho')\delta$, then $T' = \frac{\frac{T_1}{t\to v/ST_1(\gamma\cup\rho)\delta'}}{t\to v/ST}$.

- $ST = c[\gamma]\{ST_1,\ldots,ST_m\}$. As $dom(\sigma)\cap V_R = \emptyset$ then $c(\gamma\sigma)_{dom(\gamma)} = c\gamma\sigma_{ran(\gamma)}$. $c : l \to r$ *if* $\bigwedge_{j=1}^m l_j \to r_j \mid \psi$ is a rule in R and T has the form $\frac{T_1\cdots T_m}{t\to v/ST\sigma}$, where T_i has the form $\frac{T_i'}{l_i\gamma\delta'\to r_i\gamma\delta'/ST_i\sigma\delta}$, for $1 \le i \le m$, $\delta : vars(c\gamma\sigma_{ran(\gamma)}) \to \mathcal{T}_\Sigma$, $\delta' = \sigma_{ran(\gamma)}\delta$, $E_0 \vDash \psi\gamma\sigma_{ran(\gamma)}\delta$, and there are u in $\mathcal{H}_\Sigma$ and p in $pos(u)$ such that $t =_E u$, $u|_p = l\gamma\sigma_{ran(\gamma)}\delta$, and $u[r\gamma\sigma_{ran(\gamma)}\delta]_p =_E v$.
  As $\delta' = \sigma_{ran(\gamma)}\delta$, then $\delta' : vars(c\gamma) \to \mathcal{T}_\Sigma$, and $E_0 \vDash \psi\gamma\delta'$, $u|_p = l\gamma\delta'$, so there is a derivation rule $\frac{l_1\gamma\delta'\to r_1\gamma\delta'/ST_1\delta'\cdots l_m\gamma\delta'\to r_m\gamma\delta'/ST_m\delta'}{u\to u[r\gamma\delta']_p/ST}$ in $\mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$. Also $u[r\gamma\delta']_p = u[r\gamma\sigma_{ran(\gamma)}\delta]_p =_E v$.
  As $dom(\sigma)\cap V_R = \emptyset$ and $dom(\delta) \subseteq V_c \cup ran(\gamma) \subseteq V_R \cup V_{ST}$ then $dom(\delta) \cap dom(\sigma_{\setminus ran(\gamma)}) = \emptyset$ so, as $ran(\sigma)\cap(V_R\cup V_{ST}) = \emptyset$ and δ is ground, $\sigma_{\setminus ran(\gamma)}\delta = \delta\sigma_{\setminus ran(\gamma)}$ and $\sigma\delta = (\sigma_{ran(\gamma)} \uplus \sigma_{\setminus ran(\gamma)})\delta = \sigma_{ran(\gamma)}\sigma_{\setminus ran(\gamma)}\delta = \sigma_{ran(\gamma)}\delta\sigma_{\setminus ran(\gamma)} = \delta'\sigma_{\setminus ran(\gamma)}$, hence, for $1 \le i \le m$, $T_i = \frac{T_i'}{l_i\gamma\delta'\to r_i\gamma\delta'/ST_i\delta'\sigma_{\setminus ran(\gamma)}}$, and, by I.H., there is a c.p.t. T_i'' with the form $\frac{T_i'''}{l_i\gamma\delta'\to r_i\gamma\delta'/ST_i\delta'}$ and the same depth and number of nodes as T_i. Then, as $t =_E u$ and $u[r\gamma\delta']_p =_E v$, $T' = \frac{T_1''\cdots T_m''}{t\to v/ST}$.
- $ST = \mathtt{top}(c[\gamma]\{ST_1,\ldots,ST_m\})$. As $dom(\sigma)\cap V_R = \emptyset$ then $c(\gamma\sigma)_{dom(\gamma)} = c\gamma\sigma_{ran(\gamma)}$. $c : l \to r$ *if* $\bigwedge_{j=1}^m l_j \to r_j \mid \psi$ is a rule in R and T has the form $\frac{T_1\cdots T_m}{t\to v/ST\sigma}$, where T_i has the form $\frac{T_i'}{l_i\gamma\delta'\to r_i\gamma\delta'/ST_i\sigma\delta}$, for $1 \le i \le m$, $\delta : vars(c\gamma\sigma_{ran(\gamma)}) \to \mathcal{T}_\Sigma$, $\delta' = \sigma_{ran(\gamma)}\delta$, $E_0 \vDash \psi\gamma\sigma_{ran(\gamma)}\delta$, $t =_E l\gamma\sigma_{ran(\gamma)}\delta$, and $r\gamma\sigma_{ran(\gamma)}\delta =_E v$.
  As $\delta' = \sigma_{ran(\gamma)}\delta$, then $\delta' : vars(c\gamma) \to \mathcal{T}_\Sigma$ and $E_0 \vDash \psi\gamma\delta'$, then there is a derivation rule $\frac{l_1\gamma\delta'\to r_1\gamma\delta'/ST_1\delta'\cdots l_m\gamma\delta'\to r_m\gamma\delta'/ST_m\delta'}{l\gamma\delta'\to r\gamma\delta'/ST}$ in $\mathcal{D}_{\mathcal{R},Call_\mathcal{R}}$. Also $t =_E l\gamma\sigma_{ran(\gamma)}\delta = l\gamma\delta'$ and $r\gamma\delta' = r\gamma\sigma_{ran(\gamma)}\delta =_E v$.
  As in the previous case, for $1 \le i \le m$ there is a c.p.t. T_i'' with the form $\frac{T_i'''}{l_i\gamma\delta'\to r_i\gamma\delta'/ST_i\delta'}$ and the same depth and number of nodes as T_i. Then, as $t =_E l\gamma\delta'$ and $r\gamma\delta' =_E v$, $T' = \frac{T_1''\cdots T_m''}{t\to v/ST}$.
- $ST = \mathtt{matchrew}\ u\ \mathtt{s.t.}\ \bigwedge_{j=1}^m (l_j = r_j) \wedge \phi\ \mathtt{by}\ x_{s_1}^1\ \mathtt{using}\ ST_1,\ldots,x_{s_n}^n\ \mathtt{using}\ ST_n$, where $u = u[x_{s_1}^1,\ldots,x_{s_n}^n]_{p_1\ldots p_n}$ and T has the form $\frac{T_1\cdots T_m}{t\to v/ST\sigma}$, where T_i has head $x_{s_i}^i\delta \to t_i/ST_i\sigma\delta$, for $1 \le i \le n$, with $\hat{t} \subset \mathcal{T}_\Sigma$, $\delta_{V_{ST\sigma}} : \mathcal{X} \to \mathcal{T}_\Sigma(\mathcal{X}\setminus V_{ST\sigma})$ such that, $ran(\delta_{V_{ST\sigma}}) \subseteq V_{\overline{ST}\sigma\delta}$, $t =_E u\sigma\delta \in \mathcal{T}_\Sigma$, $u\sigma\delta[\bar{t}]_{\bar{p}} =_E v$, $\{l_j\sigma\delta, r_j\sigma\delta\}_{j=1}^m \subset \mathcal{T}_\Sigma$, $\bar{l}\sigma\delta =_E \bar{r}\sigma\delta$, $\phi\sigma\delta \in \mathcal{T}_\Sigma$, and $E_0 \vDash \phi\sigma\delta$.
  The fact that $ran(\delta_{V_{ST\sigma}}) \subseteq V_{\overline{ST}\sigma\delta}$ does not ensure that $ran(\delta_{V_{ST\sigma}}) \cap V_{ST} = \emptyset$. Let α be a renaming such that $dom(\alpha) = V_{\overline{ST}} \cap ran(\delta_{V_{ST\sigma}})$ and $ran(\alpha)$ is away from all know variables and call $\delta' = \sigma\delta\alpha$. Then $\delta'_{V_{ST}} : \mathcal{X} \to \mathcal{T}_\Sigma(\mathcal{X}\setminus V_{ST})$. By Lemma 5, as T_i has head $x_{s_i}^i\delta \to t_i/ST_i\sigma\delta$, there is also a c.p.t. with the form $\frac{T_i'}{x_{s_i}^i\delta\to t_i/ST_i\delta'}$, for $1 \le i \le n$.

As $\delta'_{V_{ST}} : \mathcal{X} \to \mathcal{T}_\Sigma(\mathcal{X} \backslash V_{ST})$, $t =_E u\delta = u\delta' \in \mathcal{T}_\Sigma$, $u\delta'[\bar{t}]_{\bar{p}} =_E v$, $\bar{l}\delta' = \bar{l}\delta =_E \bar{r}\delta = \bar{r}\delta'$, so $\{l_j\delta', r_j\delta'\}_{j=1}^m \subset \mathcal{T}_\Sigma$, and $\phi\delta' = \phi\delta \in \mathcal{T}_\Sigma$, so $E_0 \vDash \phi\delta'$ then there is a derivation rule $\frac{x^1_{s_1}\delta' \to t_1/ST_1\delta' \cdots x^n_{s_n}\delta' \to t_n/ST_n\delta'}{u\delta' \to u\delta'[\bar{t}]_{\bar{p}}/ST}$ in $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}$. As $u\delta'[\bar{x}\delta']_{\bar{p}} = u\delta' = u\delta = u\delta[\bar{x}\delta]_{\bar{p}}$, so also $\bar{x}\delta' = \bar{x}\delta'$, the derivation rule can be written $\frac{x^1_{s_1}\delta \to t_1/ST_1\delta' \cdots x^n_{s_n}\delta \to t_n/ST_n\delta'}{u\delta \to u\delta[\bar{t}]_{\bar{p}}/ST}$, hence

$$T' = \frac{\frac{T'_1}{x^1_{s_1}\delta \to t_1/ST_1\delta'} \cdots \frac{T'_n}{x^n_{s_n}\delta \to t_n/ST_n\delta'}}{u\delta \to u\delta[\bar{t}]_{\bar{p}}/ST}.$$

□

Proposition 10. Given a rewrite theory $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ and a set of call strategy definitions $Call_{\mathcal{R}}$, and an admissible goal G with the form

- $\bigwedge_{i=1}^n u_i \to v_i/ST_i^\nu \varrho_\nu \mid \phi \mid V, \nu$, or
- $u_1|_p \to^1 x_k, u_1[x_k]_p \to v_1/ST_1^\nu \varrho_\nu \wedge \bigwedge_{i=2}^n u_i \to v_i/ST_i^\nu \varrho_\nu \mid \phi \mid V, \nu$,

if G_0 is a goal of type (a), with substitution ν_0 ($\varrho_{\nu_0} = none$ by definition), and $G_0 \rightsquigarrow^*_\theta G$ then the following invariants hold:

1. $vars(B) \cap V = \emptyset$ and $V_{\mathcal{R}} \cap V_{Call_{\mathcal{R}}} \subseteq V$,
2. $V \cap ran(\nu) = \emptyset$ and $\nu = (\nu_0\theta)_V$, hence $dom(\nu) \subseteq V$, so $dom(\nu)$ satisfies the restrictions given for V in Definition 33.2,
3. $\varrho_\nu = \theta_{\backslash V}$, hence $dom(\varrho_\nu) \cap V = \emptyset$ and ϱ_ν is idempotent,
4. $ran(\theta) \cap (V \cup V_{\mathcal{R},Call_{\mathcal{R}}} \cup vars(\overline{ST})) = \emptyset$ and $ran(\varrho_\nu) \cap V = \emptyset$,
5. $dom(\varrho_\nu) \cap ran(\nu) = \emptyset$,
6. $dom(\varrho_\nu) \cap V^\nu = \emptyset$,
7. $V_{\mathcal{R}^\nu} \cap V_{Call_{\mathcal{R}^\nu}} \subseteq V^\nu$,
8. if $t \in \mathcal{T}_\Sigma(\mathcal{X})$ then $t^\nu \varrho_\nu = t(\nu \uplus \varrho_\nu)$,
9. u_i, v_i, $1 \le i \le n$, and each term in $\hat{\phi}$ have the form $t^\nu \varrho_\nu$,
10. $vars(\bar{u}, \bar{v}, \phi) \cap dom(\nu) = \emptyset$, and
11. G has also the form $G_1^\nu \varrho'_\nu$, where $\varrho'_\nu = \theta_{V_{G_1} \backslash V}$, so $dom(\varrho'_\nu) \subseteq V_{G_1} \setminus V$.

Proof. - If G is a goal of type (a) then we have that $G = G_0$, $\theta = none$, and $\varrho_\nu = \varrho_{\nu_0} = none$. The invariants $1-7$ and 11 are direct consequence of the definitions of reachability problem and goal of type (a), and the fact if $\theta = \sigma_1 \ldots \sigma_m$ then $ran(\sigma_i)$ is away from any known variable, for $1 \le i \le m$, by the definition of the calculus rules. We prove invariants $8-10$.

8. As $\varrho_\nu = none$, then $t^\nu \varrho_\nu = t^\nu = t\nu = t(\nu \uplus \varrho_\nu)$.
9. We have to prove $w \in \hat{u} \cup \hat{v} \cup \hat{\phi} \implies \exists t,, w = t^\nu \varrho_\nu$. As, by the previous point, $t^\nu \varrho_\nu = t\nu$, then we prove $w \in \hat{u} \cup \hat{v} \cup \hat{\phi} \implies \exists t,, w = t\nu$. Now, as G is a goal of type (a), G has the form $\bigwedge_{i=1}^n u_i^0\nu \to v_i^0\nu/ST_i^\nu \mid \phi^0\nu \mid V, \nu$, so $\bar{u} = \bar{u}^0\nu$, $\bar{v} = \bar{v}^0\nu$, $\bar{\phi} = \bar{\phi}^0\nu$, hence $w \in \hat{u} \cup \hat{v} \cup \hat{\phi} \implies \exists t,, t \in \hat{u}^0 \cup \hat{v}^0 \cup \hat{\phi}^0 \wedge w = t\nu$.
10. As G is a goal of type (a) then $dom(\nu) \cap ran(\nu) = \emptyset$. By the previous point, there exists $\hat{u}^0 \cup \hat{v}^0 \cup \hat{\phi}^0$ such that $\hat{u} \cup \hat{v} \cup \hat{\phi} = \hat{u}^0\nu \cup \hat{v}^0\nu \cup \hat{\phi}^0\nu$. As $dom(\nu) \cap ran(\nu) = \emptyset$ then $vars(\bar{u}^0\nu, \bar{v}^0\nu, \phi^0\nu) \cap dom(\nu) = \emptyset$, i.e., $vars(\bar{u}, \bar{v}, \phi) \cap dom(\nu) = \emptyset$.

- We prove the invariants for goals of type (b) by induction on the number of applied calculus rules from Figures 3 and 4 in $G_0 \rightsquigarrow^*_{\sigma'} G' \rightsquigarrow_{[r],\sigma} G$, so $\theta = \sigma'\sigma$, using the fact that the properties hold in G'. We call $\bar{u}'$, $\bar{v}'$, ϕ', ν', and $\overline{ST'}$ the structures in G' in place of $\bar{u}$, $\bar{v}$, ϕ, ν, and $\overline{ST}$, so either $\nu = \nu'$ or there is a substitution σ such that $\nu = (\nu'\sigma)_V$ where, for proper t_1 and t_2, $\sigma \in CSU_B(t_1, t_2)$ so $V \cap ran(\sigma) = \emptyset$ by definition of CSU_B. Also, as $dom(\nu_0) \cap ran(\nu_0) = \emptyset$, $\nu = (\nu_0\theta)_V$ and θ is a composition of several CSUs, so $ran(\theta)$ is away from all known variables, then $dom(\nu) \cap ran(\nu) = \emptyset$ and, as $V^\nu = (V \setminus dom(\nu)) \cup ran(\nu)$, also $dom(\nu) \cap V^\nu = \emptyset$.

1. Immediate, since the invariant holds in G', by I.H, and no rule modifies V.

2. As either $\nu = \nu'$ or $\nu = (\nu'\sigma)_V$, $V \cap ran(\sigma) = \emptyset$, and $V \cap ran(\nu') = \emptyset$, by I.H., then $V \cap ran(\nu) = \emptyset$ in either case. Also, by I.H., $\nu' = (\nu_0\sigma')_V$, so $\nu = (\nu'\sigma)_V = ((\nu_0\sigma')_V\sigma)_V = (\nu_0\sigma'\sigma)_V = (\nu_0\theta)_V$.

3. By I.H., $\varrho_{\nu'} = \sigma'_{\setminus V}$, with $dom(\varrho_{\nu'}) \cap (V \cup ran(\nu')) = \emptyset$ and $ran(\varrho_{\nu'}) \cap V = \emptyset$, i.e., $ran(\sigma'_{\setminus V}) \cap V = \emptyset$. Then:

   - If $[r]$ computes a CSU_B of two terms, say σ, then we can find in G (depending on the actual calculus $[r]$ applied):
     - open goals that are an instance with σ of one open goal in G' with the form $u' \to v'/ST^{\nu'}\varrho_{\nu'}$. The strategy of one open goal in G will be an instance with σ of part of $ST^{\nu'}\varrho_{\nu'}$ in the case of rules `if then else` and `match`,
     - new open goals with the form $(u \to v/\texttt{idle})\sigma$ which are equal to $(u \to v/\texttt{idle}\varrho_{\nu'})\sigma$, or
     - new open goals with the form $(u \to v/ST\varrho_{\nu'}\,;\,\texttt{idle})\sigma$, where $ST\varrho_{\nu'}$ is an already existing strategy in G', which are equal to $(u \to v/(ST\,;\,\texttt{idle})\varrho_{\nu'})\sigma$.

     In any of these cases, by Def. 38, $\varrho_\nu = (\varrho_{\nu'}\sigma)_{\setminus V}$, hence $\varrho_\nu = (\varrho_{\nu'}\sigma)_{\setminus V} = (\sigma'_{\setminus V}\sigma)_{\setminus V} = (\sigma'\sigma)_{\setminus V} = \theta_{\setminus V}$.
   - If $[r]$ is a `call strategy` rule, applied to a open goal with the form $u' \to v'/CS; ST^{\nu'}\varrho_{\nu'}$ or $u' \to v'/CS(\bar{t}^{\nu'}\varrho_{\nu'})\,;\,ST^{\nu'}\varrho_{\nu'}$, where CS has parameters $\bar{x}$, then $\sigma = none$, $\nu = \nu'$, $\varrho_\nu = \varrho_{\nu'} = \sigma'_{\setminus V} = (\sigma'\sigma)_{\setminus V} = \theta_{\setminus V}$, and $dom(\varrho_\nu) \cap (V \cup ran(\nu)) = \emptyset$. Apart from the rest of existing open goals, that remain unchanged, we can find in G:
     - for conditional call strategies, new open goals with the form $u \to v/\texttt{idle}$ which are equal to $u \to v/\texttt{idle}\,\varrho_\nu$, and
     - a new open goal $u \to v/ST_2^\nu\gamma; ST^\nu\varrho_\nu$, where if the call strategy has no parameters then: (i) $\gamma = none$, call $\gamma_0 = none$, or else (ii) $\gamma = \{\bar{x} \mapsto \bar{t}^\nu\varrho_\nu\}$, call $\gamma_0 = \{\bar{x} \mapsto \bar{t}\}$, and ST_2^ν is a fresh version of the strategy ST_1^ν in the call strategy definition for CS in $Call^\nu_{\mathcal{R}}$, except for $dom(\gamma) \cup V^\nu$. As $dom(\varrho_\nu) \cap (V \cup ran(\nu)) = \emptyset$ and $vars(ST_2^\nu) \cap dom(\nu) = \emptyset$ then $vars(ST_2^\nu) \cap dom(\varrho_\nu) = \emptyset$ so, if either (i) or (ii) holds, $ST_2^\nu\gamma = (ST_2\gamma_0)^\nu\varrho_\nu$.
   - $\sigma = none$ for the rest of the rules, so $\nu = \nu'$ and $\varrho_\nu = \varrho_{\nu'} = \sigma'_{\setminus V} = (\sigma'\sigma)_{\setminus V} = \theta_{\setminus V}$, and no new strategies are added. In these rules, for any open goal $u' \to v'/ST_G \in G$ there is one open goal $u' \to v'/ST^{\nu'}\varrho_{\nu'} \in G'$ such that if $ST_1 \in tokens(ST_G)$ then $ST_1 \in tokens(ST^{\nu'}\varrho_{\nu'})$, so ST_1 has the form $ST_2^{\nu'}\varrho_{\nu'}$, i.e., $ST_2^\nu\varrho_\nu$.

4. Immediate, since θ is a composition of several $CSUs$, where the range of each CSU is away from all known variables (so $dom(\theta) \cap ran(\theta) = \emptyset$), including V, and, by the previous point, $\varrho_\nu = \theta_{\setminus V}$.

5. If $\sigma = none$ there is nothing to prove. Else, as $\varrho_\nu = (\varrho_{\nu'}\sigma)_{\setminus V}$ and $\nu = (\nu'\sigma)_V$ then $dom(\varrho_\nu) = dom(\varrho_{\nu'}) \cup (dom(\sigma) \setminus (V \cup ran(\varrho_{\nu'}))$ and $ran(\nu) = ran(\sigma_V) \cup (ran(\nu') \setminus dom(\sigma))$.

As $ran(\sigma)$ is away from all known variables and, by I.H., $dom(\varrho_{\nu'}) \cap ran(\nu') = \emptyset$ then
$dom(\varrho_\nu) \cap ran(\nu) =$
$(dom(\varrho_{\nu'}) \cup (dom(\sigma) \setminus (V \cup ran(\varrho_{\nu'}))) \;\cap\; (ran(\sigma_V) \cup (ran(\nu') \setminus dom(\sigma))) =$
$(dom(\varrho_{\nu'}) \cup (dom(\sigma) \setminus (V \cup ran(\varrho_{\nu'}))) \;\cap\; (ran(\nu') \setminus dom(\sigma)) =$
$(dom(\sigma) \setminus (V \cup ran(\varrho_{\nu'})) \;\cap\; (ran(\nu') \setminus dom(\sigma)) \subseteq dom(\sigma) \;\cap\; (ran(\nu') \setminus dom(\sigma)) = \emptyset$.

6. As $dom(\varrho_\nu) \cap V = \emptyset$, $dom(\varrho_\nu) \cap ran(\nu) = \emptyset$, and $V^\mu \subseteq V \cup ran(\nu)$, then $dom(\varrho_\nu) \cap V^\nu = \emptyset$.

7. Immediate, since $V_{\mathcal{R}} \cap V_{Call_{\mathcal{R}}} \subseteq V$, in $\mathcal{R}$ and $Call_{\mathcal{R}}$ we are replacing each variable $v \in dom(\nu)$ with $v\nu$, and $V^\nu = ran(\nu) \cup (V \setminus dom(\nu))$.

8. Immediate, since $dom(\nu) \subseteq V$ and $dom(\varrho_\nu) \cap (V \cup ran(\nu)) = \emptyset$, invariant 5, imply $t^\nu \varrho_\nu = t(\nu \uplus \varrho_\nu)$

9. Let $w \in \bar{u'} \cup \bar{v'} \cup \phi'$ such that $w\sigma \in \bar{u} \cup \bar{v} \cup \phi$. By I.H., $w = t^{\nu'} \varrho_{\nu'}$, for proper t. By I.H. and the previous point, $w = t(\nu' \uplus \varrho_{\nu'})$. As, by I.H., $dom(\nu') \subseteq V$ and $dom(\varrho_{\nu'}) \cap V = \emptyset$, then $w\sigma = t(\nu' \uplus \varrho_{\nu'})\sigma = t(\nu'_V \uplus (\varrho_{\nu'})_{\setminus V})\sigma = t((\nu'\sigma)_V \uplus (\varrho_{\nu'}\sigma)_{\setminus V}) = t(\nu \uplus \varrho_\nu)$ so, by the previous point, $w\sigma = t^\nu \varrho_\nu$.

10. By I.H., $vars(\bar{u'}, \bar{v'}, \phi') \cap dom(\nu') = \emptyset$, with $dom(\nu') \subseteq V$. As $\nu = (\nu'\sigma)_V$, then $dom(\nu) = dom(\nu') \cup dom(\sigma_V)$, so $vars(\bar{u'}\sigma, \bar{v'}\sigma, \phi'\sigma) \cap dom(\nu) = \emptyset$. Then we only have to check $vars(\bar{u}, \bar{v}, \phi) \setminus vars(\bar{u'}\sigma, \bar{v'}\sigma, \phi')\sigma$, i.e., those variables introduced by the rule that do not belong to the instantiation of $vars(\bar{u'}, \bar{v'}, \phi')$ with σ.
    - Each one of the variables, say x, introduced by $abstract_{\Sigma_1}$ is new so, as $\nu = (\nu'\sigma)_V$:
      - if $x \in dom(\sigma)$ then $vars(x\sigma) \cap dom(\nu) \subseteq ran(\sigma) \cap dom(\nu) \subseteq ran(\sigma) \cap V = \emptyset$, and
      - if $x \notin dom(\sigma)$ then, as x is new (so $x \notin V$), $vars(x\sigma) \cap dom(\nu) = \{x\} \cap dom(\nu) \subseteq \{x\} \cap V = \emptyset$ (†).

      This covers all the rules in Figure 3, except rule transitivity. It also covers rule match and it partially covers the rest of rules in Figure 4.
    - Both rules transitivity and congruence introduce one new variable not in $dom(\nu)$, so (†) applies ($\sigma = none$).
    - Rule matchrew introduces one vector of new variables ($\bar{y}$) not in $dom(\nu)$, so (†) applies.
    - The next case is rule rule application, with strategy $c[\gamma]\{\overline{ST}\}$ and substitution σ. By I.H. $c[\gamma]\{\overline{ST}\}$ has the form $(c[\delta]\{\overline{ST'}\})^{\nu'} \varrho_{\nu'}$, for proper δ, so $c[\gamma] = c^{\nu'}[\delta(\nu' \uplus \varrho_{\nu'})_{ran(\delta)}]$, where $dom(\delta) = dom(\gamma)$. The calculus rule uses a version, say $c_1^{\nu'}$, of $c^{\nu'}$ where all the variables are new except for $dom(\gamma) \cup V^{\nu'}$. The new variables of $vars(c_1^{\nu'})$ are not in $dom(\nu)$, so (†) applies. We check the rest of the variables in $vars(c_1^{\nu'})$. For each $x \in vars(c_1^{\nu'}) \cap (dom(\gamma) \cup V^{\nu'})$:
      - if $x \in V^{\nu'}$ then:
        * if $x \in dom(\sigma)$ then $vars(x\sigma) \subseteq ran(\nu)$ so, as $dom(\nu) \cap ran(\nu) = \emptyset$ then $vars(x\sigma) \cap dom(\nu) = \emptyset$;
        * else $x\sigma = x$, so $x \in V^\nu$, and:
          · if $x \in V$ then $x \notin dom(\nu)$ so, as $x\sigma = x$, $vars(x\sigma) \cap dom(\nu) = \emptyset$;
          · else $x \in ran(\nu')$ so, as $x\sigma = x$, $x \in ran(\nu)$. Then, as $x\sigma = x$ and $dom(\nu) \cap ran(\nu) = \emptyset$, $vars(x\sigma) \cap dom(\nu) = \emptyset$;
      - else $x \in dom(\gamma)$ ($= dom(\delta)$), and $x\gamma\sigma = x\delta(\nu' \uplus \varrho_{\nu'})_{ran(\delta)}\sigma = x\delta(\nu \uplus \varrho_\nu)_{ran(\delta)}$, call $\alpha = (\nu \uplus \varrho_\nu)_{ran(\delta)}$. By definition of the rule application strategy, $ran(\delta) \subseteq \mathcal{T}_\Sigma(\mathcal{X} \setminus V_{\mathcal{R},Call_{\mathcal{R}}})$ so, as $dom(\delta) \subseteq V_{\mathcal{R},Call_{\mathcal{R}}}$, $ran(\delta) \cap dom(\delta) = \emptyset$. Then for each $y \in vars(x\delta)$, $y \in ran(\delta)$, $x \neq y$, and:

* if $y \notin dom(\alpha)$ then $y\alpha = y$ and $y \notin dom(\nu)_{ran(\delta)}$. In particular, as $y \in ran(\delta)$, $y \notin dom(\nu)$, so $vars(y\alpha) \cap dom(\nu) = \emptyset$;
    * else $y \in dom(\alpha)$ $(= dom((\nu \uplus \varrho_\nu)_{ran(\delta)}))$. Then:
      · if $y \in dom(\nu_{ran(\delta)})$ then $vars(y\alpha) \subseteq ran(\nu)$ so, as $dom(\nu) \cap ran(\nu) = \emptyset$, $vars(y\alpha) \cap dom(\nu) = \emptyset$, and
      · if $y \in dom((\varrho_\nu)_{ran(\delta)})$ then, as we have already proved $\varrho_\nu = \theta_{\setminus V}$ and θ is a composition of several CSUs, so $ran(\theta)$ is away from all known variables, $vars(y\alpha) \cap dom(\nu) = \emptyset$.

  In conclusion, $vars(x\gamma\sigma) \cap dom(\nu) = \emptyset$.

- The proof for rule top, with strategy $\mathsf{top}(c[\gamma]\{\overline{ST}\})$ and substitution σ, is exactly the same as the previous one.
- In rule $[c1]$ call strategy, $(\bar{u}, \bar{v}, \phi) = (\bar{u}'\sigma, \bar{v}'\sigma, \phi'\sigma)$, where $\sigma = none$, so there is nothing to prove.
- Now, we check rule $[c2]$ call strategy with strategy invocation $CS(\bar{t})$ and substitution $\gamma = \{\bar{x} \mapsto \bar{t}\}$. By I.H. $CS(\bar{t})$ has the form $(CS(\bar{w}))^{\nu'}\varrho_{\nu'}$, for proper $\bar{w}$, so $\bar{t} = \bar{w}(\nu' \uplus \varrho_{\nu'}) = \bar{w}(\nu \uplus \varrho_\nu) = \bar{w}(\nu \uplus \varrho_\nu)$ $(\sigma = none)$, hence $\gamma = \{\bar{x} \mapsto \bar{w}(\nu \uplus \varrho_\nu)\}$, call $\alpha = \nu \uplus \varrho_\nu$. The calculus rule uses a version of the condition C in the right-side of the call strategy definition, call it C', where all the variables are new except for $dom(\gamma) \cup V^{\nu'}$. The new variables in C' are not in $dom(\nu)$, so (†) applies. We check the rest of the variables in C'. For each $x \in vars(C') \cap (dom(\gamma) \cup V^{\nu'})$:
  - if $x \in V^{\nu'}$ then $x \in V^\nu$, because $\sigma = none$, and:
    * if $x \in V$ then $x \notin dom(\nu)$ so, as $x\sigma = x$, $vars(x\sigma) \cap dom(\nu) = \emptyset$;
    * else $x \in ran(\nu)$. Then, as $x\sigma = x$ and $dom(\nu) \cap ran(\nu) = \emptyset$, $vars(x\sigma) \cap dom(\nu) = \emptyset$;
  - else $x \in dom(\gamma)$ $(= \bar{x})$, say $x = x_i$, so $x\gamma = w_i\alpha$ $(\alpha = \nu \uplus \varrho_\nu)$. For every $y \in vars(w_i)$:
    * if $y \in dom(\nu)$ then $vars(y\alpha) \subseteq ran(\nu)$ so, as $dom(\nu) \cap ran(\nu) = \emptyset$, $vars(y\alpha) \cap dom(\nu) = \emptyset$,
    * if $y \in dom(\varrho_\nu)$ then, as we have already proved $\varrho_\nu = \theta_{\setminus V}$ and θ is a composition of several CSUs, so $ran(\theta)$ is away from all known variables, $vars(y\alpha) \cap dom(\nu) = \emptyset$,
    * else $y \notin (dom(\nu) \cup dom(\varrho_\nu))$, so $y\alpha = y$. Then, as $y \notin dom(\nu)$, $vars(y\alpha) \cap dom(\nu) = \emptyset$.

  In conclusion, $vars(x\gamma) \cap dom(\nu) = \emptyset$.

11. The last calculus rule applied to get G from a goal of the form $G_1^\nu\varrho_\nu$, where $G_0 \rightsquigarrow^*_{\theta'} G_1^\nu\varrho_\nu$ and, by I.H. and invariant 3, $\varrho_\nu = \theta'_{\setminus V}$:

    - may have generated G as an instance of $G_1^\nu\varrho_\nu$ with a substitution σ, so $\theta = \theta'\sigma$. Then Definition 38 ensures that $\varrho_\mu = (\varrho_\nu\sigma)_{V_G\setminus V} = (\theta'_{\setminus V}\sigma)_{V_G\setminus V} = (\theta'\sigma)_{V_G\setminus V} = \theta_{V_G\setminus V}$, and we take $\varrho'_\mu = \varrho_\mu$, or
    - it may have not generated an instance, so $\theta = \theta'$, and we take $\varrho'_\mu = (\varrho_\nu)_{V_G} = (\theta'_{\setminus V})_{V_G} = \theta'_{V_G\setminus V} = \theta_{V_G\setminus V}$.

□

Theorem 2. Given an associated rewrite theory $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ closed under B-extensions and a reachability goal G, if $\nu \mid \psi$ is a computed answer for G then for each substitution $\rho : V^\nu \to \mathcal{T}_\Sigma$ such that $\psi\rho$ is satisfiable, $\nu \cdot \rho$ is a solution for G.

Proof. By structural induction over the depth of the corresponding canonical narrowing path and the first inference rule applied. Remember that $V^\mu = (V \setminus dom(\mu)) \cup ran(\mu)$, $(\bigwedge_{i=1}^n u_i \to v_i/ST_i^\mu(\varrho_\mu)_i)\sigma = \bigwedge_{i=1}^n u_i\sigma \to v_i\sigma/ST_i^{(\mu\sigma)_V}((\varrho_\mu)_i\sigma)_{\setminus V}$, and $vars(G) = vars(\phi) \cup \bigcup_{i=1}^n vars(\{u_i, v_i\}) \cup V^\mu$ or $vars(G) = \{x_k\} \cup vars(\phi) \cup \bigcup_{i=1}^n vars(\{u_i, v_i\}) \cup V^\mu$ (for rules $[c]$ and $[r]$).

- Base case

  Rule $[d1]$ (idle):

  $G = u_1 \to v_1/\texttt{idle} \mid \psi_1 \mid V, \mu \rightsquigarrow_{[d1],\sigma} nil \mid \psi \mid V, (\mu\sigma)_V$, where $abstract_{\Sigma_1}((u, v)) = \langle\lambda(\bar{x}, \bar{y}).(u_1^\circ, v_1^\circ); (\theta_u^\circ, \theta_v^\circ); (\phi_u^\circ, \phi_v^\circ)\rangle$, $\psi = (\psi_1 \wedge \phi_u^\circ \wedge \phi_v^\circ)\sigma$, $\bar{x} = \{x_1, \ldots, x_{i_x}\}$, $u_1^\circ = u_1[\bar{x}]_{\bar{p}}$, $\phi_u^\circ = (\bigwedge_{i=1}^{i_x} x_i = u_1|_{p_i})$, $\bar{y} = \{y_1, \ldots, y_{i_y}\}$, $v_1^\circ = v_1[\bar{y}]_{\bar{q}}$, $\phi_v^\circ = (\bigwedge_{j=1}^{i_y} y_j = v_1|_{q_j})$, $\sigma \in CSU_B(u_1^\circ = v_1^\circ)$, so $u_1^\circ\sigma =_B v_1^\circ\sigma$, and ψ is satisfiable, for proper $\bar{p}$ and $\bar{q}$.

  As ρ is a ground substitution such that $dom(\rho) = vars(G\sigma)$ and $\psi\rho$ is satisfiable, i.e., $(\psi_1 \wedge \phi_u^\circ \wedge \phi_v^\circ)\sigma\rho$ is satisfiable, then $\psi_1\sigma\rho$ is ground, so $E_0 \vDash \psi_1\sigma\rho$, and $(\phi_u^\circ \wedge \phi_v^\circ)\sigma\rho$ is satisfiable, where $u_1\sigma\rho$ and $v_1\sigma\rho$ are ground terms, so there exists a substitution $\rho' : V_{\bar{x}\sigma\rho, \bar{y}\sigma\rho} \to \mathcal{T}_\Sigma$ such that $\bar{x}\sigma\rho\rho' =_{E_0} u_1|_{\bar{p}}\sigma\rho\rho' = u_1|_{\bar{p}}\sigma\rho$ and $\bar{y}\sigma\rho\rho' =_{E_0} v_1|_{\bar{q}}\sigma\rho\rho' = v_1|_{\bar{q}}\sigma\rho$.

  Let $\gamma = \sigma\rho\rho'$. As $u_1\sigma\rho$ and $v_1\sigma\rho$ are terms in $\mathcal{T}_\Sigma$, the theory inclusion $(\Sigma_0, E_0) \subseteq (\Sigma, E)$ is protecting, and $u_1^\circ\sigma\rho =_B v_1^\circ\sigma\rho$, then $u_1\sigma\rho = u_1\sigma\rho[u_1|_{\bar{p}}\sigma\rho]_{\bar{p}} =_{E_0} u_1\sigma\rho[\bar{x}\gamma]_{\bar{p}} = u_1\gamma[\bar{x}\gamma]_{\bar{p}} = u_1^\circ\gamma =_B v_1^\circ\gamma = v_1\gamma[\bar{y}\gamma]_{\bar{q}} = v_1\sigma\rho[\bar{y}\gamma]_{\bar{q}} =_{E_0} v_1\sigma\rho[v_1|_{\bar{q}}\sigma\rho]_{\bar{q}} = v_1\sigma\rho$, so $u_1\sigma\rho =_E v_1\sigma\rho$. As $vars(\{u_1, v_1, \psi_1\}) \subseteq vars(G)$ then $u_1\sigma_{vars(G)}\rho = u_1\sigma\rho =_E v_1\sigma\rho = v_1\sigma_{vars(G)}\rho$ and $E_0 \vDash \psi_1\sigma\rho$ implies $E_0 \vDash \psi_1\sigma_{vars(G)}\rho$ so, as in example 10, $[v_1\sigma_{vars(G)}\rho]_E \in \texttt{idle}@[u_1\sigma_{vars(G)}\rho]_E$, and $\sigma_{vars(G)}\rho$ is a solution of G.

- Inductive step

  $G = \bigwedge_{i=1}^n u_i \to v_i/ST_i^\mu(\varrho_\mu)_i \mid \psi_1 \mid V, \mu$ or $G = u_1|_p \to^1 x, u_1[x]_p \to v_1/ST_1^\mu(\varrho_\mu)_1 \wedge \bigwedge_{i=2}^n u_i \to v_i/ST_i^\mu(\varrho_\mu)_i \mid \psi_1 \mid V, \mu$. We let $\Delta = \bigwedge_{i=2}^n u_i \to v_i/ST_i^\mu(\varrho_\mu)_i$. When the substitution applied in the first narrowing step is *none*, Δ, ψ_1, and μ remain unchanged, so I.H. ensures that Δ and ψ_1 comply with the thesis of the theorem, as it is shown in the proof for the second subcase. We will omit this proof in the rest of related subcases, as the proof is always the same.

  1. Rule $[d1]$ (idle):
     $G = u_1 \to v_1/\texttt{idle} \wedge \Delta \mid \psi_1 \mid V, \mu \rightsquigarrow_{[d1],\sigma_1} \Delta \circ \sigma_1 \mid \psi_1\sigma_1 \wedge \phi^\circ\sigma_1 \mid V, (\mu\sigma_1)_V = G'\sigma_1$, with $G' = \Delta \mid \psi_1 \wedge \phi^\circ \mid V, \mu$, where $abstract_{\Sigma_1}(v_1) = \langle\lambda\bar{x}.v_1^\circ; \theta^\circ; \phi^\circ\rangle$, $\bar{x} = \{x_1, \ldots, x_l\}$, $v_1^\circ = v_1[x_1, \ldots, x_l]_{q_1 \ldots q_l}$, $\phi^\circ = (\bigwedge_{i=1}^l x_i = v_1|_{q_i})$, $\sigma_1 \in CSU_B(u_1 = v_1^\circ)$, $\psi_1\sigma_1 \wedge \phi^\circ\sigma_1$ is satisfiable, and $G'\sigma_1 \rightsquigarrow_{\sigma'}^+ nil \mid \psi \mid V, \nu$, call $\sigma = \sigma_1\sigma'$, so $\sigma_{vars(G)} \mid \psi$ is a computed answer for G, and $\sigma'_{vars(G'\sigma_1)} \mid \psi$ is a computed answer for $G'\sigma_1$.
     If $\rho : vars(G\sigma) \to \mathcal{T}_\Sigma$ is a substitution such that $\psi\rho$ is satisfiable, then let $\rho_1 = \rho_{vars(G'\sigma)}$, so also $\psi\rho_1$ is satisfiable. As $dom(\rho) = vars(G\sigma)$ then $dom(\rho_1) = vars(G\sigma) \cap vars(G'\sigma)$. Let $\rho_2 = \rho_{vars(G\sigma) \setminus vars(G'\sigma)}$, so $\rho = \rho_1 \uplus \rho_2$, and let $\rho_1' : vars(G'\sigma) \setminus vars(G\sigma) \to \mathcal{T}_\Sigma$, so $dom(\rho_1) \cap dom(\rho_1') = \emptyset$ and $dom(\rho_1) \cup dom(\rho_1') = vars(G'\sigma)$, such that $\psi(\rho_1 \uplus \rho_1')$ is satisfiable, and call $\rho' = \rho_1 \uplus \rho_1'$, so $\rho' : vars(G'\sigma) \to \mathcal{T}_\Sigma$.
     As $dom(\rho_1') = vars(G'\sigma) \setminus vars(G\sigma)$ and $dom(\rho_1) = vars(G\sigma) \cap vars(G'\sigma) \subseteq vars(G\sigma)$, then $\rho'_{vars(G\sigma)} = (\rho_1 \uplus \rho_1')_{vars(G\sigma)} = (\rho_1)_{vars(G\sigma)} = \rho_1$, so by I.H., as $\rho' : vars(G'(\sigma_1\sigma')) \to \mathcal{T}_\Sigma$ and $\psi\rho'$ is satisfiable, $\sigma'_{vars(G'\sigma_1)}\rho'$ is a solution for $G'\sigma_1$, meaning that $E_0 \vDash (\psi_1 \wedge \phi^\circ)\sigma_1\sigma'_{vars(G'\sigma_1)}\rho'$ and there are closed proof trees for each open goal in $\Delta\sigma_1\sigma'_{vars(G'\sigma_1)}\rho'$ with respect to the instantiation $\mathcal{D}_{\mathcal{R}, Call_\mathcal{R}}^{(\mu\sigma_1\sigma'_{vars(G'\sigma_1)}\rho')_V}$. We prove (a) $\Delta\sigma_1\sigma'_{vars(G'\sigma_1)}\rho' = \Delta\sigma_{vars(G)}\rho$ and (b) $(\mu\sigma_1\sigma'_{vars(G'\sigma_1)}\rho')_V = (\mu\sigma_{vars(G)}\rho)_V$:

     (a) As Δ appears both in G and G' then $vars(\Delta\sigma_1) \subseteq vars(G\sigma_1) \cap vars(G'\sigma_1) \subseteq vars(G'\sigma_1)$, so $\Delta\sigma_1\sigma'_{vars(G'\sigma_1)} = \Delta\sigma_1\sigma'$ and $vars(\Delta\sigma_1\sigma'_{vars(G'\sigma_1)}) = vars(\Delta\sigma_1\sigma') \subseteq$

$vars(G\sigma_1\sigma') \cap vars(G'\sigma_1\sigma') = vars(G\sigma) \cap vars(G'\sigma)$, hence $\Delta\sigma_1{\sigma'}_{vars(G'\sigma_1)}\rho' = \Delta\sigma_1\sigma'\rho' = \Delta\sigma_1\sigma'\rho_1 = \Delta\sigma_1\sigma'\rho = \Delta\sigma\rho = \Delta\sigma_{vars(G)}\rho$.

(b) If $v \in V$ then either
   - $v \notin dom(\mu)$ and $v\mu = v$, so $vars(v\mu) \subseteq V \setminus dom(\mu) \subseteq vars(G)$, or
   - $v \in dom(\mu)$, so $vars(v\mu) \subseteq ran(\mu) \setminus dom(\mu) \subseteq vars(G)$.

   Also, either
   - $v \notin dom(\mu\sigma_1)$ and $v\mu\sigma_1 = v$, so $vars(v\mu\sigma_1) \subseteq V \setminus dom(\mu\sigma_1) \subseteq vars(G\sigma_1) \cap vars(G'\sigma_1)$, or
   - $v \in dom(\mu\sigma_1)$, so $vars(v\mu\sigma_1) \subseteq ran(\mu\sigma_1) \setminus dom(\mu\sigma_1)$, and also $vars(v\mu\sigma_1) \subseteq vars(G\sigma_1) \cap vars(G'\sigma_1)$.

   As in the previous case, then $v\mu\sigma_1{\sigma'}_{vars(G'\sigma_1)} = v\mu\sigma_1\sigma'$, $vars(v\mu\sigma) = vars(v\mu\sigma_1\sigma') = vars(v\mu\sigma_1{\sigma'}_{vars(G'\sigma_1)}) \subseteq vars(G\sigma_1{\sigma'}_{vars(G'\sigma_1)}) \cap vars(G'\sigma_1{\sigma'}_{vars(G'\sigma_1)}) = vars(G\sigma_1\sigma') \cap vars(G'\sigma_1\sigma') = vars(G\sigma) \cap vars(G'\sigma)$. Then $v\mu\sigma_1{\sigma'}_{vars(G'\sigma_1)}\rho' = v\mu\sigma\rho' = v\mu\sigma\rho_1 = v\mu\sigma\rho = v\mu\sigma_{vars(G)}\rho$ hence $(\mu\sigma_1{\sigma'}_{vars(G'\sigma_1)}\rho')_V = (\mu\sigma_{vars(G)}\rho)_V$.

Then, from (a) and (b), the same closed proof trees are also valid for each open goal in $\Delta\sigma_{vars(G)}\rho$ with respect to the instantiation $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}^{(\mu\sigma_{vars(G)}\rho)_V}$.
As $vars(\psi_1 \wedge \phi^\circ) \subseteq vars(G')$ then $(\psi_1 \wedge \phi^\circ)\sigma_1{\sigma'}_{vars(G'\sigma_1)} = (\psi_1 \wedge \phi^\circ)\sigma_1\sigma' = (\psi_1 \wedge \phi^\circ)\sigma$, so $E_0 \vDash (\psi_1 \wedge \phi^\circ)\sigma\rho'$, hence $E_0 \vDash \psi_1\sigma\rho'$ and $E_0 \vDash \phi^\circ\sigma\rho'$, where $(\psi_1 \wedge \phi^\circ)\sigma\rho'$ is ground, because $vars((\psi_1 \wedge \phi^\circ)\sigma) \subseteq vars(G'\sigma)$ and $\rho' : vars(G'\sigma) \to \mathcal{T}_\Sigma$. Now, $dom(\rho_1) = vars(G\sigma) \cap vars(G'\sigma)$, and $vars(v_1|_{q_i}) \subseteq vars(G\sigma) \cap vars(G'\sigma)$ implies $v_1|_{q_i}\sigma\rho_1 \in \mathcal{T}_\Sigma$ so, as $\rho' = \rho_1 \uplus \rho'_1$, $v_1|_{q_i}\sigma\rho' = v_1|_{q_i}\sigma(\rho_1 \uplus \rho'_1) = v_1|_{q_i}\sigma\rho_1 = v_1|_{q_i}\sigma(\rho_1 \uplus \rho_2) = v_1|_{q_i}\sigma\rho$, for $1 \leq i \leq l$, hence $\phi^\circ\sigma\rho' = (\bigwedge_{i=1}^{l} x_i\sigma\rho' = v_1|_{q_i}\sigma\rho)$. As also $vars(\psi_1\sigma) \subseteq vars(G\sigma) \cap vars(G'\sigma)$ then, reasoning exactly in the same way, $\psi_1\sigma\rho' = \psi_1\sigma\rho$, so $E_0 \vDash \psi_1\sigma\rho$.
Let $\gamma = \sigma(\rho_1 \uplus \rho_2 \uplus \rho'_1)$, where $\rho_1 \uplus \rho_2 \uplus \rho'_1 = \rho \uplus \rho'_1 = \rho' \uplus \rho_2$. As $u_1\sigma_1 =_B v_1^\circ\sigma_1$ then $u_1\sigma =_B v_1^\circ\sigma$, so $u_1\gamma =_B v_1^\circ\gamma$. Also, $u_1\sigma\rho$ and $v_1\sigma\rho \in \mathcal{T}_\Sigma$, because $vars(\{u_1\sigma, v_1\sigma\}) \subseteq dom(\rho) = vars(G\sigma)$, so $u_1\gamma = u_1\sigma\rho$ and $v_1\gamma = v_1\sigma\rho$. Finally, $\phi^\circ\sigma\rho'$ ground implies $x_i\sigma\rho'$ ground, so $x_i\sigma\rho' = x_i\gamma$, for $1 \leq i \leq l$. Then, $u_1\sigma\rho = u_1\gamma =_B v_1^\circ\gamma = v_1\gamma[x_1\gamma, \ldots, x_l\gamma]_{q_1 \ldots q_l} = v_1\sigma\rho[x_1\sigma\rho', \ldots, x_l\sigma\rho']_{q_1 \ldots q_l} =_{E_0} v_1\sigma\rho[v_1|_{q_1}\sigma\rho, \ldots, v_1|_{q_l \ldots q_l}] = v_1\sigma\rho$, so, as $E = B \cup E_0$, $u_1\sigma\rho =_E v_1\sigma\rho$, and, as $vars(\{u_1, v_1\}) \subseteq vars(G)$, $u_1\sigma_{vars(G)}\rho =_E v_1\sigma_{vars(G)}\rho$. Then, as in example 10, $[v_1\sigma_{vars(G)}\rho]_E \in \texttt{idle}@[u_1\sigma_{vars(G)}\rho]_E$. As also $E_0 \vDash \psi_1\sigma_{vars(G)}\rho$, and there are closed proof trees for each open goal in $\Delta\sigma_{vars(G)}\rho$ with respect to $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}^{(\mu\sigma_{vars(G)}\rho)_V}$, then $\sigma_{vars(G)}\rho$ is a solution of G.

2. Rule $[d2]$ (idle):
   $G = u_1 \to v_1/\texttt{idle}; ST^\mu\varrho_\mu \wedge \Delta \mid \psi_1 \mid V, \mu \rightsquigarrow_{[d2],none} u_1 \to v_1/ST^\mu\varrho_\mu \wedge \Delta \mid \psi_1 \mid V, \mu = G'$ and $G' \rightsquigarrow_\sigma^+ nil \mid \psi \mid V, \nu$, where $\nu = (\mu\sigma)_V$, so $\sigma_{vars(G)} \mid \psi$ is a computed answer for both G and G', since $vars(G) = vars(G')$. For any substitution ρ that satisfies the premises of the theorem, by I.H., $\sigma_{vars(G)}\rho$ is a solution for G', call $\delta = \sigma_{vars(G)}\rho$, $\nu' = (\mu\delta)_V$, and $\varrho_{\nu'} = (\varrho_\mu\delta)_{\setminus V}$, so $E_0 \vDash \psi_1\delta$, there are closed proof trees for each open goal in $\Delta\delta$, and also a c.p.t. $\frac{F}{u_1\delta \to v_1\delta/ST^{\nu'}\varrho_{\nu'}}$, all of them with respect to $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}^{\nu'}$. As there is a rule $\frac{u_1\delta \to u_1\delta/\texttt{idle} \quad u_1\delta \to v_1\delta/ST^{\nu'}\varrho_{\nu'}}{u_1\delta \to v_1\delta/\texttt{idle}\,;\, ST^{\nu'}\varrho_{\nu'}} \in \mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}^{\nu'}$, then $\frac{\overline{u_1\delta \to u_1\delta/\texttt{idle}} \quad \frac{F}{u_1\delta \to v_1\delta/ST^{\nu'}\varrho_{\nu'}}}{u_1\delta \to v_1\delta/\texttt{idle}\,;\, ST^{\nu'}\varrho_{\nu'}}$ is also a c.p.t., with respect to $\mathcal{D}_{\mathcal{R}}^{\nu'}$, so $\sigma_{vars(G)}\rho$, is also a solution of G.

3. Rules $[o1]$ and $[o2]$ (or):

we prove $[o1]$; the proof for $[o2]$ is exactly the same, with ST_2 instead of ST_1. $G = u_1 \to v_1/((ST_1^\mu \mid ST_2^\mu) \,;\, ST^\mu)\varrho_\mu \wedge \Delta \mid \psi_1 \mid V, \mu \rightsquigarrow_{[o1],none} u_1 \to v_1/(ST_1^\mu \,;\, ST^\mu)\varrho_\mu \wedge \Delta \mid \psi_1 \mid V, \mu = G'$, so $vars(G) = vars(G')$, and $G' \rightsquigarrow_\sigma^+ nil \mid \psi \mid V, \nu$, where $\nu = (\mu\sigma)_V$, so $\sigma_{vars(G)}|\psi$ is a computed answer for G and $\sigma_{vars(G')}|\psi$ is a computed answer for G'. Call $\Delta_1 = u_1 \to v_1/(ST_1^\mu \,;\, ST^\mu)\varrho_\mu$. By I.H., for any substitution $\rho : vars(G'\sigma) \to \mathcal{T}_\Sigma$ such that $\psi\rho$ is satisfiable, $\sigma_{vars(G')}\rho$ is a solution for G', call $\delta = \sigma_{vars(G')}\rho$ $(= \sigma_{vars(G)}\rho)$, $\nu' = (\mu\delta)_V$, and $\varrho_{\nu'} = (\varrho_\mu\delta)_{\backslash V}$, so there is a c.p.t. for $\Delta_1\delta$ with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$. The c.p.t. has the form $\frac{\frac{F_1}{u_1\delta \to t/ST_1^{\nu'}\varrho_{\nu'}} \quad \frac{F_2}{t \to v_1\delta/ST^{\nu'}\varrho_{\nu'}}}{u_1\delta \to v_1\delta/(ST_1^{\nu'} \,;\, ST^{\nu'})\varrho_{\nu'}}$ for some term $t \in \mathcal{H}_\Sigma$. As there are rules $\frac{u_1\delta \to t/(ST_1^{\nu'}|ST_2^{\nu'})\varrho_{\nu'} \quad t \to v_1\delta/ST^{\nu'}\varrho_{\nu'}}{u_1\delta \to v_1\delta/((ST_1^{\nu'}|ST_2^{\nu'}) \,;\, ST^{\nu'})\varrho_{\nu'}}$ and $\frac{u_1\delta \to t/ST_1^{\nu'}\varrho_{\nu'}}{u_1\delta \to t/(ST_1^{\nu'}|ST_2^{\nu'})\varrho_{\nu'}}$ in $\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$, then the proof tree

$$\frac{\frac{\frac{F_1}{u_1\delta \to t/ST_1^{\nu'}\varrho_{\nu'}}}{u_1\delta \to t/(ST_1^{\nu'}|ST_2^{\nu'})\varrho_{\nu'}} \quad \frac{F_2}{t \to v_1\delta/ST^{\nu'}\varrho_{\nu'}}}{u_1\delta \to v_1\delta/((ST_1^{\nu'} \mid ST_2^{\nu'}) \,;\, ST^{\nu'})\varrho_{\nu'}}$$

is closed, so, as $vars(G) = vars(G')$, $\rho : vars(G\sigma) \to \mathcal{T}_\Sigma$, $\psi\rho$ is satisfiable, and $\sigma_{vars(G)}\rho$ is a solution of G.

4. Rule $[p1]$ (plus):
$G = u_1 \to v_1/(ST_1^\mu + \,;\, ST^\mu)\varrho_\mu \wedge \Delta \mid \psi_1 \mid V, \mu \rightsquigarrow_{[p1],none} u_1 \to v_1/(ST_1^\mu \,;\, ST^\mu)\varrho_\mu \wedge \Delta \mid \psi_1 \mid V, \mu = G'$, so $vars(G) = vars(G')$, and $G' \rightsquigarrow_\sigma^+ nil \mid \psi \mid V, \nu$, where $\nu = (\mu\sigma)_V$, hence $\sigma_{vars(G)}|\psi$ is a computed answer for both G and G'. Call $\Delta_1 = u_1 \to v_1/(ST_1^\mu \,;\, ST^\mu)\varrho_\mu$. By I.H., for any substitution $\rho : vars(G\sigma) \to \mathcal{T}_\Sigma$ such that $\psi\rho$ is satisfiable, $\sigma_{vars(G)}\rho$ is a solution for G', call $\delta = \sigma_{vars(G')}\rho$ $(= \sigma_{vars(G)}\rho)$, $\nu' = (\mu\delta)_V$, and $\varrho_{\nu'} = (\varrho_\mu\delta)_{\backslash V}$, so there is a c.p.t. for $\Delta_1\delta$ with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$. The c.p.t. has the form $\frac{\frac{F_1}{u_1\delta \to t/ST_1^{\nu'}\varrho_{\nu'}} \quad \frac{F_2}{t \to v_1\delta/ST^{\nu'}\varrho_{\nu'}}}{u_1\delta \to v_1\delta/(ST_1^{\nu'} \,;\, ST^{\nu'})\varrho_{\nu'}}$ for some term $t \in \mathcal{H}_\Sigma$. As there are rules $\frac{u_1\delta \to t/(ST_1^{\nu'}\varrho_{\nu'})+ \quad t \to v_1\delta/ST^{\nu'}\varrho_{\nu'}}{u_1\delta \to v_1\delta/(ST_1^{\nu'}+ \,;\, ST^{\nu'})\varrho_{\nu'}}$ and $\frac{u_1\delta \to t/ST_1^{\nu'}\varrho_{\nu'}}{u_1\delta \to t/(ST_1^{\nu'}\varrho_{\nu'})+}$ in $\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$, then

$$\frac{\frac{\frac{F_1}{u_1\delta \to t/ST_1^{\nu'}\varrho_{\nu'}}}{u_1\delta \to t/(ST_1^{\nu'}\varrho_{\nu'})+} \quad \frac{F_2}{t \to v_1\delta/ST^{\nu'}\varrho_{\nu'}}}{u_1\delta \to v_1\delta/(ST_1^{\nu'}+ \,;\, ST^{\nu'})\varrho_{\nu'}}$$

is a c.p.t., so $\rho : vars(G\sigma) \to \mathcal{T}_\Sigma$, $\psi\rho$ is satisfiable, and $\sigma_{vars(G)}\rho$ is a solution of G.

5. Rule $[p2]$ (plus):
$G = u_1 \to v_1/(ST_1^\mu+; ST^\mu)\varrho_\mu \wedge \Delta \mid \psi_1 \mid V, \mu \rightsquigarrow_{[p2],none} u_1 \to v_1/(ST_1^\mu; ST_1^\mu+; ST^\mu)\varrho_\mu \wedge \Delta \mid \psi_1 \mid V, \mu = G'$, so $vars(G) = vars(G')$, and $G' \rightsquigarrow_\sigma^+ nil \mid \psi \mid V, \nu$, where $\nu = (\mu\sigma)_V$, hence $\sigma_{vars(G)}|\psi$ is a computed answer for both G and G'. Call $\Delta_1 = u_1 \to v_1/(ST_1^\mu; ST_1^\mu+)\varrho_\mu; ST^\mu$. By I.H., for any substitution $\rho : vars(G\sigma) \to \mathcal{T}_\Sigma$ such that $\psi\rho$ is satisfiable, $\sigma_{vars(G)}\rho$ is a solution for G', call $\delta = \sigma_{vars(G')}\rho$ $(= \sigma_{vars(G)}\rho)$, $\nu' = (\mu\delta)_V$, and $\varrho_{\nu'} = (\varrho_\mu\delta)_{\backslash V}$, so there is a c.p.t. for $\Delta_1\delta$ with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$. The c.p.t. has the form $\frac{\frac{F_1}{u_1\delta \to t_1/ST_1^{\nu'}\varrho_{\nu'}} \quad \frac{\frac{F_2}{t_1 \to t_2/(ST_1^{\nu'}\varrho_{\nu'})+} \quad \frac{F_3}{t_2 \to v_1\delta/ST^{\nu'}\varrho_{\nu'}}}{t_1 \to v_1\delta/(ST_1^{\nu'}+ \,;\, ST^{\nu'})\varrho_{\nu'}}}{u_1\delta \to v_1\delta/(ST_1^{\nu'} \,;\, ST_1^{\nu'}+ \,;\, ST^{\nu'})\varrho_{\nu'}}$, for terms t_1 and

$t_2 \in \mathcal{H}_\Sigma$. As there are rules $\frac{u_1\delta \to t_2/(ST_1^{\nu'}\varrho_{\nu'})+ \quad t_2 \to v_1\delta/ST^{\nu'}\varrho_{\nu'}}{u_1\delta \to v_1\delta/(ST_1^{\nu'}+\,;\,ST^{\nu'})\varrho_{\nu'}}$, $\frac{u_1\delta \to t_2/(ST_1^{\nu'}\,;\,ST_1^{\nu'}+)\varrho_{\nu'}}{u_1\delta \to t_2/(ST_1^{\nu'}\varrho_{\nu'})+}$, and $\frac{u_1\delta \to t_1/ST_1^{\nu'}\varrho_{\nu'} \quad t_1 \to t_2/(ST_1^{\nu'}\varrho_{\nu'})+}{u_1\delta \to t_2/ST_1^{\nu'}\varrho_{\nu'}\,;\,ST_1^{\nu'}+}$ in $\mathcal{D}^{\nu'}_{\mathcal{R},Call_{\mathcal{R}}}$, then

$$\frac{\dfrac{\dfrac{\dfrac{F_1}{u_1\delta \to t_1/ST_1^{\nu'}\varrho_{\nu'}} \quad \dfrac{F_2}{t_1 \to t_2/(ST_1^{\nu'}\varrho_{\nu'})+}}{u_1\delta \to t_2/(ST_1^{\nu'}\,;\,ST_1^{\nu'}+)\varrho_{\nu'}}}{u_1\delta \to t_2/(ST_1^{\nu'}\varrho_{\nu'})+} \quad \dfrac{F_3}{t_2 \to v_1\delta/ST^{\nu'}\varrho_{\nu'}}}{u_1\delta \to v_1\delta/(ST_1^{\nu'}+\,;\,ST^{\nu'})\varrho_{\nu'}}$$

is a c.p.t., so $\rho : vars(G\sigma) \to \mathcal{T}_\Sigma$, $\psi\rho$ is satisfiable, and $\sigma_{vars(G)}\rho$ is a solution of G.

6. Rule $[s1]$ (star):
$G = u_1 \to v_1/(ST_1^{\mu}*\,;\,ST^{\mu})\varrho_\mu \wedge \Delta \mid \psi_1 \mid V, \mu \leadsto_{[s1],none} u_1 \to v_1/ST^{\mu}\varrho_\mu \wedge \Delta \mid \psi_1 \mid V, \mu = G'$, so $vars(G) = vars(G')$, and $G' \leadsto^+_\sigma nil \mid \psi \mid V, \nu$, where $\nu = (\mu\sigma)_V$, so $\sigma_{vars(G)}|\psi$ is a computed answer for G and $\sigma_{vars(G')}|\psi$ is a computed answer for G'. Call $\Delta_1 = u_1 \to v_1/ST^{\mu}\varrho_\mu$. By I.H., for any substitution $\rho : vars(G'\sigma) \to \mathcal{T}_\Sigma$ such that $\psi\rho$ is satisfiable, $\sigma_{vars(G')}\rho$ is a solution for G', call $\delta = \sigma_{vars(G')}\rho$ $(= \sigma_{vars(G)}\rho)$, $\nu' = (\mu\delta)_V$, and $\varrho_{\nu'} = (\varrho_\mu\delta)_{\setminus V}$, so there is a c.p.t. for $\Delta_1\delta$ with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_{\mathcal{R}}}$. The c.p.t. has the form $\frac{F_1}{u_1\delta \to v_1\delta/ST^{\nu'}\varrho_{\nu'}}$.
As, by definition, $(ST_1^{\nu'}\varrho_{\nu'})* = \mathtt{idle} \mid (ST_1^{\nu'}\varrho_{\nu'})+$ and there are rules $\frac{}{u_1\delta \to u_1\delta/\mathtt{idle}}$, $\frac{u_1\delta \to u_1\delta/\mathtt{idle}}{u_1\delta \to u_1\delta/\mathtt{idle}|(ST_1^{\nu'}\varrho_{\nu'})+}$, and $\frac{u_1\delta \to u_1\delta/\mathtt{idle}|(ST_1^{\nu'}\varrho_{\nu'})+ \quad u_1\delta \to v_1\delta/ST^{\nu'}\varrho_{\nu'}}{u_1\delta \to v_1\delta/(\mathtt{idle}|ST_1^{\nu'}+\,;\,ST^{\nu'})\varrho_{\nu'}}$ in $\mathcal{D}^{\nu'}_{\mathcal{R},Call_{\mathcal{R}}}$, then the proof tree

$$\frac{\dfrac{\dfrac{}{u_1\delta \to u_1\delta/\mathtt{idle}}}{u_1\delta \to u_1\delta/\mathtt{idle}|(ST_1^{\nu'}\varrho_{\nu'})+} \quad \dfrac{F_1}{u_1\delta \to v_1\delta/ST^{\nu'}\varrho_{\nu'}}}{u_1\delta \to v_1\delta/((\mathtt{idle} \mid ST_1^{\nu'}+)\,;\,ST^{\nu'})\varrho_{\nu'}}$$

is closed, so $\rho : vars(G\sigma) \to \mathcal{T}_\Sigma$, $\psi\rho$ is satisfiable, and $\sigma_{vars(G)}\rho$ is a solution of G.

7. Rule $[s2]$ (star):
$G = u_1 \to v_1/(ST_1^{\mu}*; ST^{\mu})\varrho_\mu \wedge \Delta \mid \psi_1 \mid V, \mu \leadsto_{[s2],none} u_1 \to v_1/(ST_1^{\mu}+; ST^{\mu})\varrho_\mu \wedge \Delta \mid \psi_1 \mid V, \mu = G'$, so $vars(G) = vars(G')$, and $G' \leadsto^+_\sigma nil \mid \psi \mid V, \nu$, where $\nu = (\mu\sigma)_V$, hence $\sigma_{vars(G)}|\psi$ is a computed answer for both G and G'. Call $\Delta_1 = (ST_1^{\mu}+; ST^{\mu})\varrho_\mu$. By I.H., for any substitution $\rho : vars(G\sigma) \to \mathcal{T}_\Sigma$ such that $\psi\rho$ is satisfiable, $\sigma_{vars(G)}\rho$ is a solution for G', call $\delta = \sigma_{vars(G')}\rho$ $(= \sigma_{vars(G)}\rho)$, $\nu' = (\mu\delta)_V$, and $\varrho_{\nu'} = (\varrho_\mu\delta)_{\setminus V}$, so there is a c.p.t. for the goal $\Delta_1\delta$ with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_{\mathcal{R}}}$.
The c.p.t. has the form $\frac{\frac{F_1}{u_1\delta \to t/(ST_1^{\nu'}\varrho_{\nu'})+} \quad \frac{F_2}{t \to v_1\delta/ST^{\nu'}\varrho_{\nu'}}}{u_1\delta \to v_1\delta/(ST_1^{\nu'}\varrho_{\nu'})+\,;\,ST^{\nu'}\varrho_{\nu'}}$ for some term $t \in \mathcal{H}_\Sigma$. As, by definition, $(ST_1^{\nu'}\varrho_{\nu'})* = \mathtt{idle} \mid (ST_1^{\nu'}\varrho_{\nu'})+$ and there are rules $\frac{u_1\delta \to t/(ST_1^{\nu'}\varrho_{\nu'})+}{u_1\delta \to t/\mathtt{idle}|(ST_1^{\nu'}\varrho_{\nu'})+}$ and $\frac{u_1\delta \to t/\mathtt{idle}|(ST_1^{\nu'}\varrho_{\nu'})+ \quad t \to v_1\delta/ST_1\delta}{u_1\delta \to v_1\delta/((\mathtt{idle}|ST_1^{\nu'}+)\,;\,ST^{\nu'})\varrho_{\nu'}}$ in $\mathcal{D}^{\nu'}_{\mathcal{R},Call_{\mathcal{R}}}$, then

$$\frac{\dfrac{\dfrac{F_1}{u_1\delta \to t/(ST_1^{\nu'}\varrho_{\nu'})+}}{u_1\delta \to t/\mathtt{idle}|(ST_1^{\nu'}\varrho_{\nu'})+} \quad \dfrac{F_2}{t \to v_1\delta/ST^{\nu'}\varrho_{\nu'}}}{u_1\delta \to v_1\delta/((\mathtt{idle} \mid ST_1^{\nu'}+)\,;\,ST^{\nu'})\varrho_{\nu'}}$$

is a c.p.t., so $\rho : vars(G\sigma) \to \mathcal{T}_\Sigma$, $\psi\rho$ is satisfiable, and $\sigma_{vars(G)}\rho$ is a solution of G.

8. Rule $[i1]$ (if then else):
$G = u_1 \to v_1 \,/\, (\texttt{match}\; t_1\; \texttt{s.t.}\; \phi_1 \;?\; ST_1 : ST_2 ; ST)^{\mu}\varrho_\mu\,(\wedge\Delta) \mid \psi_1 \mid V,\mu \rightsquigarrow_{[i1],\sigma_1} (u_1 \to v_1/(ST_1\,;ST)^{\mu}\varrho_\mu\,(\wedge\,\Delta) \mid \psi_2 \mid V,\mu)\sigma_1 = G'\sigma_1$, call $t = t_1^{\mu}\varrho_\mu$ and $\phi = \phi_1^{\mu}\varrho_\mu$, where $abstract_{\Sigma_1}(t) = \langle \lambda\bar{x}.t^\circ;\sigma^\circ;\phi^\circ\rangle$, $t^\circ = t[\bar{x}]_{\bar{q}}$, with $\bar{x} = x_1,\ldots,x_l$ and $\bar{q} = q_1,\ldots,q_l$, $\phi^\circ = (\bigwedge_{i=1}^{l} x_i = t|_{q_i})$, hence $V_{t^\circ}\cup V_{\phi^\circ} = V_t\cup\hat{x}$, $\sigma_1 \in CSU_B(u_1 = t^\circ)$, $\psi_2 = \psi_1\wedge\phi\wedge\phi^\circ$, so $V_G \subseteq V_{G'}$, $\psi_2\sigma_1$ is satisfiable, and $G'\sigma_1 \rightsquigarrow^{+}_{\sigma'} nil \mid \psi \mid V,\nu$, call $\sigma = \sigma_1\sigma'$, where $\nu = (\mu\sigma)_V = (\mu\sigma_1\sigma')_V$, so $\sigma_{V_G} \mid \psi$ is a computed answer for G and $\sigma'_{V_{G'\sigma_1}} \mid \psi$ is a computed answer for $G'\sigma_1$.

Let $\rho : V_{G\sigma} \to \mathcal{T}_\Sigma$ be a substitution such that $\psi\rho$ is satisfiable, call $\delta = \sigma_{V_G}\rho$, $\nu' = (\mu\delta)_V$, so $dom(\nu') = V$ and $ran(\nu') = \emptyset$, and $\varrho_{\nu'} = (\varrho_\mu\delta)_{\setminus V}$, so $\delta : V_G \to \mathcal{T}_\Sigma$. As $dom(\rho) = V_{G\sigma}$ and $V_G \subseteq V_{G'}$, so $V_{G\sigma} \subseteq V_{G'\sigma}$, then $dom(\rho) \subseteq V_{G'\sigma}$. Let $\rho'_1 : V_{G'\sigma}\setminus V_{G\sigma} \to \mathcal{T}_\Sigma$, so $dom(\rho)\cup dom(\rho'_1) = V_{G'\sigma}$, such that $\psi(\rho\uplus\rho'_1)$ is satisfiable, and call $\rho' = \rho\uplus\rho'_1$, so $\rho' : V_{G'\sigma} \to \mathcal{T}_\Sigma$ and $\rho'_{V_{G\sigma}} = \rho$.

By I.H., as $\rho' : V_{G'\sigma_1\sigma'} \to \mathcal{T}_\Sigma$ and $\psi\rho'$ is satisfiable, $\sigma'_{V_{G'\sigma_1}}\rho'$ is a solution for $G'\sigma_1$, call $\delta' = \sigma_1\sigma'_{V_{G'\sigma_1}}\rho'$, $\varrho' = (\varrho_\mu\delta')_{\setminus V}$, and $\rho'' = \delta'_{V_{t,\phi}\setminus V_G}$.

We prove several intermediate results:

- $(\mu\delta)_V = (\mu\delta')_V$.
  We prove the equivalent fact, $x \in vars(V\mu) \implies x\delta = x\delta'$: as $V^\mu = (V \setminus dom(\mu))\cup ran(\mu)$ then $V_{V\mu} = V^\mu$ so if $x \in V_{V\mu} = V^\mu \subseteq V_G$ then $x \in V_G$, $x\sigma_1 \in V_{G\sigma_1} \subseteq V_{G'\sigma_1}$, and $x(\sigma_1\sigma')_{V_G} = x\sigma_1\sigma'_{V_{G'\sigma_1}}$. Now, as $x\delta$ $(= x\sigma_{V_G}\rho)$ is ground, $x\delta = x\sigma_{V_G}\rho = x\sigma_{V_G}(\rho\uplus\rho'_1) = x\sigma_{V_G}\rho' = x(\sigma_1\sigma')_{V_G}\rho' = x\sigma_1\sigma'_{V_{G'\sigma_1}}\rho' = x\delta'$.
- $V_{(t\sigma,\phi\sigma)} \subseteq V_{G'\sigma}$.
  As $V_{t^\circ}\cup V_{\phi^\circ} = V_t\cup\hat{x}$, $\psi_2 = \psi_1\wedge\phi\wedge\phi^\circ$, and $\sigma_1 \in CSU_B(u_1 = t^\circ)$, so $V_{t^\circ\sigma_1} = V_{u_1\sigma_1} \subseteq V_{G\sigma_1}$, because B is regular, hence $V_{G\sigma_1}\cup V_{t^\circ\sigma_1} = V_{G\sigma_1}$, then $V_{G'\sigma_1} = V_{G\sigma_1}\cup V_{\phi^\circ\sigma_1}\cup V_{\phi\sigma_1} = V_{G\sigma_1}\cup V_{t^\circ\sigma_1}\cup V_{\phi^\circ\sigma_1}\cup V_{\phi\sigma_1} = V_{G\sigma_1}\cup V_{t\sigma_1}\cup V_{\bar{x}\sigma_1}\cup V_{\phi\sigma_1} = V_{G\sigma_1}\cup V_{(t\sigma_1,\phi\sigma_1)}\cup V_{\bar{x}\sigma_1}$, so $V_{(t\sigma_1,\phi\sigma_1)} \subseteq V_{G'\sigma_1}$, hence $V_{(t\sigma,\phi\sigma)} \subseteq V_{G'\sigma}$.
- $V_{(t_1^{\nu'},\phi_1^{\nu'})} \subseteq V_{(t_1^{\mu},\phi_1^{\mu})}$.
  This is immediate since $dom(\mu) \subseteq V$, $\nu' = (\mu\delta)_V$, so $dom(\mu) \subseteq dom(\nu')$, and $\nu' : V \to \mathcal{T}_\Sigma$.
- $V_{(t_1^{\mu},\phi_1^{\mu})} \setminus V_{(t_1^{\nu'},\phi_1^{\nu'})} \subseteq V^\mu$.
  As $dom(\mu) \subseteq V$ and $\nu' = (\mu\delta)_V$ then the variables in $V_{(t_1^{\mu},\phi_1^{\mu})}$ instantiated in $V_{(t_1^{\nu'},\phi_1^{\nu'})}$ must belong either to $V\setminus dom(\mu)$ or to $ran(\mu)$, i.e., to V^μ. Since $\nu' : V \to \mathcal{T}_\Sigma$ then $V_{(t_1^{\nu'},\phi_1^{\nu'})} \setminus V_{(t_1^{\mu},\phi_1^{\mu})} = \emptyset$ and the result follows.
- $\phi\sigma\rho' = \phi_1^{\nu'}\varrho_{\nu'}\rho''$.
  As $(\mu\delta)_V = (\mu\delta')_V$ then $\phi\sigma\rho' = \phi\delta' = (\phi_1^{\mu}\varrho_\mu)\delta' = \phi_1^{(\mu\delta')_V}(\varrho_\mu\delta')_{\setminus V} = \phi_1^{(\mu\delta)_V}\varrho' = \phi_1^{\nu'}\varrho'$, so we prove the equivalent $\phi_1^{\nu'}\varrho' = \phi_1^{\nu'}\varrho_{\nu'}\rho''$ by proving $x \in V_{\phi_1} \implies x^{\nu'}\varrho' = x^{\nu'}\varrho_{\nu'}\rho''$. We consider two cases:
  * if $x \in V$ then $x^{\nu'}$ is ground, so $x^{\nu'}\varrho' = x^{\nu'}\varrho_{\nu'}\rho''$.
  * if $x \notin V$ then $x^{\nu'} = x$, so $x^{\nu'} \notin V$. Also, as $x \notin V$, $x^\mu = x$ so, as $x \in V_{\phi_1}$, $x \in V_{\phi_1^{\mu}}$. As $x \notin V$ and $x^{\nu'} = x$ then $x^{\nu'}\varrho' = x\varrho' = x(\varrho_\mu\delta')_{\setminus V} = x\varrho_\mu\delta'$ and $x^{\nu'}\varrho_{\nu'}\rho'' = x\varrho_{\nu'}\rho'' = x(\varrho_\mu\delta)_{\setminus V}\rho'' = x\varrho_\mu\delta\rho''$, so we check $x\varrho_\mu\delta' = x\varrho_\mu\delta\rho''$ by checking $y \in V_{x\varrho_\mu} \implies y\delta' = y\delta\rho''$:
    · as $x \in V_{\phi_1^{\mu}}$ and $y \in V_{x\varrho_\mu}$ then $y \in V_{\phi_1^{\mu}\varrho_\mu}$, i.e., $y \in V_\phi$;
    · again, we consider two cases:
    (a) if $y \in V_G$ then $y\delta$ is ground, so $y\delta\rho'' = y\delta = y\sigma_{V_G}\rho = y\sigma\rho = y\sigma_1\sigma'\rho$. Also, as $V_G \subseteq V_{G'}$, $y \in V_{G'}$ and $V_{y\sigma_1} \subseteq V_{G'\sigma_1}$, so $y\delta' = y\sigma_1\sigma'_{V_{G'\sigma_1}}\rho = y\sigma_1\sigma'\rho = y\delta\rho''$;

(b) if $y \notin V_G$ then, as $y \in V_\phi$, $y \in V_{\phi \setminus G} \subseteq V_{(\phi,t)\setminus G}$ so, as also $y \notin V_G$ and $\delta : V_G \to \mathcal{T}_\Sigma$, $y\delta\rho'' = y\rho'' = y\delta'_{V_{(\phi_1,t)\setminus G}} = y\delta'$.

– $t\sigma\rho' = t_1^{\nu'} \varrho_{\nu'} \rho''$.
The proof is the same as the previous one, just exchanging ϕ and t everywhere, even when they appear with subscripts and/or superscripts.

As $\sigma'_{V_{G'\sigma_1}} \rho'$ is a solution for $G'\sigma_1$ then, by I.H.:

(a) $E_0 \vDash \psi_2\delta'$, i.e., $E_0 \vDash (\psi_1 \wedge \phi \wedge \phi^\circ)\delta'$,

(b) there are closed proof trees for each open goal in $\Delta\delta'$, with respect to $\mathcal{D}^{(\mu\delta')_V}_{\mathcal{R},Call_\mathcal{R}}$ ($=\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$, we use ν' instead of $(\mu\delta')_V$ in (c)), and

(c) $[v_1\delta']_E \in (ST_1\,;ST)^{\nu'} \varrho' @ [u_1\delta']_E$,

so:

(a) i. $V_{\psi_2} \subseteq V_{G'}$ implies $\psi_2\sigma_1\sigma'_{V_{G'\sigma_1}} = \psi_2\sigma_1\sigma' = \psi_2\sigma$, so $E_0 \vDash \psi_2\sigma\rho'$, where $\psi_2\sigma\rho'$ is ground, because $V_{\psi_2\sigma} \subseteq V_{G'\sigma}$ and $\rho' : V_{G'\sigma} \to \mathcal{T}_\Sigma$, hence $E_0 \vDash \psi_1\sigma\rho'$, $E_0 \vDash \phi^\circ\sigma\rho'$, and $E_0 \vDash \phi\sigma\rho'$, all ground expressions.
   ii. $V_{\psi_1\sigma} \subseteq V_{G\sigma}$ and $dom(\rho) = V_{G\sigma}$ implies $\psi_1\sigma\rho \in \mathcal{T}_\Sigma$ so, as $\rho' = \rho \uplus \rho'_1$, $\psi_1\sigma\rho' = \psi_1\sigma(\rho \uplus \rho'_1) = \psi_1\sigma\rho = \psi_1\delta$, hence $E_0 \vDash \psi_1\delta$ (†).

(b) As in subcase (a)-ii, $V_\Delta \subseteq V_G$ implies $\Delta\delta' = \Delta\delta$, and the same closed proof trees are valid for each open goal in $\Delta\delta$ with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$ (††).

(c) Again, $V_{v_1,u_1} \subseteq V_G$ implies that $v_1\delta' = v_1\delta$ and $u_1\delta' = u_1\delta$. Then there is a c.p.t. of the form $\dfrac{\dfrac{F_1}{u_1\delta \to w / ST_1^{\nu'} \varrho'} \quad \dfrac{F_2}{w \to v_1\delta / ST^{\nu'} \varrho_{\nu'}}}{u_1\delta \to v_1\delta / (ST_1;ST)^{\nu'} \varrho'}$, for some term $w \in \mathcal{H}_\Sigma$, with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$.

We prove (a) $ST^{\nu'} \varrho_{\nu'} \rho'' = ST^{\nu'} \varrho'$ and (b) $dom(\rho'') = V_{t,\phi} \setminus V_G$:

(a) As $\varrho_{\nu'} = (\varrho_\mu\delta)_{\setminus V}$, $\delta = \sigma_{V_G}\rho$, $\sigma = \sigma_1\sigma'$, $\delta' = \sigma_1\sigma'_{V_{G'\sigma_1}}\rho'$, $\varrho' = (\varrho_\mu\delta')_{\setminus V}$, and $V_{ST^{\nu'}} \cap V = \emptyset$ this is the same as $ST^{\nu'} \varrho_\mu (\sigma_1\sigma')_{V_G}\rho\rho'' = ST^{\nu'} \varrho_\mu \sigma_1\sigma'_{V_{G'\sigma_1}}\rho'$.
Let $y \in V_{ST^{\nu'} \varrho_\mu}$, so $y \notin V$. There are two options:
   i. $y \in V_G$. Then $V_{y\sigma_1} \subseteq V_{G\sigma_1} \subseteq V_{G'\sigma_1}$, so $y(\sigma_1\sigma')_{V_G} = y\sigma_1\sigma' = y\sigma_1\sigma'_{V_{G'\sigma_1}}$. Also $y(\sigma_1\sigma')_{V_G} = y\sigma$, hence $V_{y(\sigma_1\sigma')_{V_G}} \subseteq V_{G\sigma}$. Then, as $\rho : V_{G\sigma} \to \mathcal{T}_\Sigma$, $y(\sigma_1\sigma')_{V_G}\rho$ is ground, so $y(\sigma_1\sigma')_{V_G}\rho\rho'' = y(\sigma_1\sigma')_{V_G}\rho = y\sigma_1\sigma'_{V_{G'\sigma_1}}\rho = y\sigma_1\sigma'_{V_{G'\sigma_1}}(\rho \cup \rho'_1) = y\sigma_1\sigma'_{V_{G'\sigma_1}}\rho'$;
   ii. $y \notin V_G$, so $y(\sigma_1\sigma')_{V_G} = y$. As $ran(\sigma) \cap V_{ST^\mu \varrho_\mu} = \emptyset$ and $V_{ST^{\nu'} \varrho_\mu} \subseteq V_{ST^\mu \varrho_\mu}$ then $ran(\sigma) \cap V_{ST^{\nu'} \varrho_\mu} = \emptyset$ so $y \notin V_{G\sigma}$ and, as $dom(\rho) = V_{G\sigma}$, $y(\sigma_1\sigma')_{V_G}\rho = y$. Then:
      A. if $y \in V_{t,\phi}$ then $y(\sigma_1\sigma')_{V_G}\rho\rho'' = y\rho'' = y\delta'_{V_{t,\phi}\setminus V_G} = y\delta' = y\sigma_1\sigma'_{V_{G'\sigma_1}}\rho'$, ground term because $V_{y\sigma_1} \subseteq V_{t\sigma_1,\phi\sigma_1} \subseteq V_{G'\sigma_1}$ and $\rho' : V_{G'\sigma_1\sigma'} \to \mathcal{T}_\Sigma$;
      B. if $y \notin V_{t,\phi}$ then $y(\sigma_1\sigma')_{V_G}\rho\rho'' = y\rho'' = y\delta'_{V_{t,\phi}\setminus V_G} = y$. As $dom(\sigma_1) \subseteq (V_{u_1} \cup V_{t^\circ}) \subseteq (V_G \cup V_{t,\phi} \cup \bar{x})$ and $y \notin (V_G \cup V_{t,\phi})$ then $y\sigma_1 = y$ so, as $ran(\sigma_1) \cap V_{ST^{\nu'} \varrho_\mu} = \emptyset$, $y\sigma_1 \notin V_{G\sigma_1}$, and $y\sigma_1\sigma'_{V_{G'\sigma_1}} = y \notin V_{G\sigma_1\sigma'_{V_{G'\sigma_1}}}$ so, as $\rho' : V_{G'\sigma_1\sigma'} \to \mathcal{T}_\Sigma$, $y\sigma_1\sigma'_{V_{G'\sigma_1}}\rho' = y = y(\sigma_1\sigma')_{V_G}\rho\rho''$.

(b) As $dom(\rho'') \subseteq (V_{t,\phi} \setminus V_G)$ and, from (a.ii.A), $y \in (V_{t,\phi} \setminus V_G) \implies V_{y\rho''} = \emptyset$ then $dom(\rho'') = V_{t,\phi} \setminus V_G$, hence $\rho'' : V_{t,\phi} \setminus V_G \to \mathcal{T}_\Sigma$.

In exactly the same way as the proof for (a), $ST_1^{\nu'} \varrho_{\nu'} \rho'' = ST_1^{\nu'} \varrho'$ and $ST_2^{\nu'} \varrho_{\nu'} \rho'' = ST_2^{\nu'} \varrho'$.

Now, we prove (a) $dom(\rho'') = V_{(t_1^{\nu'}\varrho_{\nu'}, \phi_1^{\nu'}\varrho_{\nu'})}$, (b) $E_0 \vDash \phi_1^{\nu'}\varrho_{\nu'}\rho''$, and (c) $u_1\delta =_E t_1^{\nu'}\varrho_{\nu'}\rho''$:

(a) As $dom(\rho'') = V_{t,\phi} \setminus V_G$, $V_{(t_1^{\nu'},\phi_1^{\nu'})} \subseteq V_{(t_1^{\mu},\phi_1^{\mu})}$, $V_{(t_1^{\mu},\phi_1^{\mu})} \setminus V_{(t_1^{\nu'},\phi_1^{\nu'})} \subseteq V^{\mu} \subseteq V_G$, and $dom(\varrho_\mu) \cap V^{\mu} = \emptyset$, then $dom(\rho'') = V_{(t,\phi)} \setminus V_G = V_{(t_1^{\mu}\varrho_\mu, \phi_1^{\mu}\varrho_\mu)} \setminus V_G = V_{(t_1^{\nu'}\varrho_\mu, \phi_1^{\nu'}\varrho_\mu)} \setminus V_G$.
As $\varrho_{\nu'} = (\varrho_\mu\delta)_{\setminus V}$ and $V_{(t_1^{\nu'},\phi_1^{\nu'})} \cap V = \emptyset$, then $V_{(t_1^{\nu'}\varrho_{\nu'}, \phi_1^{\nu'}\varrho_{\nu'})} = V_{(t_1^{\nu'}(\varrho_\mu\delta)_{\setminus V}, \phi_1^{\nu'}(\varrho_\mu\delta)_{\setminus V})} = V_{(t_1^{\nu'}\varrho_\mu\delta, \phi_1^{\nu'}\varrho_\mu\delta)}$, so we prove $V_{(t_1^{\nu'}\varrho_\mu\delta, \phi_1^{\nu'}\varrho_\mu\delta)} = V_{(t_1^{\nu'}\varrho_\mu, \phi_1^{\nu'}\varrho_\mu)} \setminus V_G$, which is trivial, since $\delta : V_G \to \mathcal{T}_\Sigma$.

(b) Immediate, since $E_0 \vDash \phi\sigma\rho'$ and $\phi\sigma\rho' = \phi_1^{\nu'}\varrho_{\nu'}\rho''$.

(c) $u_1\sigma_1 =_B t^\circ\sigma_1$ and $\sigma = \sigma_1\sigma'$ imply $u_1\sigma =_B t^\circ\sigma$ so, as $V_{u_1} \subseteq V_G$, $u_1\sigma_{vars(G)} = u_1\sigma =_B t^\circ\sigma$. As $\rho : V_{G\sigma} \to \mathcal{T}_\Sigma$, so $u_1\sigma_{V_G}\rho$ is a ground term, and $\rho' = \rho \uplus \rho'_1$ then $u_1\delta = u_1\sigma_{V_G}\rho = u_1\sigma_{V_G}\rho' =_B t^\circ\sigma\rho' = t[\bar{x}]_{\bar{q}}\sigma\rho' = t\sigma\rho'[\bar{x}\sigma\rho']_{\bar{q}}$.
As $E_0 \vDash \phi^\circ\sigma\rho'$ then $t\sigma\rho'[\bar{x}\sigma\rho']_{\bar{q}} =_{E_0} t\sigma\rho'[t|_{q_1}\sigma\rho', \ldots, t|_{q_l}\sigma\rho']_{\bar{q}} = t\sigma\rho'[t\sigma\rho'|_{\bar{q}}]_{\bar{q}} = t\sigma\rho' = t_1^{\nu'}\varrho_{\nu'}\rho''$, because $t\sigma\rho' = t_1^{\nu'}\varrho_{\nu'}\rho''$, so $u_1\delta =_B t^\circ\sigma\rho' =_{E_0} t_1^{\nu'}\varrho_{\nu'}\rho''$, i.e., $u_1\delta =_E t_1^{\nu'}\varrho_{\nu'}\rho''$.

Then, as $\rho'' : V_{(t_1^{\nu'}\varrho_{\nu'}, \phi_1^{\nu'}\varrho_{\nu'})} \to \mathcal{T}_\Sigma$, $E_0 \vDash \phi_1^{\nu'}\varrho_{\nu'}\rho''$, $u_1\delta =_E t_1^{\nu'}\varrho_{\nu'}\rho''$, and $ST_1^{\nu'}\varrho_{\nu'}\rho'' = ST_1^{\nu'}\varrho'$, there is a derivation rule $\frac{u_1\delta \to w/ST_1^{\nu'}\varrho'}{u_1\delta \to w/\texttt{match}\, t_1^{\nu'}\varrho_{\nu'}\ \texttt{s.t.}\ \phi_1^{\nu'}\varrho_{\nu'}\ ?\ ST_1^{\nu'}\varrho_{\nu'} : ST_2^{\nu'}\varrho_{\nu'}} \in \mathcal{D}^{\nu'}_{\mathcal{R},Call_{\mathcal{R}}}$.
Now,

$$\frac{\dfrac{\dfrac{F_1}{u_1\delta \to w/ST_1^{\nu'}\varrho'}}{u_1\delta \to w/\texttt{match}\, t_1^{\nu'}\varrho_{\nu'}\ \texttt{s.t.}\ \phi_1^{\nu'}\varrho_{\nu'}\ ?\ ST_1^{\nu'}\varrho_{\nu'} : ST_2^{\nu'}\varrho_{\nu'}} \quad \dfrac{F_2}{w \to v_1\delta/ST^{\nu'}\varrho_{\nu'}}}{u_1\delta \to v_1\delta/(\texttt{match}\, t_1^{\nu'}\varrho_{\nu'}\ \texttt{s.t.}\ \phi_1^{\nu'}\varrho_{\nu'}\ ?\ ST_1^{\nu'}\varrho_{\nu'} : ST_2^{\nu'}\varrho_{\nu'}); ST^{\nu'}\varrho_{\nu'}}$$

is a c.p.t., $\rho : vars(G\sigma) \to \mathcal{T}_\Sigma$, $\psi\rho$ is satisfiable, $E_0 \vDash \psi_1\delta$ (†), and there are closed proof trees for each open goal in $\Delta\delta$ with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_{\mathcal{R}}}$ (††), hence $\sigma_{vars(G)}\rho$ is a solution of G.

9. Rule $[i2]$ (**if then else**):
$G = u_1 \to v_1 / (\texttt{match}\, t_1\ \texttt{s.t.}\ \phi_1\ ?\ ST_1 : ST_2\, ; ST)^{\mu}\varrho_\mu\, (\wedge\, \Delta) \mid \psi_1 \mid V, \mu \rightsquigarrow_{[i1],\sigma_1} (u_1 \to v_1/(ST_2\, ; ST)^{\mu}\varrho_\mu\, (\wedge\, \Delta) \mid \psi_2 \mid V, \mu)\sigma_1 = G'\sigma_1$, call $t = t_1^{\mu}\varrho_\mu$ and $\phi = \phi_1^{\mu}\varrho_\mu$, where $abstract_{\Sigma_1}(t) = \langle \lambda\bar{x}.t^\circ; \sigma^\circ; \phi^\circ \rangle$, $t^\circ = t[\bar{x}]_{\bar{q}}$, with $\bar{x} = x_1, \ldots, x_l$ and $\bar{q} = q_1, \ldots, q_l$, $\phi^\circ = (\bigwedge_{i=1}^{l} x_i = t|_{q_i})$, hence $V_{t^\circ} \cup V_{\phi^\circ} = V_t \cup \hat{x}$, $\sigma_1 \in CSU_B(u_1 = t^\circ)$, $\psi_2 = \psi_1 \wedge \neg\phi \wedge \phi^\circ$, so $V_G \subseteq V_{G'}$, $\psi_2\sigma_1$ is satisfiable, and $G'\sigma_1 \rightsquigarrow^{+}_{\sigma'} nil \mid \psi \mid V, \nu$.
The proof is the same as the one for rule $[i1]$, just replacing ϕ with $\neg\phi$, and exchanging ST_1 and ST_2 everywhere except in the $\texttt{match}$ strategy at the beginning "$\texttt{match}\, t_1\ \texttt{s.t.}\ \phi_1\ ?\ ST_1 : ST_2\, ; ST$".

10. Rule $[t]$ (**transitivity**):
$G = u_1 \to v_1/(RA; ST)^{\mu}\varrho_\mu(\wedge\Delta) \mid \psi_1 \mid V, \mu \rightsquigarrow_{[t]} u_1 \to^1 x_k, x_k \to v_1/(RA; ST)^{\mu}\varrho_\mu(\wedge\Delta) \mid \psi_1 \mid V, \mu = G'$, so $V_G \subseteq V_G \cup \{x_k\} = V_{G'}$, and $G' \rightsquigarrow^{+}_{\sigma} nil \mid \psi \mid V, \nu$, where $\nu = (\mu\sigma)_V$, hence $\sigma_{V_G}|\psi$ is a computed answer for G and $\sigma_{V_{G'}}|\psi$ is a computed answer for G'. Let $\rho : V_{G\sigma} \to \mathcal{T}_\Sigma$ such that $\psi\rho$ is satisfiable, call $\delta = \sigma_{V_G}\rho$, $\nu' = (\mu\delta)_V$, and $\varrho_{\nu'} = (\varrho_\mu\delta)_{\setminus V}$, where $dom(\nu') = V$ and $ran(\nu') = \emptyset$, let $\varrho : V_{G'\sigma} \setminus V_{G\sigma} \to \mathcal{T}_\Sigma$, such that $\psi(\rho \uplus \varrho)$ is satisfiable, let $\rho' = \rho \uplus \varrho$, and call $\delta' = \sigma_{V_{G'}}\rho'$. As $V_G \subseteq V_{G'}$ then $\rho' : V_{G'\sigma} \to \mathcal{T}_\Sigma$ and $G\delta' = G\delta$.
By I.H., as $\rho' : V_{G'\sigma} \to \mathcal{T}_\Sigma$ and $\psi\rho'$ is satisfiable, δ' is a solution for G', so $[x_k\delta']_E \in RA^{\mu}\varrho_\mu\delta'\ @\ [u_1\delta']_E$ and $[v_1\delta']_E \in ST^{\mu}\varrho_\mu\delta' @ [x_k\delta']_E$. This is equivalent, since $G\delta' =$

$G\delta$, to $[x_k\delta']_E \in RA^{\mu}\varrho_\mu\delta \;@\; [u_1\delta]_E$ and $[v_1\delta]_E \in ST^{\mu}\varrho_\mu\delta @[x_k\delta']_E$, i.e., $[x_k\delta']_E \in RA^{\nu'}\varrho_{\nu'} \;@\; [u_1\delta]_E$ and $[v_1\delta]_E \in ST^{\nu'}\varrho_{\nu'}@[x_k\delta']_E$, so there are closed proof trees of the forms $\frac{F_1}{u_1\delta \to x_k\delta'/RA^{\nu'}\varrho_{\nu'}}$ and $\frac{F_2}{x_k\delta' \to v_1\delta/ST^{\nu'}\varrho_{\nu'}}$ with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_{\mathcal{R}}}$. As there is a rule $\frac{u_1\delta \to x_k\delta'/RA^{\nu'}\varrho_{\nu'} \quad x_k\delta' \to v_1\delta/ST^{\nu'}\varrho_{\nu'}}{u_1\delta \to v_1\delta/(RA\,;\,ST)^{\nu'}\varrho_{\nu'}} \in \mathcal{D}^{\nu'}_{\mathcal{R},Call_{\mathcal{R}}}$, then

$$\frac{\frac{F_1}{u_1\delta \to x_k\delta'/RA^{\nu'}\varrho_{\nu'}} \quad \frac{F_2}{x_k\delta' \to v_1\delta/ST^{\nu'}\varrho_{\nu'}}}{u_1\delta \to v_1\delta/(RA\,;\,ST)^{\nu'}\varrho_{\nu'}}$$

is a c.p.t. with $\rho : vars(G\sigma) \to \mathcal{T}_\Sigma$ and $\psi\rho$ satisfiable, so $\sigma_{vars(G)}\rho$ is a solution of G.

11. Rule $[c]$ (congruence):
$G = u_1|_p \to^1 x_k, u_1[x_k]_p \to v_1/(RA\,;\,ST)^{\mu}\varrho_\mu\ (\wedge\ \Delta) \mid \psi_1 \mid V, \mu \leadsto_{[t],\sigma_1} u'_i \to^1 y_{k'}, u_1[y_{k'}]_{p.i} \to v_1/(RA\,;\,ST)^{\mu}\varrho_\mu\ (\wedge\ \Delta) \mid \psi_1 \mid V, \mu = G'$, where $u_1|_p = f(u'_1, \ldots, u'_m)$, $u'_i \in \mathcal{H}_\Sigma(\mathcal{X}) \setminus \mathcal{X}$, $y_{k'}$ fresh variable, and $\sigma_1 = \{x_k \mapsto u_1|_p[y_{k'}]_i\}$, so $(\mu\sigma_1)_V = \mu$ and $V_{G\sigma_1} = V_{G'}$, and $G' \leadsto^+_{\sigma'} nil \mid \psi \mid V, \nu$, call $\sigma = \sigma_1\sigma'$, where $\nu = (\mu\sigma')_V = (\mu\sigma)_V$, hence $\sigma_{V_G}|\psi$ is a computed answer for G and $\sigma'_{V_{G'}}|\psi$ is a computed answer for G'.
Let $\rho : V_{G\sigma} \to \mathcal{T}_\Sigma$ such that $\psi\rho$ is satisfiable, call $\delta = \sigma_{V_G}\rho$, $\nu' = (\mu\delta)_V$, and $\varrho_{\nu'} = (\varrho_\mu\delta)_{\setminus V}$, where $dom(\nu') = V$ and $ran(\nu') = \emptyset$. As $V_{G\sigma_1} = V_{G'}$ then $V_{G\sigma} = V_{G\sigma_1\sigma'} = V_{G'\sigma'}$, so also $\rho : V_{G'\sigma'} \to \mathcal{T}_\Sigma$, call $\delta' = \sigma'_{V_{G'}}\rho$.
For every variable $z \in V_G \cap V_{G'}$, as $dom(\sigma_1) = \{x_k\}$ and $x_k \notin V_{G'}$, $z\delta = z\sigma_{V_G}\rho = z\sigma\rho = z\sigma_1\sigma'\rho = z\sigma'\rho = z\sigma'_{V_{G'}}\rho = z\delta'$. As $vars(u_1|_p) \subseteq V_G$ and $vars(u_1[y_{k'}]_{p.i}) \subseteq V_{G'}$ then $vars(u_1|_p[]_i) \subseteq V_G \cap V_{G'}$ so $u_1|_p\delta[]_i = u_1|_p\delta'[]_i$.
By I.H., as $\rho : V_{G'\sigma'} \to \mathcal{T}_\Sigma$ and $\psi\rho$ is satisfiable, $\sigma'_{V_{G'}}\rho$ is a solution for G', so $[y'_k\delta']_E \in RA^{\mu}\varrho_\mu\delta' \;@\; [u'_i\delta']_E$ and $[v_1\delta']_E \in ST^{\mu}\varrho_\mu\delta'@[u_1[y'_k]_{p.i}\delta']_E$. As $V_{G'} = \{y_{k'}\} \cup V_G \setminus \{x_k\}$ and $V_{RA^{\mu}\varrho_\mu} \cap \{x_k, y_{k'}\} = \emptyset$, so $RA^{\mu}\varrho_\mu(\sigma_1\sigma')_{V_G} = RA^{\mu}\varrho_\mu\sigma'_{V_G}$, then $RA^{\mu}\varrho_\mu\delta' = RA^{\mu}\varrho_\mu\sigma'_{V_{G'}}\rho = RA^{\mu}\varrho_\mu\sigma'_{V_G}\rho = RA^{\mu}\varrho_\mu(\sigma_1\sigma')_{V_G}\rho = RA^{\mu}\varrho_\mu\sigma_{V_G}\rho = RA^{\mu}\varrho_\mu\delta = RA^{\nu'}\varrho_{\nu'}$.
In the same way, $ST^{\mu}\varrho_\mu\delta' = ST^{\nu'}\varrho_{\nu'}$. Then, $[y'_k\delta']_E \in RA^{\nu'}\varrho_{\nu'} \;@\; [u'_i\delta']_E$, $[v_1\delta']_E \in ST^{\nu'}\varrho_{\nu'}@[u_1[y'_k]_{p.i}\delta']_E$, and there are closed proof trees of the forms (1) $\frac{}{u'_i\delta' \to y'_k\delta'/RA^{\nu'}\varrho_{\nu'}}$ or (2) $\frac{F_1}{u'_i\delta' \to y'_k\delta'/RA^{\nu'}\varrho_{\nu'}}$, and (3) $\frac{F_2}{u_1[y'_k]_{p.i}\delta' \to v_1\delta'/ST^{\nu'}\varrho_{\nu'}}$ with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_{\mathcal{R}}}$.

- Case (1): $RA^{\nu'} = c^{\nu'}[\gamma]$, so $RA^{\nu'}\varrho_{\nu'} = c^{\nu'}[\gamma(\varrho_{\nu'})_{ran(\gamma)}]$, $c^{\nu'} : l \to r\ if\ \phi$ and there exist a substitution η, a position q, and terms $t, t' \in \mathcal{H}_\Sigma$ such that $E_0 \vDash \phi\gamma(\varrho_{\nu'})_{ran(\gamma)}\eta$, $t \xrightarrow[c^{\nu'}\gamma(\varrho_{\nu'})_{ran(\gamma)}, q, \eta]{}{}^{1}_{R} t'$, so $\frac{}{t \to t'/RA^{\nu'}\varrho_{\nu'}}$ is a derivation rule in $\mathcal{D}^{\nu'}_{\mathcal{R},Call_{\mathcal{R}}}$, $u'_i\delta' =_E t$, and $t' =_E y'_k\delta'$. By definition of $\to^1_R$, also $u_1\delta|_p[t]_i \xrightarrow[c^{\nu'}\gamma(\varrho_{\nu'})_{ran(\gamma)}, i.q, \eta]{}{}^{1}_{R} u_1\delta|_p[t']_i$ so there is a derivation rule $\frac{}{u_1\delta|_p[t]_i \to u_1\delta|_p[t']_i/RA^{\nu'}\varrho_{\nu'}} \in \mathcal{D}^{\nu'}_{\mathcal{R},Call_{\mathcal{R}}}$.
  Now, $u'_i\delta = u'_i\delta' =_E t$, so $u_1\delta|_p[t]_i =_E u_1\delta|_p[u'_i\delta]_i = u_1|_p[u'_i]_i\delta = u_1|_p\delta$, and $x_k \in V_G$, so $x_k\delta = x_k\sigma_{vars(G)}\rho = x_k\sigma\rho = x_k\sigma_1\sigma'\rho = u_1|_p[y_{k'}]_i\sigma'\rho = u_1\sigma'|_p[y_{k'}\sigma']_i\rho = u_1\sigma|_p[y_{k'}\sigma']_i\rho = u_1\sigma\rho|_p[y_{k'}\sigma'\rho]_i = u_1\delta|_p[y_{k'}\delta']_i =_E u_1\delta|_p[t']_i$.
  If we apply the previous derivation rule, with $u_1\delta|_p[t]_i =_E u_1|_p\delta$ and $x_k\delta =_E u_1\delta|_p[t']_i$, then we get the c.p.t. $\frac{}{u_1|_p\delta \to x_k\delta/RA^{\nu'}\varrho_{\nu'}}$, so $[x_k\delta]_E \in RA^{\nu'}\varrho_{\nu'}@[u_1|_p\delta]_E$.
- Case (2): $RA^{\nu'} = c^{\nu'}[\gamma]\{ST^{\nu'}_1, \ldots, ST^{\nu'}_m\}$, $RA^{\nu'}\varrho_{\nu'} = c^{\nu'}[\gamma(\varrho_{\nu'})_{ran(\gamma)}]\{\overline{ST}^{\nu'}\varrho_{\nu'}\}$, $c^{\nu'}\gamma(\varrho_{\nu'})_{ran(\gamma)}$ has the form $l \to r\,if \bigwedge_{j=1}^{m} l_j \to r_j \mid \phi$ and there exist a substitution η, a term $t \in \mathcal{H}_\Sigma$, and a position $q \in pos(t)$ such that $t|_q = l\eta$ and $E_0 \vDash \phi\eta$, so there is a derivation rule $\frac{l_1\eta \to r_1\eta/ST^{\nu'}_1\varrho_{\nu'}\eta \cdots l_m\eta \to r_m\eta/ST^{\nu'}_m\varrho_{\nu'}\eta}{t \to t[r\eta]_q/RA^{\nu'}\varrho_{\nu'}} \in \mathcal{D}^{\nu'}_{\mathcal{R},Call_{\mathcal{R}}}$, T_j is a

c.p.t. with root $l_j\eta \to r_j\eta/ST_j^{\nu'}\varrho_{\nu'}\eta$ with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_{\mathcal{R}}}$, for $1 \leq j \leq m$, $u'_i\delta' =_E t$, and $t[r\eta]_q =_E y'_k\delta'$.
We take $w = u_1|_p\delta[t]_i$ and the position $i.q$. Then, as $E_0 \vDash \phi\eta$ and $w|_{i.q} = t|_q = l\eta$, there is also a derivation rule $\frac{l_1\eta \to r_1\eta/ST_1^{\nu'}\varrho_{\nu'}\eta \cdots l_m\eta \to r_m\eta/ST_m^{\nu'}\varrho_{\nu'}\eta}{w \to w[r\eta]_{i.q}/RA^{\nu'}\varrho_{\nu'}} \in \mathcal{D}^{\nu'}_{\mathcal{R},Call_{\mathcal{R}}}$.
We have $u'_i\delta' =_E t$ and $t[r\eta]_q =_E y'_k\delta'$. From the previous subcase we also know that $u'_i\delta = u'_i\delta'$ and $x_k\delta = u_1|_p\delta[y_{k'}\delta']_i$. Then $w = u_1|_p\delta[t]_i =_E u_1|_p\delta[u'_i\delta']_i = u_1|_p\delta[u'_i\delta]_i = u_1|_p\delta$ and $w[r\eta]_{i.q} = (u_1|_p\delta[t]_i)[r\eta]_{i.q} = u_1|_p\delta[t[r\eta]_q]_i =_E u_1|_p\delta[y'_k\delta']_i$. As $\sigma_1 = \{x_k \mapsto u_1|_p[y_{k'}]_i\}$, $x_k \in V_G$, $vars(u_1[y_{k'}]_{p.i}) \subseteq V_{G'}$, so $vars(u_1|_p[y_{k'}]_i) \subseteq V_{G'}$, and $u_1|_p\delta[]_i = u_1|_p\delta'[]_i$, then $x_k\delta = x_k\sigma_{V_G}\rho = x_k(\sigma_1\sigma')_{V_G}\rho = u_1|_p[y_{k'}]_i\sigma'\rho = u_1|_p[y_{k'}]_i\sigma'_{V_{G'}}\rho = u_1|_p[y_{k'}]_i\delta' = u_1|_p\delta'[y_{k'}\delta']_i = u_1|_p\delta[y_{k'}\delta']_i$ then $w[r\eta]_{i.q} =_E x_k\delta$ so, as $w =_E u_1|_p\delta$, we can apply the derivation rule with $u_1|_p\delta$ and $x_k\delta$ and complete a c.p.t. with $T_1, \ldots, T_m$, yielding $\frac{F_1}{u_1|_p\delta \to x_k\delta/RA^{\nu'}\varrho_{\nu'}}$, hence $[x_k\delta]_E \in RA^{\nu'}\varrho_{\nu'}@[u_1|_p\delta]_E$.

As $V_{(v_1,u_1)} \subseteq V_G \cap V_{G'}$ then $v_1\delta' = v_1\delta$ and $v_1\delta' = v_1\delta$, so $u_1[x_k]_p\delta = u_1\delta[x_k\delta]_p = u_1\delta[u_1|_p\delta[y_{k'}\delta']_i]_p = u_1\delta[y_{k'}\delta']_{p.i} = u_1\delta'[y_{k'}\delta']_{p.i}$, and the c.p.t. (3) can also be written as $\frac{F_2}{u_1[x_k]_p\delta \to v_1\delta/ST^{\nu'}\varrho_{\nu'}}$, hence $[v_1\delta]_E \in ST^{\nu'}\varrho_{\nu'}@[u_1[x_k]_p\delta]_E$. As also $[x_k\delta]_E \in RA^{\nu'}\varrho_{\nu'}@[u_1|_p\delta]_E$, either for case (1) or (2), and $\psi\rho$ is satisfiable then $\sigma_{vars(G)}\rho$ is a solution of G.

12. Rule $[r]$ (rule application):
We prove this case for conditional rules. For rules without rewrite conditions, the proof is the same just with the part dealing with the conditions removed from it.
$G = u|_p \to^1 x_k, u[x_k]_p \to v/(c[\gamma_r]\{ST_1, \ldots, ST_m\}\ ;\ ST)^\mu\varrho_\mu\ (\wedge\ \Delta) \mid \psi_1 \mid V, \mu \rightsquigarrow_{[r],\sigma_1}$ $(\bigwedge_{i=1}^n(l_i\gamma \to r_i\gamma/ST_i^\mu\varrho_\mu; \texttt{idle}) \wedge u[r\gamma]_p \to v/ST^\mu\varrho_\mu(\wedge\Delta) \mid \psi_2)\sigma_1 \mid V, (\mu\sigma_1)_V = G'\sigma_1$, where:
   - $\gamma = (\gamma_r^\mu\varrho_\mu)_{dom(\gamma_r^\mu)}$ (so $ran(\gamma) \subseteq V_G$), $c \in R$, $c_0 \in c_B \subseteq R_B$ has the form $c : l^c \to r^c\ if\ \bigwedge_{i=1}^n(l_i^c \to r_i^c) \mid \phi^c$, $c_{\gamma'} : l \to r\ if\ \bigwedge_{i=1}^n(l_i \to r_i) \mid \phi$ is a fresh version with some renaming γ' of $c_0^\mu \in R_B^\mu$, with $dom(\gamma') = vars(c_0^\mu) \setminus (dom(\gamma_r) \uplus V^\mu)$, so $c_{\gamma'} = c_0^\mu\gamma'$, call $l' = l\gamma$;
   - $abstract_{\Sigma_1}(u|_p) = \langle\lambda\bar{u}.u^\circ; \sigma_u^\circ; \phi_u^\circ\rangle$, $u^\circ = u|_p[\bar{x}]_{\bar{p}}$, with $\bar{x} = x_1, \ldots, x_u$ and $\bar{p} = p_1, \ldots, p_u$, $\phi_u^\circ = (\bigwedge_{j=1}^u x_j = u|_{p.p_j})$;
   - $abstract_{\Sigma_1}(l') = \langle\lambda\bar{y}.l^\circ; \sigma^\circ; \phi^\circ\rangle$, $l^\circ = l'[\bar{y}]_{\bar{q}}$, with $\bar{y} = y_1, \ldots, y_l$ and $\bar{q} = q_1, \ldots, q_l$, $\phi^\circ = (\bigwedge_{i=1}^l y_i = l'|_{q_i})$;
   - $\sigma'_1 \in CSU_B(u^\circ = l^\circ)$, $\sigma_1 = \sigma'_1 \cup \{x_k \mapsto r\gamma\sigma_1\sigma'_1\}$, $\psi_2 = \psi_1 \wedge \phi^\circ \wedge \phi_u^\circ \wedge \phi\gamma$, $\psi_2\sigma_1$ is satisfiable;

Then $G'\sigma_1 \rightsquigarrow^+_{\sigma'} nil \mid \psi \mid V, \nu$, call $\sigma = \sigma_1\sigma'$, where $\nu = (\mu\sigma)_V = (\mu\sigma_1\sigma')_V = (\mu\sigma'_1\sigma')_V$, so $\sigma_{V_G} \mid \psi$ is a computed answer for G and $\sigma'_{V_{G'\sigma_1}} \mid \psi$ is a computed answer for $G'\sigma_1$.
As $\gamma = (\gamma_r^\mu\varrho_\mu)_{dom(\gamma_r^\mu)}$ then $dom(\gamma_r) = dom(\gamma_r^\mu) = dom(\gamma)$.
Let $\rho : V_{G\sigma} \to \mathcal{T}_\Sigma$ be a substitution such that $\psi\rho$ is satisfiable, call $\delta = \sigma_{V_G}\rho$ and $\varrho_{\nu'} = (\varrho_\mu\delta)_{\setminus V}$, so $\delta : V_G \to \mathcal{T}_\Sigma$, $\rho_1 = \rho_{V_{G'\sigma}}$, so also $\psi\rho_1$ is satisfiable, and call $\nu' = (\nu\rho)_V$, where $dom(\nu') = V$ and $ran(\nu') = \emptyset$. As $dom(\rho) = V_{G\sigma}$ then $dom(\rho_1) = V_{G\sigma} \cap V_{G'\sigma}$. Let $\rho_2 = \rho_{V_{G\sigma} \setminus V_{G'\sigma}}$, so $\rho = \rho_1 \uplus \rho_2$, and let $\rho'_1 : V_{G'\sigma} \setminus V_{G\sigma} \to \mathcal{T}_\Sigma$, so $dom(\rho_1) \cap dom(\rho'_1) = \emptyset$ and $dom(\rho_1) \cup dom(\rho'_1) = V_{G'\sigma}$, such that $\psi(\rho_1 \uplus \rho'_1)$ is satisfiable, and call $\rho' = \rho_1 \uplus \rho'_1$, so $\rho' : V_{G'\sigma} \to \mathcal{T}_\Sigma$.

We prove several intermediate results:

- As $\nu = (\mu\sigma)_V$, $V^\mu \subseteq V_G \cap V_{G'}$, and $dom(\rho_1) = V_{G\sigma} \cap V_{G'\sigma}$ then $V^\nu \subseteq dom(\rho_1)$ so, as $dom(\nu') = V$ and $ran(\nu') = \emptyset$, $\nu' = (\nu\rho)_V = (\nu\rho_1)_V = (\nu\rho')_V$. Also, as $dom(\mu) \subseteq V$, $\nu' = (\nu\rho)_V = ((\mu\sigma)_V\rho)_V = (\mu\sigma\rho)_V = \mu(\sigma\rho)_{V^\mu}$.
- As $dom(\rho'_1) = V_{G'\sigma} \setminus V_{G\sigma}$ and $dom(\rho_1) = V_{G\sigma} \cap V_{G'\sigma} \subseteq V_{G\sigma}$, then $\rho'_{V_{G\sigma}} = (\rho_1 \uplus \rho'_1)_{V_{G\sigma}} = (\rho_1)_{V_{G\sigma}} = \rho_1$.
- As $\delta_{V^\mu} = (\sigma_{V_G}\rho)_{V^\mu}$, $\rho : V_{G\sigma} \to \mathcal{T}_\Sigma$, and $V^\mu \subseteq V_G$, then $\delta_{V^\mu} = (\sigma\rho)_{V^\mu}$, $ran(\delta_{V^\mu}) = \emptyset$, and $dom(\delta_{V^\mu}) = V^\mu$ $(= (V \setminus dom(\mu)) \cup ran(\mu)$, so $ran(\mu) \subseteq dom(\delta_{V^\mu}))$. Then $\nu' = (\mu\sigma_{V_G}\rho)_V = \mu\delta_{V^\mu}$ and $c_0^{\nu'} = c_0\nu' = c_0\mu\delta_{V^\mu}$.

As $\sigma'_{V_{G'\sigma_1}} \mid \psi$ is a computed answer for $G'\sigma_1$, $\rho' : V_{G'\sigma_1\sigma'} \to \mathcal{T}_\Sigma$, and $\psi\rho'$ is satisfiable then, by I.H., $\sigma'_{V_{G'\sigma_1}}\rho'$ is a solution for $G'\sigma_1$, call $\delta' = \sigma_1\sigma'_{V_{G'\sigma_1}}\rho'$ and $\varrho' = (\varrho_\mu\delta')_{\setminus V}$, meaning that:

(a) $E_0 \vDash \psi_2\delta'$,

(b) there are closed proof trees for each open goal in $\Delta\delta'$, with respect to $\mathcal{D}^{(\nu\rho')_V}_{\mathcal{R},Call_\mathcal{R}}$ ($=\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$, we use ν' instead of $(\nu\rho')_V$ in (c) and (d)),

(c) $[v\delta']_E \in ST^{\nu'}\varrho'@[u[r\gamma]_p\delta']_E$, i.e., $[v\delta']_E \in ST^{\nu'}\varrho'@[u\delta'[r\gamma\delta']_p]_E$, and

(d) $[r_i\gamma\delta']_E \in ST_i^{\nu'}\varrho'@[l_i\gamma\delta']_E$, for $1 \le i \le n$.

Then:

(a) i. $V_{\psi_2} \subseteq V_{G'}$ implies $V_{\psi_2\sigma_1} \subseteq V_{G'\sigma_1}$ and $V_{\psi_2\sigma} \subseteq V_{G'\sigma}$, so $\psi_2\delta' = \psi_2\sigma_1\sigma'_{V_{G'\sigma_1}}\rho' = \psi_2\sigma_1\sigma'\rho' = \psi_2\sigma\rho'$, hence $E_0 \vDash \psi_2\sigma\rho'$, where $\psi_2\sigma\rho'$ is ground, because $V_{\psi_2\sigma} \subseteq V_{G'\sigma}$ and $\rho' : V_{G'\sigma} \to \mathcal{T}_\Sigma$. As $\psi_2 = \psi_1 \wedge \phi^\circ \wedge \phi_u^\circ \wedge \phi\gamma$, then $\psi_1\delta' = \psi_1\sigma\rho'$, $E_0 \vDash \psi_1\sigma\rho'$, $E_0 \vDash \phi^\circ\sigma\rho'$, $E_0 \vDash \phi_u^\circ\sigma\rho'$, and $E_0 \vDash \phi\gamma\sigma\rho'$, all ground formulas.

ii. Also as $\psi_2 = \psi_1 \wedge \phi^\circ \wedge \phi\gamma$, so $V_{\psi_1} \subseteq V_G \cap V_{G'}$ hence $V_{\psi_1\sigma} \subseteq V_{G\sigma} \cap V_{G'\sigma}$, and $dom(\rho_1) = V_{G\sigma} \cap V_{G'\sigma}$ imply $\psi_1\sigma\rho_1 \in \mathcal{T}_\Sigma$. Then, as $\rho' = \rho_1 \uplus \rho'_1$, we have $\psi_1\sigma\rho' = \psi_1\sigma(\rho_1 \uplus \rho'_1) = \psi_1\sigma\rho_1 = \psi_1\sigma(\rho_1 \uplus \rho_2) = \psi_1\sigma\rho = \psi_1\delta$, so $E_0 \vDash \psi_1\delta$ (1).

iii. As $\psi_1\delta' = \psi_1\sigma\rho'$ and $\psi_1\sigma\rho' = \psi_1\delta$ then $\psi_1\delta' = \psi_1\delta$.

(b) As in subcases (a)-ii and (a)-iii, $V_\Delta \subseteq V_G \cap V_{G'}$ implies $\Delta\delta' = \Delta\delta$, and the same closed proof trees are valid for each open goal in $\Delta\delta$ with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$ (2).

(c) i. Again, $V_{v,u[\,]_p} \subseteq V_G \cap V_{G'}$ implies that $v\delta' = v\delta$ and $u\delta'[\,]_p = u\delta[\,]_p$.

ii. We prove that $ST^{\nu'}\varrho' = ST^{\nu'}\varrho_{\nu'}$.
As $\varrho_{\nu'} = (\varrho_\mu\delta)_{\setminus V}$, $\delta = \sigma_{V_G}\rho$, $\sigma = \sigma_1\sigma'$, $\delta' = \sigma_1\sigma'_{V_{G'\sigma_1}}\rho'$, $\varrho' = (\varrho_\mu\delta')_{\setminus V}$, and $V_{ST^{\nu'}} \cap V = \emptyset$, this is the same as $ST^{\nu'}\varrho_\mu(\sigma_1\sigma')_{V_G}\rho = ST^{\nu'}\varrho_\mu\sigma_1\sigma'_{V_{G'\sigma_1}}\rho'$.
Let $x \in V_{ST^{\nu'}\varrho_\mu}$. As $V_{ST^{\nu'}\varrho_\mu} \subseteq V_{ST^\mu\varrho_\mu} \subseteq V_G \cap V_{G'}$, then $x \in V_G \cap V_{G'}$ and $V_{x\sigma_1} \subseteq V_{G\sigma_1} \cap V_{G'\sigma_1} \subseteq V_{G'\sigma_1}$, so $x(\sigma_1\sigma')_{V_G} = x\sigma_1\sigma' = x\sigma_1\sigma'_{V_{G'\sigma_1}}$. Also $x(\sigma_1\sigma')_{V_G} = x\sigma$, hence $V_{x(\sigma_1\sigma')_{V_G}} \subseteq V_{G\sigma}$. Then, as $\rho : V_{G\sigma} \to \mathcal{T}_\Sigma$, $x(\sigma_1\sigma')_{V_G}\rho$ is ground, so $x(\sigma_1\sigma')_{V_G}\rho = x\sigma_1\sigma'_{V_{G'\sigma_1}}\rho = x\sigma_1\sigma'_{V_{G'\sigma_1}}(\rho \cup \rho'_1) = x\sigma_1\sigma'_{V_{G'\sigma_1}}\rho'$.

iii. As in subcase (a)-i, $V_{r\gamma} \subseteq V_{G'}$ implies $r\gamma\delta' = r\gamma\sigma\rho'$.
As $x_k\sigma_1 = r\gamma\sigma_1$ and $\sigma = \sigma_1\sigma'$ then $x_k\sigma = r\gamma\sigma$ so, as $x_k \in V_G$, $r\gamma\sigma = x_k\sigma_{V_G}$ and $r\gamma\sigma\rho' = x_k\sigma_{V_G}\rho'$, ground terms. But, as $V_{x_k\sigma_{V_G}} \subseteq V_{G\sigma}$ then $x_k\sigma_{V_G}\rho' = x_k\sigma_{V_G}(\rho_1 \uplus \rho'_1) = x_k\sigma_{V_G}\rho_1 = x_k\sigma_{V_G}(\rho_1 \uplus \rho_2) = x_k\sigma_{V_G}\rho = x_k\delta$, so $r\gamma\sigma\rho' = x_k\delta$ (3).

From (i)-(iii), $[v\delta]_E \in ST^{\nu'}\varrho_{\nu'}@[u\delta[x_k\delta]_p]_E$, i.e, $[v\delta]_E \in ST^{\nu'}\varrho_{\nu'}@[u[x_k]_p\delta]_E$ (4), holds.

(d) Using the same proof as in the previous case, $[r_i\gamma\delta']_E \in ST_i^{\nu'}\varrho'@[l_i\gamma\delta']_E$, $V_{l_i\gamma,r_i\gamma} \subseteq V_{G'}$, and $V_{ST_i^{\nu'}\varrho'} \subseteq V_G \cap V_{G'}$ imply $[r_i\gamma\sigma\rho']_E \in ST_i^{\nu'}\varrho_{\nu'}@[l_i\gamma\sigma\rho']_E$, for $1 \le i \le n$, where each term and strategy are ground (5).

Now:

(a) $V_{u|_p} \subseteq V_G$ imply $u|_p\sigma_{V_G} = u|_p\sigma$, hence $u|_p\sigma_{V_G}\theta = u|_p\sigma\theta$, and $u^\circ\sigma'_1 =_B l^\circ\sigma'_1$ imply $u^\circ\sigma\theta =_B l^\circ\sigma\theta$.

(b) As $E_0 \models \phi^\circ_u\sigma\theta$, ground formula, then $u^\circ\sigma\theta = u|_p[\bar{x}]_{\bar{p}}\sigma\theta = u|_p\sigma\theta[\bar{x}\sigma\theta]_{\bar{p}} =_{E_0} u|_p\sigma\theta[u|_{p.\bar{p}}\sigma\theta]_{\bar{p}} = u|_p\sigma\theta$, all ground terms.

(c) As $E_0 \models \phi^\circ\sigma\theta$, ground formula, then $l^\circ\sigma\theta = l'[\bar{y}]_{\bar{q}}\sigma\theta = l'\sigma\theta[\bar{y}\sigma\theta]_{\bar{q}} =_{E_0} l'\sigma\theta[l'|_{\bar{q}}\sigma\theta]_{\bar{q}} = l'\sigma\theta$, all ground terms (6).

(d) As $\rho : V_{G\sigma} \to \mathcal{T}_\Sigma$, so $u|_p\sigma_{V_G}\rho$ $(= u|_p\delta)$ is a ground term, then $u|_p\delta = u|_p\sigma_{V_G}\rho = u|_p\sigma_{V_G}\theta = u|_p\sigma\theta =_{E_0} u^\circ\sigma\theta =_B l^\circ\sigma\theta =_{E_0} l'\sigma\theta = l\gamma\sigma\theta$ (7).

Recall that $c[\gamma_r]\{\overline{ST}\}^{\nu'}\varrho_{\nu'} = c\nu'[(\gamma_r^{\nu'}\varrho_{\nu'})_{dom(\gamma_r^{\nu'})}]\{\overline{ST}^{\nu'}\varrho_{\nu'}\}$. Now, we prove $[x_k\delta]_E \in c[\gamma_r]\{\overline{ST}\}^{\nu'}\varrho_{\nu'} \,@\, [u|_p\delta]_E$.
As $dom(\gamma') = vars(c_0^\mu) \setminus (dom(\gamma) \uplus V^\mu)$, $c_0^\mu = c_0\mu$, and $dom(\delta_{V^\mu}) = V^\mu$, then $V_{c_0\mu} \subseteq dom(\delta_{V^\mu}) \uplus dom(\gamma) \uplus dom(\gamma')$. Then, as $c_0\nu' = c_0\mu\delta_{V^\mu}$ and δ_{V^μ} is a ground substitution, it follows that $V_{c_0\nu'} = dom(\gamma) \uplus dom(\gamma')$, hence $V_{c_0\nu'(\gamma\delta)_{dom(\gamma)}} = V_{ran(\gamma)\delta_{ran(\gamma)}} \cup V_{dom(\gamma')(\gamma\delta)_{dom(\gamma)}}$.
Then:

- As $(\gamma\delta)_{dom(\gamma)}$ is a ground substitution, if z is a variable in $ran(\gamma)$ then $z\delta_{ran(\gamma)}$ is a ground term, so $V_{ran(\gamma)\delta_{ran(\gamma)}} = \emptyset$.
- As $dom(\gamma) \cap dom(\gamma') = \emptyset$, if z is a variable in $dom(\gamma')$ then $z(\gamma\delta)_{dom(\gamma)} = z$, so $V_{dom(\gamma')(\gamma\delta)_{dom(\gamma)}} = dom(\gamma')$.

In conclusion, $V_{c_0\nu'\gamma\delta_{ran(\gamma)}} = dom(\gamma')$.
Call $\nu'' = \nu'(\gamma\delta)_{dom(\gamma)}$ $(= \nu' \uplus (\gamma\delta)_{dom(\gamma)}$ because $dom(\nu') \cap dom(\gamma) = V \cap dom(\gamma) = \emptyset)$. We must find a substitution $\tau : V_{c_0\nu''} \to \mathcal{T}_\Sigma$ such that $E_0 \models \phi^c\nu''\tau$. Let $\theta = \rho_2 \uplus \rho_1 \uplus \rho'_1 (= \rho_2 \uplus \rho')$, so $dom(\theta) = V_{G\sigma} \cup V_{G'\sigma}$. We choose $\tau = (\gamma'\sigma\theta)_{dom(\gamma')} = \gamma'(\sigma\theta)_{ran(\gamma')}$, so $dom(\tau) = dom(\gamma') = V_{c_0\nu''}$ and $(c_0\nu'')\tau = (c_0\nu'')\gamma'\sigma\theta$.
We prove that τ is a ground substitution by proving that $(c_0\nu'')\gamma'\sigma\theta$ is ground. Call $\delta'' = \delta_{V^\mu}\gamma\delta_{ran(\gamma)}$. As δ_{V^μ} and $\gamma\delta_{ran(\gamma)}$ are ground substitutions, $dom(\delta_{V^\mu}) \cap (dom(\gamma') \cup ran(\gamma')) = \emptyset$, and $V_{c_0\nu'} = dom(\gamma) \uplus dom(\gamma')$, then $(c_0\nu'')\gamma' = c_0\nu'(\gamma\delta_{ran(\gamma)} \uplus \gamma') = c_0^\mu\delta_{V^\mu}(\gamma\delta_{ran(\gamma)} \uplus \gamma') = c_0^\mu\delta_{V^\mu}\gamma'\gamma\delta_{ran(\gamma)} = c_0^\mu\gamma'\delta_{V^\mu}\gamma\delta_{ran(\gamma)} = c_0^\mu\gamma'\delta'' = c_{\gamma'}\delta''$. If $z \in V_{c_{\gamma'}\delta''}$ then, as δ_{V^μ} is ground, either $z \in V_{G'}$ or $z \in V_{l'} \setminus V_{G'}$, because l' is the only term of $c_{\gamma'}\gamma$ that does not appear in G'. then:

- If $z \in V_{G'}$ then $V_{z\sigma} \subseteq V_{G'\sigma}$, so $z\sigma\theta$ is a ground term because $dom(\theta) = V_{G\sigma} \cup V_{G'\sigma}$.
- If $z \in V_{l'} \setminus V_{G'}$, as $z \in V_{l'}$ and, by (6), $l'\sigma\theta$ is ground, then $z\sigma\theta$ is a ground term.

Now, we prove $E_0 \models \phi^c\nu''\tau$.

- As $ran(\gamma) \subseteq V_G$ and δ is a ground substitution, then $\gamma\delta_{ran(\gamma)}$ is a ground substitution so, as $c_0^{\nu'} = c_0\mu\delta_{V^\mu}$ and $\nu'' = \nu'\gamma\delta_{ran(\gamma)}$, $\phi^c\nu''\tau = \phi^c\mu\delta_{V^\mu}\gamma\delta_{ran(\gamma)}\tau = \phi^c\mu\delta_{V^\mu}(\gamma\delta_{ran(\gamma)} \uplus \tau)$.
- As δ_{V^μ} is a ground substitution, $V_{\phi^c\mu\delta_{V^\mu}} \subseteq V_{c_0^\mu\delta_{V^\mu}} = dom(\gamma) \uplus dom(\gamma')$, $dom(\tau) = dom(\gamma')$, and $dom(\gamma\delta_{ran(\gamma)}) = dom(\gamma)$ then $\phi^c\mu\delta_{V^\mu}(\gamma\delta_{ran(\gamma)} \uplus \tau) = \phi^c\mu(\delta_{V^\mu} \uplus \gamma\delta_{ran(\gamma)} \uplus \tau) = \phi^c\mu((\sigma\rho)_{V^\mu} \uplus \gamma(\sigma\rho)_{ran(\gamma)} \uplus \tau) = \phi^c\mu((\sigma\theta)_{V^\mu} \uplus \gamma(\sigma\theta)_{ran(\gamma)} \uplus \gamma'(\sigma\theta)_{ran(\gamma')})$,
because as $\phi^c\mu\delta''\tau$ is ground, it remains the same if we substitute the appearances of ρ, ground substitution, with $\theta = \rho \uplus \rho'$.
- As $(\sigma\theta)_{V^\mu}$ is ground then $\phi^c\mu((\sigma\theta)_{V^\mu} \uplus \gamma(\sigma\theta)_{ran(\gamma)} \uplus \gamma'(\sigma\theta)_{ran(\gamma')}) = \phi^c\mu(\gamma' \uplus \gamma)\sigma\theta$, the last equality because as the formula is ground, no new instantiation will come from an unrestricted substitution.

- As $dom(\gamma) \cap dom(\gamma') = \emptyset$ and $dom(\gamma) \cap ran(\gamma') = \emptyset$, we can apply the substitutions one after the other, so $\phi^c\mu(\gamma' \uplus \gamma)\sigma\theta = \phi^c\mu\gamma'\gamma\sigma\theta = \phi\gamma\sigma\theta$.
- As $V_{\phi\gamma\sigma} \subseteq V_{G'\sigma}$, $\rho' : V_{G'\sigma} \to \mathcal{T}_\Sigma$, and $\theta = \rho_2 \uplus \rho'$ then $\phi\gamma\sigma\theta = \phi\gamma\sigma\rho'$.

Joining all the equalities, we get $\phi^c\nu''\tau = \phi\gamma\sigma\rho'$. Then, as $E_0 \vDash \phi\gamma\sigma\rho'$, also $E_0 \vDash \phi^c\nu''\tau$.

Now, we prove the existence of a needed derivation rule in $\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$. As $\varrho_{\nu'} = (\varrho_\mu\delta)_{\setminus V}$ and $\nu' = (\mu\delta)_V$, both ground, $\bigcup_{i=1}^m V^\mu_{ST_i}\varrho_\mu \subseteq V_G$, and $\delta : V_G \to \mathcal{T}_\Sigma$, then $ST^\mu_i\varrho_\mu\delta = ST^{\nu'}_i\varrho_{\nu'}$ and $V_{ST^{\nu'}_i\varrho_{\nu'}} = \emptyset$, for $1 \leq i \leq m$, and $(c[\gamma_r])^\mu\varrho_\mu\delta = c^\mu[(\gamma^\mu_r\varrho_\mu)_{dom(\gamma^\mu_r)}]\delta = c^\mu[\gamma]\delta = c^{\nu'}[(\gamma\delta)_{dom(\gamma)}]$.

Recall that $c_0 \in R_B$ has the form $c : l^c \to r^c \ if \ \bigwedge_{i=1}^n(l^c_i \to r^c_i) \mid \phi^c$ and $\nu'' = \nu'(\gamma\delta)_{dom(\gamma)}$. There are two cases to consider now:

(a) $c_0 \in R$:
as $\tau : V_{c_0\nu''} \to \mathcal{T}_\Sigma$, $E_0 \vDash \phi^c\nu''\tau$, $l^c\nu''\tau$ and $r^c\nu''\tau$ are terms in $\mathcal{H}_\Sigma$, ϵ is a position in $pos(l^c\nu''\tau)$ such that $(l^c\nu''\tau)|_\epsilon = l^c\nu''\tau$, and $\overline{ST}^{\nu'}\varrho_{\nu'}$ are ground strategies, then there is a derivation rule

$$\frac{l^c_1\nu''\tau \to r^c_1\nu''\tau/ST^{\nu'}_1\varrho_{\nu'} \cdots l^c_m\nu''\tau \to r^c_m\nu''\tau/ST^{\nu'}_m\varrho_{\nu'}}{l^c\nu''\tau \to r^c\nu''\tau/c[(\gamma\delta)_{dom(\gamma)}]\{\overline{ST}^{\nu'}\varrho_{\nu'}\}}$$

in $\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$.

(b) $c_0 \notin R$:
then there is a rule $c_1 : f(t,t') \to t'' \ if \ C \in R$ such that c_0 has the form $c : f(x_s, f(t,t')) \to f(x_s, t'') \ if \ C$, where $dom(\gamma_r) \subseteq V_{c_1}$ and $C = \bigwedge_{i=1}^n(l^c_i \to r^c_i) \mid \phi^c$. Let $\tau' = \tau_{V_{c_1\nu''}}$. As $V_{c_1} \subset V_{c_0}$ and $\tau : V_{c_0\nu''} \to \mathcal{T}_\Sigma$ then $\tau' : V_{c_1\nu''} \to \mathcal{T}_\Sigma$. Also, as $V_{c_1} \subset V_{c_0}$ and $E_0 \vDash \phi^c\nu''\tau$ then $E_0 \vDash \phi^c\nu''\tau'$.
As $l^c\nu''\tau$ is a term in $\mathcal{H}_\Sigma$, 2 is a position in $pos(l^c\nu''\tau)$ such that $(l^c\nu''\tau)|_2 = f(t,t')\nu''\tau$, $E_0 \vDash \phi^c\nu''\tau'$, $t''\nu''\tau' = t''\nu''\tau$, and $\overline{ST}^{\nu'}\varrho_{\nu'}$ are ground strategies, then there is a derivation rule

$$\frac{l^c_1\nu''\tau \to r^c_1\nu''\tau/ST^{\nu'}_1\varrho_{\nu'} \cdots l^c_m\nu''\tau \to r^c_m\nu''\tau/ST^{\nu'}_m\varrho_{\nu'}}{l^c\nu''\tau \to l^c\nu''\tau[t''\nu''\tau]_2/c[(\gamma\delta)_{dom(\gamma)}]\{\overline{ST}^{\nu'}\varrho_{\nu'}\}}$$

in $\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$. As $l^c[]_2 = r^c[]_2 = f(x_s, [])$, and $r^c\nu''\tau[t''\nu''\tau]_2 = r^c[t'']_2\nu''\tau = r^c\nu''\tau$, this is the same as

$$\frac{l^c_1\nu''\tau \to r^c_1\nu''\tau/ST^{\nu'}_1\varrho_{\nu'} \cdots l^c_m\nu''\tau \to r^c_m\nu''\tau/ST^{\nu'}_m\varrho_{\nu'}}{l^c\nu''\tau \to r^c\nu''\tau/c[(\gamma\delta)_{dom(\gamma)}]\{\overline{ST}^{\nu'}\varrho_{\nu'}\}},$$

so in both cases we have the same derivation rule. Now, as:

- $\nu'' = \nu' \uplus (\gamma\delta)_{dom(\gamma)}$ is ground, $\nu' = \mu\delta_{V^\mu}$, $\delta = \sigma_{V_G}\rho$, $\theta = \rho \uplus \rho'_1$, and $dom(\delta_{V^\mu}) = V^\mu$,
- $\tau = \gamma'(\sigma\theta)_{ran(\gamma')}$ and $\delta_{V^\mu} \uplus (\gamma\delta)_{dom(\gamma)}$ are ground substitutions,
- $c_0 : l^c \to r^c \ if \ \bigwedge_{i=1}^n(l^c_i \to r^c_i) \mid \phi^c$ and $c_0\nu''\tau$ is ground,
- $c_{\gamma'} : l \to r \ if \ \bigwedge_{i=1}^n(l_i \to r_i) \mid \phi$, and
- $c_{\gamma'}$ is a fresh version of c^μ_0 except for $dom(\gamma) \uplus dom(\delta_{V^\mu})$, with renaming $\gamma' : vars(c^\mu_0) \setminus (dom(\gamma) \uplus dom(\delta_{V^\mu})) \to vars(c_{\gamma'}) \setminus (dom(\gamma) \uplus dom(\delta_{V^\mu}))$,

then, $c_0\nu''\gamma' = c_0(\nu'' \uplus \gamma') = c_0(\nu' \uplus (\gamma\delta)_{dom(\gamma)} \uplus \gamma') = c_0((\mu\delta_{V^\mu}) \uplus (\gamma\delta)_{dom(\gamma)} \uplus \gamma') = c^\mu_0(\delta_{V^\mu} \uplus (\gamma\delta)_{dom(\gamma)} \uplus \gamma') = c_{\gamma'}(\delta_{V^\mu} \uplus (\gamma\delta)_{dom(\gamma)})$, so $c_0\nu''\tau = c_0\nu''\gamma'(\sigma\theta)_{ran(\gamma')} = c_{\gamma'}(\delta_{V^\mu} \uplus (\gamma\delta)_{dom(\gamma)})(\sigma\theta)_{ran(\gamma')} = c_{\gamma'}(\delta_{V^\mu} \uplus (\gamma\delta)_{dom(\gamma)} \uplus \sigma\theta) = c_{\gamma'}(\delta \uplus \gamma\delta \uplus \sigma\theta) =$

$c_{\gamma'}((\sigma_{V_G}\rho)\uplus(\gamma\sigma_{V_G}\rho)\uplus\sigma\theta) = c_{\gamma'}((\sigma\rho)\uplus(\gamma\sigma\rho)\uplus\sigma\theta) = c_{\gamma'}((\sigma\theta)\uplus(\gamma\sigma\theta)\uplus\sigma\theta) = c_{\gamma'}\gamma\sigma\theta$, all because $c_0\nu''\tau$ is ground, and we can write the derivation rule as

$$\frac{l_1\gamma\sigma\theta \to r_1\gamma\sigma\theta/ST_1^{\nu'}\varrho_{\nu'} \cdots l_m\gamma\sigma\theta \to r_m\gamma\sigma\theta/ST_m^{\nu'}\varrho_{\nu'}}{l\gamma\sigma\theta \to r\gamma\sigma\theta/c[(\gamma\delta)_{dom(\gamma)}]\{\overline{ST}^{\nu'}\varrho_{\nu'}\}} \quad (8)$$

Also, as $\delta' = \sigma_1\sigma'_{V_{G'\sigma_1}}\rho'$, $V_{r_i\gamma\sigma_1, l_i\gamma\sigma_1} \subseteq V_{G'\sigma_1}$, $[r_i\gamma\delta']_E \in ST_i^{\nu'}\varrho_{\nu'}@[l_i\gamma\delta']_E$, for $1 \le i \le n$, where each term is ground, $\sigma = \sigma_1\sigma'$, and $\theta = \rho' \uplus \rho_2$, then $r_i\gamma\delta' = r_i\gamma\sigma_1\sigma'_{V_{G'\sigma_1}}\rho' = r_i\gamma\sigma_1\sigma'\rho' = r_i\gamma\sigma\rho' = r_i\gamma\sigma\theta$ (and $l_i\gamma\delta' = l_i\gamma\sigma\theta$), so $[r_i\gamma\sigma\theta]_E \in ST_i^{\nu'}\varrho_{\nu'}@[l_i\gamma\sigma\theta]_E$, and there are closed proof trees of the form $\frac{F_i}{l_i\gamma\sigma\theta \to r_i\gamma\sigma\theta/ST_i^{\nu'}\varrho_{\nu'}}$, with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_{\mathcal{R}}}$.
As $dom(\gamma') = vars(c_0^{\mu}) \setminus (dom(\gamma) \uplus V^{\mu})$ then $dom(\gamma) \cap dom(\gamma') = \emptyset$, so $\gamma'\gamma = \gamma' \uplus \gamma = \gamma\gamma'$. We already know that $r\gamma\sigma\rho' = x_k\delta$ (3) and $u|_p\delta =_E l\gamma\sigma\theta$ (7) so, as $r\gamma\sigma\rho'$ is ground and $\theta = \rho' \uplus \rho_2$, then also $r\gamma\sigma\theta = r\gamma\sigma\rho' = x_k\delta$, and we can apply the derivation rule (8) to $u|_p\delta$ and $x_k\delta$ and construct the c.p.t. for $[x_k\delta]_E \in (c[\gamma_r]\{\overline{ST}\})^{\mu}\varrho_{\mu}\delta \,@\, [u|_p\delta]_E$, i.e., $[x_k\delta]_E \in c[\gamma\delta_{ran(\gamma)}]\{\overline{ST}^{\nu'}\varrho_{\nu'}\} \,@\, [u|_p\delta]_E$, with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_{\mathcal{R}}}$:

$$\frac{\frac{F_1}{l_1\gamma\sigma\theta \to r_1\gamma\sigma\theta/ST_1\delta} \cdots \frac{F_m}{l_m\gamma\sigma\theta \to r_m\gamma\sigma\theta/ST_m\delta}}{u|_p\delta \to x_k\delta/c[(\gamma\delta)_{dom(\gamma)}]\{\overline{ST}^{\nu'}\varrho_{\nu'}\}}.$$

As we have shown before that $E_0 \vDash \psi_1\delta$ (1), that there are closed proof trees for each open goal in $\Delta\delta$ with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_{\mathcal{R}}}$ (2), and that $[v\delta]_E \in ST^{\nu'}\varrho_{\nu'}@[u[x_k]_p\delta]_E$ (4), then $\delta = \sigma_{vars(G)}\rho$ is a solution of G.

13. Rule $[tp]$ (`top`):
Again, we prove this case for conditional rules. For unconditional rules the proof is the same, just with the part dealing with the conditions removed from it.
$G = u \to v/(\texttt{top}(c[\gamma_r]\{ST_1,\ldots,ST_m\}); ST)^{\mu}\varrho_{\mu}(\wedge\Delta) \mid \psi_1 \mid V,\mu \leadsto_{[tp],\sigma_1} (\bigwedge_{i=1}^{n}(l_i\gamma \to r_i\gamma/ST_i^{\mu}\varrho_{\mu};\texttt{idle}) \wedge r\gamma \to v\,/ST^{\mu}\varrho_{\mu}\,(\wedge\,\Delta) \mid \psi_2)\sigma_1 \mid V,(\mu\sigma_1)_V = G'\sigma_1$, where:
    - $\gamma = (\gamma_r^{\mu}\varrho_{\mu})_{dom(\gamma_r^{\mu})}$ (so $ran(\gamma) \subseteq V_G$), $c \in R$, $c_0 \in c_B \subseteq R_B$ has the form $c : l^c \to r^c$ *if* $\bigwedge_{i=1}^{n}(l_i^c \to r_i^c) \mid \phi^c$, $c_{\gamma'} : l \to r$ *if* $\bigwedge_{i=1}^{n}(l_i \to r_i) \mid \phi$ is a fresh version with some renaming γ' of $c_0^{\mu} \in R_B^{\mu}$, with $dom(\gamma') = vars(c_0^{\mu}) \setminus (dom(\gamma_r) \uplus V^{\mu})$, so $c_{\gamma'} = c_0^{\mu}\gamma'$, call $l' = l\gamma$;
    - $abstract_{\Sigma_1}(u|_p) = \langle\lambda\bar{u}.u^{\circ};\sigma_u^{\circ};\phi_u^{\circ}\rangle$, $u^{\circ} = u|_p[\bar{x}]_{\bar{p}}$, with $\bar{x} = x_1,\ldots,x_u$ and $\bar{p} = p_1,\ldots,p_u$, $\phi_u^{\circ} = (\bigwedge_{j=1}^{u} x_j = u|_{p.p_j})$;
    - $abstract_{\Sigma_1}(l') = \langle\lambda\bar{y}.l^{\circ};\sigma_1^{\circ};\phi^{\circ}\rangle$, $l^{\circ} = l'[\bar{y}]_{\bar{q}}$, with $\bar{y} = y_1,\ldots,y_l$ and $\bar{q} = q_1,\ldots,q_l$, $\phi^{\circ} = (\bigwedge_{i=1}^{l} y_i = l'|_{q_i})$;
    - $\sigma_1 \in CSU_B(u^{\circ} = l^{\circ})$, $\psi_2 = \psi_1 \wedge \phi^{\circ} \wedge \phi_u^{\circ} \wedge \phi\gamma$, $\psi_2\sigma_1$ is satisfiable;

    Then $G'\sigma_1 \leadsto^{+}_{\sigma'} nil \mid \psi \mid V,\nu$, call $\sigma = \sigma_1\sigma'$, where $\nu = (\mu\sigma)_V = (\mu\sigma_1\sigma')_V = (\mu\sigma_1'\sigma')_V$, so $\sigma_{V_G} \mid \psi$ is a computed answer for G and $\sigma'_{V_{G'\sigma_1}} \mid \psi$ is a computed answer for $G'\sigma_1$.
    As $\gamma = \gamma_r^{\mu}(\varrho_{\mu})_{ran(\gamma_r^{\mu})}$ then $dom(\gamma_r) = dom(\gamma_r^{\mu}) = dom(\gamma)$.
    Let $\rho : V_{G\sigma} \to \mathcal{T}_{\Sigma}$ be a substitution such that $\psi\rho$ is satisfiable, call $\delta = \sigma_{V_G}\rho$ and $\varrho_{\nu'} = (\varrho_{\mu}\delta)_{\setminus V}$, so $\delta : V_G \to \mathcal{T}_{\Sigma}$, $\rho_1 = \rho_{V_{G'\sigma}}$, so also $\psi\rho_1$ is satisfiable, and call $\nu' = (\nu\rho)_V$, where $dom(\nu') = V$ and $ran(\nu') = \emptyset$. As $dom(\rho) = V_{G\sigma}$ then $dom(\rho_1) = V_{G\sigma} \cap V_{G'\sigma}$. Let $\rho_2 = \rho_{V_{G\sigma}\setminus V_{G'\sigma}}$, so $\rho = \rho_1 \uplus \rho_2$, and let $\rho_1' : V_{G'\sigma} \setminus V_{G\sigma} \to \mathcal{T}_{\Sigma}$, so $dom(\rho_1) \cap dom(\rho_1') = \emptyset$ and $dom(\rho_1) \cup dom(\rho_1') = V_{G'\sigma}$, such that $\psi(\rho_1 \uplus \rho_1')$ is satisfiable, and call $\rho' = \rho_1 \uplus \rho_1'$, so $\rho' : V_{G'\sigma} \to \mathcal{T}_{\Sigma}$.
    We prove several intermediate results:

– As $\nu = (\mu\sigma)_V$, $V^\mu \subseteq V_G \cap V_{G'}$, and $dom(\rho_1) = V_{G\sigma} \cap V_{G'\sigma}$ then $V^\nu \subseteq dom(\rho_1)$ so, as $dom(\nu') = V$ and $ran(\nu') = \emptyset$, $\nu' = (\nu\rho)_V = (\nu\rho_1)_V = (\nu\rho')_V$. Also, as $dom(\mu) \subseteq V$, $\nu' = (\nu\rho)_V = ((\mu\sigma)_V\rho)_V = (\mu\sigma\rho)_V = \mu(\sigma\rho)_{V^\mu}$.
– As $dom(\rho'_1) = V_{G'\sigma} \setminus V_{G\sigma}$ and $dom(\rho_1) = V_{G\sigma} \cap V_{G'\sigma} \subseteq V_{G\sigma}$, then $\rho'_{V_{G\sigma}} = (\rho_1 \uplus \rho'_1)_{V_{G\sigma}} = (\rho_1)_{V_{G\sigma}} = \rho_1$.
– As $\delta_{V^\mu} = (\sigma_{V_G}\rho)_{V^\mu}$, $\rho : V_{G\sigma} \to \mathcal{T}_\Sigma$, and $V^\mu \subseteq V_G$, then $\delta_{V^\mu} = (\sigma\rho)_{V^\mu}$, $ran(\delta_{V^\mu}) = \emptyset$, and $dom(\delta_{V^\mu}) = V^\mu$ $(= (V \setminus dom(\mu)) \cup ran(\mu)$, so $ran(\mu) \subseteq dom(\delta_{V^\mu}))$. Then $\nu' = (\mu\sigma_{V_G}\rho)_V = \mu\delta_{V^\mu}$ and $c_0^{\nu'} = c_0\nu' = c_0\mu\delta_{V^\mu}$.

As $\sigma'_{V_{G'\sigma_1}} \mid \psi$ is a computed answer for $G'\sigma_1$, $\rho' : V_{G'\sigma_1\sigma'} \to \mathcal{T}_\Sigma$, and $\psi\rho'$ is satisfiable then, by I.H., $\sigma'_{V_{G'\sigma_1}}\rho'$ is a solution for $G'\sigma_1$, call $\delta' = \sigma_1\sigma'_{V_{G'\sigma_1}}\rho'$ and $\varrho' = (\varrho_\mu\delta')_{\setminus V}$, meaning that:

(a) $E_0 \vDash \psi_2\delta'$,
(b) there are closed proof trees for each open goal in $\Delta\delta'$, with respect to $\mathcal{D}^{(\nu\rho')_V}_{\mathcal{R},Call_\mathcal{R}}$ $(=\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$, we use ν' instead of $(\nu\rho')_V$ in (c) and (d)),
(c) $[v\delta']_E \in ST^{\nu'}\varrho'@[r\gamma\delta']_E$, and
(d) $[r_i\gamma\delta']_E \in ST_i^{\nu'}\varrho'@[l_i\gamma\delta']_E$, for $1 \le i \le n$.

Then:

(a) i. $V_{\psi_2} \subseteq V_{G'}$ implies $V_{\psi_2\sigma_1} \subseteq V_{G'\sigma_1}$ and $V_{\psi_2\sigma} \subseteq V_{G'\sigma}$, so $\psi_2\delta' = \psi_2\sigma_1\sigma'_{V_{G'\sigma_1}}\rho' = \psi_2\sigma_1\sigma'\rho' = \psi_2\sigma\rho'$, hence $E_0 \vDash \psi_2\sigma\rho'$, where $\psi_2\sigma\rho'$ is ground, because $V_{\psi_2\sigma} \subseteq V_{G'\sigma}$ and $\rho' : V_{G'\sigma} \to \mathcal{T}_\Sigma$. As $\psi_2 = \psi_1 \wedge \phi^\circ \wedge \phi\gamma$, then $\psi_1\delta' = \psi_1\sigma\rho'$, $E_0 \vDash \psi_1\sigma\rho'$, $E_0 \vDash \phi^\circ\sigma\rho'$, $E_0 \vDash \phi^\circ_u\sigma\rho'$, and $E_0 \vDash \phi\gamma\sigma\rho'$, all ground formulas.
ii. Also as $\psi_2 = \psi_1 \wedge \phi^\circ \wedge \phi\gamma$, so $V_{\psi_1} \subseteq V_G \cap V_{G'}$ hence $V_{\psi_1\sigma} \subseteq V_{G\sigma} \cap V_{G'\sigma}$, and $dom(\rho_1) = V_{G\sigma} \cap V_{G'\sigma}$ imply $\psi_1\sigma\rho_1 \in \mathcal{T}_\Sigma$. Then, as $\rho' = \rho_1 \uplus \rho'_1$, we have $\psi_1\sigma\rho' = \psi_1\sigma(\rho_1 \uplus \rho'_1) = \psi_1\sigma\rho_1 = \psi_1\sigma(\rho_1 \uplus \rho_2) = \psi_1\sigma\rho = \psi_1\delta$, so $E_0 \vDash \psi_1\delta$ (1).
iii. As $\psi_1\delta' = \psi_1\sigma\rho'$ and $\psi_1\sigma\rho' = \psi_1\delta$ then $\psi_1\delta' = \psi_1\delta$.

(b) As in subcases (a)-ii and (a)-iii, $V_\Delta \subseteq V_G \cap V_{G'}$ implies $\Delta\delta' = \Delta\delta$, and the same closed proof trees are valid for each open goal in $\Delta\delta$ with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$ (2).

(c) i. Again, $V_v \subseteq V_G \cap V_{G'}$ implies that $v\delta' = v\delta$.
ii. We prove that $ST^{\nu'}\varrho' = ST^{\nu'}\varrho_{\nu'}$.
As $\varrho_{\nu'} = (\varrho_\mu\delta)_{\setminus V}$, $\delta = \sigma_{V_G}\rho$, $\sigma = \sigma_1\sigma'$, $\delta' = \sigma_1\sigma'_{V_{G'\sigma_1}}\rho'$, $\varrho' = (\varrho_\mu\delta')_{\setminus V}$, and $V_{ST^{\nu'}} \cap V = \emptyset$, this is the same as $ST^{\nu'}\varrho_\mu(\sigma_1\sigma')_{V_G}\rho = ST^{\nu'}\varrho_\mu\sigma_1\sigma'_{V_{G'\sigma_1}}\rho'$.
Let $x \in V_{ST^{\nu'}\varrho_\mu}$. As $V_{ST^{\nu'}\varrho_\mu} \subseteq V_{ST^\mu\varrho_\mu} \subseteq V_G \cap V_{G'}$, then $x \in V_G \cap V_{G'}$ and $V_{x\sigma_1} \subseteq V_{G\sigma_1} \cap V_{G'\sigma_1} \subseteq V_{G'\sigma_1}$, so $x(\sigma_1\sigma')_{V_G} = x\sigma_1\sigma' = x\sigma_1\sigma'_{V_{G'\sigma_1}}$. Also $x(\sigma_1\sigma')_{V_G} = x\sigma$, hence $V_{x(\sigma_1\sigma')_{V_G}} \subseteq V_{G\sigma}$. Then, as $\rho : V_{G\sigma} \to \mathcal{T}_\Sigma$, $x(\sigma_1\sigma')_{V_G}\rho$ is ground, so $x(\sigma_1\sigma')_{V_G}\rho = x\sigma_1\sigma'_{V_{G'\sigma_1}}\rho = x\sigma_1\sigma'_{V_{G'\sigma_1}}(\rho \cup \rho'_1) = x\sigma_1\sigma'_{V_{G'\sigma_1}}\rho'$.
iii. As in subcase (a)-i, $V_{r\gamma} \subseteq V_{G'}$ implies $r\gamma\delta' = r\gamma\sigma\rho'$.
Joining all the results, we get $[v\delta]_E \in ST^{\nu'}\varrho_{\nu'}@[r\gamma\sigma\rho']_E$, so there is a c.p.t. of the form $\frac{F}{r\gamma\sigma\rho' \to v\delta/ST^{\nu'}\varrho_{\nu'}}$ with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$ (3).

(d) Using the same proof as in the previous case, $[r_i\gamma\delta']_E \in ST_i^{\nu'}\varrho'@[l_i\gamma\delta']_E$, $V_{l_i\gamma,r_i\gamma} \subseteq V_{G'}$, and $V_{ST_i^{\nu'}\varrho'} \subseteq V_G \cap V_{G'}$ imply $[r_i\gamma\sigma\rho']_E \in ST_i^{\nu'}\varrho_{\nu'}@[l_i\gamma\sigma\rho']_E$, for $1 \le i \le n$, where each term and strategy are ground (4).

Now:

(a) $V_u \subseteq V_G$ imply $u\sigma_{V_G} = u\sigma$, hence $u\sigma_{V_G}\theta = u\sigma\theta$, and $u^\circ\sigma'_1 =_B l^\circ\sigma'_1$ imply $u^\circ\sigma\theta =_B l^\circ\sigma\theta$.
(b) As $E_0 \vDash \phi^\circ_u\sigma\theta$, ground formula, then $u^\circ\sigma\theta = u[\bar{x}]_{\bar{p}}\sigma\theta = u\sigma\theta[\bar{x}\sigma\theta]_{\bar{p}} =_{E_0} u\sigma\theta[u|_{\bar{p}}\sigma\theta]_{\bar{p}} = u\sigma\theta$, all ground terms.
(c) As $E_0 \vDash \phi^\circ\sigma\theta$, ground formula, then $l^\circ\sigma\theta = l'[\bar{y}]_{\bar{q}}\sigma\theta = l'\sigma\theta[\bar{y}\sigma\theta]_{\bar{q}} =_{E_0} l'\sigma\theta[l'|_{\bar{q}}\sigma\theta]_{\bar{q}} = l'\sigma\theta$, all ground terms (5).
(d) As $\rho : V_{G\sigma} \to \mathcal{T}_\Sigma$, so $u\sigma_{V_G}\rho$ $(= u\delta)$ is a ground term, then $u\delta = u\sigma_{V_G}\rho = u\sigma_{V_G}\theta = u\sigma\theta =_{E_0} u^\circ\sigma\theta =_B l^\circ\sigma\theta =_{E_0} l'\sigma\theta = l\gamma\sigma\theta$ (6).

We need to prove:
$[r\gamma\sigma\rho']_E \in c[\gamma_r]\{\overline{ST}\}^{\nu'}\varrho_{\nu'}@[u\delta]_E$, where $c[\gamma_r]\{\overline{ST}\}^{\nu'}\varrho_{\nu'} = c\nu'[(\gamma_r^{\nu'}\varrho_{\nu'})_{dom(\gamma_r^{\nu'})}]\{\overline{ST}^{\nu'}\varrho_{\nu'}\}$.
As $dom(\gamma') = vars(c_0^\mu) \setminus (dom(\gamma) \uplus V^\mu)$, $c_0^\mu = c_0\mu$, and $dom(\delta_{V^\mu}) = V^\mu$, then $V_{c_0\mu} \subseteq dom(\delta_{V^\mu}) \uplus dom(\gamma) \uplus dom(\gamma')$. Then, as $c_0\nu' = c_0\mu\delta_{V^\mu}$ and δ_{V^μ} is a ground substitution, it follows that $V_{c_0\nu'} = dom(\gamma) \uplus dom(\gamma')$, hence $V_{c_0\nu'\gamma} = ran(\gamma) \cup dom(\gamma')$ and $V_{c_0\nu'(\gamma\delta)_{dom(\gamma)}} = V_{ran(\gamma)\delta_{ran(\gamma)}} \cup V_{dom(\gamma')(\gamma\delta)_{dom(\gamma)}}$. Then:

- As $(\gamma\delta)_{dom(\gamma)}$ is a ground substitution, if z is a variable in $ran(\gamma)$ then $z\delta_{ran(\gamma)}$ is a ground term, so $V_{ran(\gamma)\delta_{ran(\gamma)}} = \emptyset$.
- As $dom(\gamma) \cap dom(\gamma') = \emptyset$, if z is a variable in $dom(\gamma')$ then $z(\gamma\delta)_{dom(\gamma)} = z$, so $V_{dom(\gamma')(\gamma\delta)_{dom(\gamma)}} = dom(\gamma')$.

In conclusion, $V_{c_0\nu'(\gamma\delta)_{dom(\gamma)}} = dom(\gamma')$.
Call $\nu'' = \nu'(\gamma\delta)_{dom(\gamma)}$ $(= \nu' \uplus (\gamma\delta)_{dom(\gamma)}$ because $dom(\nu') \cap dom(\gamma) = V \cap dom(\gamma) = \emptyset)$. We must find a substitution $\tau : V_{c_0\nu''} \to \mathcal{T}_\Sigma$ such that $E_0 \vDash \phi^c\nu''\tau$. Let $\theta = \rho_2 \uplus \rho_1 \uplus \rho'_1 (= \rho_2 \uplus \rho')$, so $dom(\theta) = V_{G\sigma} \cup V_{G'\sigma}$. We choose $\tau = (\gamma'\sigma\theta)_{dom(\gamma')} = \gamma'(\sigma\theta)_{ran(\gamma')}$, so $dom(\tau) = dom(\gamma') = V_{c_0\nu''}$ and $(c_0\nu'')\tau = (c_0\nu'')\gamma'\sigma\theta$.
We prove that τ is a ground substitution by proving that $(c_0\nu'')\gamma'\sigma\theta$ is ground. Call $\delta'' = \delta_{V^\mu}\gamma\delta_{ran(\gamma)}$. As δ_{V^μ} and $\gamma\delta_{ran(\gamma)}$ are ground substitutions, $dom(\delta_{V^\mu}) \cap (dom(\gamma') \cup ran(\gamma')) = \emptyset$, and $V_{c_0\nu'} = dom(\gamma) \uplus dom(\gamma')$, then $(c_0\nu'')\gamma' = c_0\nu'(\gamma\delta_{ran(\gamma)} \uplus \gamma') = c_0^\mu\delta_{V^\mu}(\gamma\delta_{ran(\gamma)} \uplus \gamma') = c_0^\mu\delta_{V^\mu}\gamma'\gamma\delta_{ran(\gamma)} = c_0^\mu\gamma'\delta_{V^\mu}\gamma\delta_{ran(\gamma)} = c_0^\mu\gamma'\delta'' = c_{\gamma'}\delta''$. If $z \in V_{c_{\gamma'}\delta''}$ then, as δ_{V^μ} is ground, either $z \in V_{G'}$ or $z \in V_{l'} \setminus V_{G'}$, because l' is the only term of $c_{\gamma'}\gamma$ that does not appear in G'. We check each case:

- If $z \in V_{G'}$ then $V_{z\sigma} \subseteq V_{G'\sigma}$, so $z\sigma\theta$ is a ground term because $dom(\theta) = V_{G\sigma} \cup V_{G'\sigma}$.
- If $z \in V_{l'} \setminus V_{G'}$, as $z \in V_{l'}$ and, by (5), $l'\sigma\theta$ is ground, then $z\sigma\theta$ is a ground term.

We prove $E_0 \vDash \phi^c\nu''\tau$.

- As $ran(\gamma) \subseteq V_G$ and δ is a ground substitution, then $\gamma\delta_{ran(\gamma)}$ is a ground substitution so, as $c_0^{\nu'} = c_0\mu\delta_{V^\mu}$ and $\nu'' = \nu'\gamma\delta_{ran(\gamma)}$, $\phi^c\nu''\tau = \phi^c\mu\delta_{V^\mu}\gamma\delta_{ran(\gamma)}\tau = \phi^c\mu\delta_{V^\mu}(\gamma\delta_{ran(\gamma)} \uplus \tau)$.
- As δ_{V^μ} is a ground substitution, $V_{\phi^c\mu\delta_{V^\mu}} \subseteq V_{c_0^\mu\delta_{V^\mu}} = dom(\gamma) \uplus dom(\gamma')$, $dom(\tau) = dom(\gamma')$, and $dom(\gamma\delta_{ran(\gamma)}) = dom(\gamma)$ then $\phi^c\mu\delta_{V^\mu}(\gamma\delta_{ran(\gamma)} \uplus \tau) = \phi^c\mu(\delta_{V^\mu} \uplus \gamma\delta_{ran(\gamma)} \uplus \tau) = \phi^c\mu((\sigma\rho)_{V^\mu} \uplus \gamma(\sigma\rho)_{ran(\gamma)} \uplus \tau) = \phi^c\mu((\sigma\theta)_{V^\mu} \uplus \gamma(\sigma\theta)_{ran(\gamma)} \uplus \gamma'(\sigma\theta)_{ran(\gamma')})$,
  because as $\phi^c\mu\delta''\tau$ is ground, it remains the same if we substitute the appearances of ρ, ground substitution, with $\theta = \rho \uplus \rho'$.
- As $(\sigma\theta)_{V^\mu}$ is ground then $\phi^c\mu((\sigma\theta)_{V^\mu} \uplus \gamma(\sigma\theta)_{ran(\gamma)} \uplus \gamma'(\sigma\theta)_{ran(\gamma')}) = \phi^c\mu(\gamma' \uplus \gamma)\sigma\theta$, the last equality because as the formula is ground, no new instantiation will come from an unrestricted substitution.
- As $dom(\gamma) \cap dom(\gamma') = \emptyset$ and $dom(\gamma) \cap ran(\gamma') = \emptyset$, we can apply the substitutions one after the other, so $\phi^c\mu(\gamma' \uplus \gamma)\sigma\theta = \phi^c\mu\gamma'\gamma\sigma\theta = \phi\gamma\sigma\theta$.

– As $V_{\phi\gamma\sigma} \subseteq V_{G'\sigma}$, $\rho' : V_{G'\sigma} \to \mathcal{T}_\Sigma$, and $\theta = \rho_2 \uplus \rho'$ then $\phi\gamma\sigma\theta = \phi\gamma\sigma\rho'$.

Joining all the equalities, we get $\phi^c\nu''\tau = \phi\gamma\sigma\rho'$. Then, as $E_0 \vDash \phi\gamma\sigma\rho'$, also $E_0 \vDash \phi^c\nu''\tau$.

Now, we prove the existence of a needed derivation rule in $\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$. As $\varrho_{\nu'} = (\varrho_\mu\delta)_{\setminus V}$ and $\nu' = (\mu\delta)_V$, both ground, $\bigcup_{i=1}^m V^\mu_{ST_i}\varrho_\mu \subseteq V_G$, and $\delta : V_G \to \mathcal{T}_\Sigma$, then $ST^\mu_i\varrho_\mu\delta = ST^{\nu'}_i\varrho_{\nu'}$ and $V_{ST^{\nu'}_i\varrho_{\nu'}} = \emptyset$, for $1 \le i \le m$, and $(c[\gamma_r])^\mu\varrho_\mu\delta = c^\mu[(\gamma^\mu_r\varrho_\mu)_{dom(\gamma^\mu_r)}]\delta = c^\mu[\gamma]\delta = c^{\nu'}[(\gamma\delta)_{dom(\gamma)}]$.

Recall that $c_0 \in R$ has the form $c : l^c \to r^c\ \mathit{if}\ \bigwedge_{i=1}^n(l^c_i \to r^c_i) \mid \phi^c$ and $\nu'' = \nu'\gamma\delta_{ran(\gamma)}$. As $\tau : V_{c_0\nu''} \to \mathcal{T}_\Sigma$, $E_0 \vDash \phi^c\nu''\tau$, $l^c\nu''\tau$ and $r^c\nu''\tau$ are terms in $\mathcal{H}_\Sigma$, ϵ is a position in $pos(l^c\nu''\tau)$ such that $(l^c\nu''\tau)|_\epsilon = l^c\nu''\tau$, and $\overline{ST}^{\nu'}\varrho_{\nu'}$ are ground strategies, then there is a derivation rule

$$\frac{l^c_1\nu''\tau \to r^c_1\nu''\tau/ST^{\nu'}_1\varrho_{\nu'} \cdots l^c_m\nu''\tau \to r^c_m\nu''\tau/ST^{\nu'}_m\varrho_{\nu'}}{l^c\nu''\tau \to r^c\nu''\tau/c[(\gamma\delta)_{dom(\gamma)}]\{\overline{ST}^{\nu'}\varrho_{\nu'}\}}$$

in $\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$.

Now, as:

- $\nu'' = \nu' \uplus (\gamma\delta)_{dom(\gamma)}$ is ground, $\nu' = \mu\delta_{V^\mu}$, $\delta = \sigma_{V_G}\rho$, $\theta = \rho \uplus \rho'_1$, and $dom(\delta_{V^\mu}) = V^\mu$,
- $\tau = \gamma'(\sigma\theta)_{ran(\gamma')}$ and $\delta_{V^\mu} \uplus (\gamma\delta)_{dom(\gamma)}$ are ground substitutions,
- $c_0 : l^c \to r^c\ \mathit{if}\ \bigwedge_{i=1}^n(l^c_i \to r^c_i) \mid \phi^c$ and $c_0\nu''\tau$ is ground,
- $c_{\gamma'} : l \to r\ \mathit{if}\ \bigwedge_{i=1}^n(l_i \to r_i) \mid \phi$, and
- $c_{\gamma'}$ is a fresh version of c^μ_0 except for $dom(\gamma) \uplus dom(\delta_{V^\mu})$, with renaming $\gamma' : vars(c^\mu_0) \setminus (dom(\gamma) \uplus dom(\delta_{V^\mu})) \to vars(c_{\gamma'}) \setminus (dom(\gamma) \uplus dom(\delta_{V^\mu}))$,

then, $c_0\nu''\gamma' = c_0(\nu'' \uplus \gamma') = c_0(\nu' \uplus (\gamma\delta)_{dom(\gamma)} \uplus \gamma') = c_0((\mu\delta_{V^\mu}) \uplus (\gamma\delta)_{dom(\gamma)} \uplus \gamma') = c^\mu_0(\delta_{V^\mu} \uplus (\gamma\delta)_{dom(\gamma)} \uplus \gamma') = c_{\gamma'}(\delta_{V^\mu} \uplus (\gamma\delta)_{dom(\gamma)})$, so $c_0\nu''\tau = c_0\nu''\gamma'(\sigma\theta)_{ran(\gamma')} = c_{\gamma'}(\delta_{V^\mu} \uplus (\gamma\delta)_{dom(\gamma)})(\sigma\theta)_{ran(\gamma')} = c_{\gamma'}(\delta_{V^\mu} \uplus (\gamma\delta)_{dom(\gamma)} \uplus \sigma\theta) = c_{\gamma'}(\delta \uplus (\gamma\delta) \uplus \sigma\theta) = c_{\gamma'}((\sigma_{V_G}\rho) \uplus (\gamma\sigma_{V_G}\rho) \uplus \sigma\theta) = c_{\gamma'}((\sigma\rho) \uplus (\gamma\sigma\rho) \uplus \sigma\theta) = c_{\gamma'}((\sigma\theta) \uplus (\gamma\sigma\theta) \uplus \sigma\theta) = c_{\gamma'}\gamma\sigma\theta$, all because $c_0\nu''\tau$ is ground, and we can write the derivation rule as

$$\frac{l_1\gamma\sigma\theta \to r_1\gamma\sigma\theta/ST^{\nu'}_1\varrho_{\nu'} \cdots l_m\gamma\sigma\theta \to r_m\gamma\sigma\theta/ST^{\nu'}_m\varrho_{\nu'}}{l\gamma\sigma\theta \to r\gamma\sigma\theta/c[(\gamma\delta)_{dom(\gamma)}]\{\overline{ST}^{\nu'}\varrho_{\nu'}\}} \quad (7)$$

Also, as $\delta' = \sigma_1\sigma'_{V_{G'\sigma_1}}\rho'$, $V_{r_i\gamma\sigma_1,l_i\gamma\sigma_1} \subseteq V_{G'\sigma_1}$, $[r_i\gamma\delta']_E \in ST^{\nu'}_i\varrho_{\nu'}@[l_i\gamma\delta']_E$, for $1 \le i \le n$, where each term is ground (4), $\sigma = \sigma_1\sigma'$, and $\theta = \rho' \uplus \rho_2$, then $r_i\gamma\delta' = r_i\gamma\sigma_1\sigma'_{V_{G'\sigma_1}}\rho' = r_i\gamma\sigma_1\sigma'\rho' = r_i\gamma\sigma\rho' = r_i\gamma\sigma\theta$ (and $l_i\gamma\delta' = l_i\gamma\sigma\theta$), so $[r_i\gamma\sigma\theta]_E \in ST^{\nu'}_i\varrho_{\nu'}@[l_i\gamma\sigma\theta]_E$, and there are closed proof trees of the form $\frac{F_i}{l_i\gamma\sigma\theta \to r_i\gamma\sigma\theta/ST^{\nu'}_i\varrho_{\nu'}}$, with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$.

There is also a derivation rule $\frac{u\delta \to r\gamma\sigma\theta/c[(\gamma\delta)_{dom(\gamma)}]\{\overline{ST}^{\nu'}\varrho_{\nu'}\} \quad r\gamma\sigma\rho' \to v\delta/ST^{\nu'}\varrho_{\nu'}}{u\delta \to v\delta/(c[(\gamma\delta)_{dom(\gamma)}]\{\overline{ST}^{\nu'}\varrho_{\nu'}\}\,;ST^{\nu'}\varrho_{\nu'}}$ in $\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$, as seen in subsection 5.2.7.

We already know that there is a c.p.t. of the form $\frac{F}{r\gamma\sigma\rho' \to v\delta/ST^{\nu'}\varrho_{\nu'}}$ with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$ (3). As $r\gamma\sigma\rho'$ is ground and $\theta = \rho' \uplus \rho_2$ then $r\gamma\sigma\rho' = r\gamma\sigma\theta$, hence $\frac{F}{r\gamma\sigma\theta \to v\delta/ST^{\nu'}\varrho_{\nu'}}$ is a c.p.t. with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$.

We also know that $u\delta =_E l\gamma\sigma\theta$ (6), so we can apply the derivation rule (7) to $u\delta$ and $r\gamma\sigma\theta$, and construct the c.p.t. for $[v\delta]_E \in (c[\gamma_r]\{\overline{ST}\})^\mu\varrho_\mu\delta\ @\ [u\delta]_E$, i.e.,

$[v\delta]_E \in c[\gamma\delta_{ran(\gamma)}]\{\overline{ST}^{\nu'}\varrho_{\nu'}\} \mathbin{@} [u\delta]_E$ with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_{\mathcal{R}}}$:

$$\dfrac{\dfrac{\dfrac{F_1}{l_1\gamma\sigma\theta \to r_1\gamma\sigma\theta / ST_1^{\nu'}\varrho_{\nu'}} \cdots \dfrac{F_m}{l_m\gamma\sigma\theta \to r_m\gamma\sigma\theta / ST_m^{\nu'}\varrho_{\nu'}}}{u\delta \to r\gamma\sigma\theta / c[(\gamma\delta)_{dom(\gamma)}]\{\overline{ST}^{\nu'}\varrho_{\nu'}\}} \quad \dfrac{F}{r\gamma\sigma\theta \to v\delta / ST^{\nu'}\varrho_{\nu'}}}{u\delta \to v\delta / c[(\gamma\delta)_{dom(\gamma)}]\{\overline{ST}^{\nu'}\varrho_{\nu'}\} \,;\, ST^{\nu'}\varrho_{\nu'}}.$$

As we have shown before that $E_0 \vDash \psi_1\delta$ (1) and that there are closed proof trees for each open goal in $\Delta\delta$ with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_{\mathcal{R}}}$ (2), then $\delta = \sigma_{vars(G)}\rho$ is a solution of G.

14. Rule $[c1]$ (call strategy):
There are two versions of the rule where in $Call_{\mathcal{R}}$ we have either (a) `sd` $CS := ST_1$ or (b) `sd` $CS(\bar{x}) := ST_1$.

(a) $G = u_1 \to v_1 / CS^{\mu}\varrho_{\mu} \,;\, ST^{\mu}\varrho_{\mu} \,(\wedge\, \Delta) \mid \psi_1 \mid V, \mu$, where $CS^{\mu}\varrho_{\mu} = CS^{\mu}$, $G \leadsto_{[c1]} u_1 \to v_1 / ST_2 \,;\, ST^{\mu}\varrho_{\mu} \,(\wedge\, \Delta) \mid \psi_1 \mid V, \mu = G'$ and $G' \leadsto^{+}_{\sigma} nil \mid \psi \mid V, \nu$, where `sd` $CS := ST_1 \in Call^{\mu}_{\mathcal{R}}$, $\nu = (\mu\sigma)_V$, and ST_2 is a fresh version of ST_1, with some renaming γ', where $dom(\gamma') = V_{ST_1} \setminus V^{\mu}$, so $ST_2 = ST_1\gamma'$, hence $\sigma_{V_G}|\psi$ is a computed answer for G and $\sigma_{V_{G'}}|\psi$ is a computed answer for G'. As $V_{CS} = \emptyset$ then $V_G \subseteq V_{G'}$, so $ran(\sigma_{V_G}) \subseteq ran(\sigma_{V_{G'}})$. Then:

i. as `sd` $CS := ST_1 \in Call^{\mu}_{\mathcal{R}}$ then there is `sd` $CS := ST_0 \in Call_{\mathcal{R}}$ such that $ST_0^{\mu} = ST_1$, hence $ST_2 = ST_1\gamma' = ST_0^{\mu}\gamma' = (ST_0\gamma')^{\mu}$, since $dom(\gamma') \cap V^{\mu} = \emptyset$ and $ran(\gamma') \cap dom(\mu) = \emptyset$, and

ii. as $dom(\varrho_{\mu}) \cap V^{\mu} = \emptyset$, invariant for admissible goals, and ST_2 has only new variables except for V^{μ}, then $ST_2 = ST_2\varrho_{\mu} = (ST_0\gamma')^{\mu}\varrho_{\mu}$.

Let $\rho : V_{G\sigma} \to \mathcal{T}_{\Sigma}$ such that $\psi\rho$ is satisfiable, call $\delta = \sigma_{V_G}\rho$, so $\delta : V_G \to \mathcal{T}_{\Sigma}$, and call $\nu' = (\nu\rho)_V$, where $dom(\nu') = V$ and $ran(\nu') = \emptyset$. Let $\rho_1 : V_{G'\sigma} \setminus V_{G\sigma} \to \mathcal{T}_{\Sigma}$, so $dom(\rho) \cap dom(\rho_1) = \emptyset$ and $dom(\rho) \cup dom(\rho_1) = V_{G'\sigma}$, such that $\psi(\rho \uplus \rho_1)$ is satisfiable. Call $\rho' = \rho \uplus \rho_1$, so $\rho' : V_{G'\sigma} \to \mathcal{T}_{\Sigma}$, and call $\delta' = \sigma_{V_{G'}}\rho'$, so $\delta' : V_{G'} \to \mathcal{T}_{\Sigma}$. As $dom(\nu') = V$ and $ran(\nu') = \emptyset$ then $(\nu\rho')_V = (\nu\rho)_V = \nu'$. Then $G\delta' = G\sigma_{V_{G'}}\rho' = G\sigma_{V_{G'}}(\rho \uplus \rho_1) = G(\sigma_{V_G} \uplus \sigma_{V_{G'} \setminus V_G})(\rho \uplus \rho_1) = G(\sigma_{V_G}\rho \uplus \sigma_{V_{G'} \setminus V_G}\rho_1) = G\sigma_{V_G}\rho = G\delta$.
By I.H, $E_0 \vDash \psi_1\delta'$ and there is a c.p.t. of the form $\dfrac{\dfrac{F_1}{u_1\delta' \to t / (ST_0\gamma')^{\nu'}\varrho_{\nu'}} \quad \dfrac{F_2}{t \to v_1\delta' / ST^{\nu'}\varrho_{\nu'}}}{u_1\delta' \to v_1\delta' / (ST_0\gamma' \,; ST))^{\nu'}\varrho_{\nu'}}$, for some term $t \in \mathcal{H}_{\Sigma}$, with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_{\mathcal{R}}}$. By Lemma 6, there is also a c.p.t. of the form $\dfrac{F_3}{u_1\delta' \to t / (ST_0\gamma')^{\nu'}}$.
As $CS^{\nu'}\varrho_{\nu'} = CS$, $(ST_0\gamma')^{\nu'} = ST_0^{\nu'}\gamma'$, since $(dom(\gamma') \cup ran(\gamma')) \cap V = \emptyset$, and there are derivation rules $\dfrac{u_1\delta' \to t / CS \quad t \to v_1\delta' / ST^{\nu'}\varrho_{\nu'}}{u_1\delta' \to v_1\delta' / (CS \,; ST)^{\nu'}\varrho_{\nu'}}$ and $\dfrac{u_1\delta' \to t / ST_0^{\nu'}\gamma'}{u_1\delta' \to t / CS}$, i.e., $\dfrac{u_1\delta' \to t / (ST_0\gamma')^{\nu'}}{u_1\delta' \to t / CS}$ in $\mathcal{D}^{\nu'}_{\mathcal{R},Call_{\mathcal{R}}}$, then $\dfrac{\dfrac{\dfrac{F_3}{u_1\delta' \to t / (ST_0\gamma')^{\nu'}}}{u_1\delta' \to t / CS} \quad \dfrac{F_2}{t \to v_1\delta' / ST^{\nu'}\varrho_{\nu'}}}{u_1\delta' \to v_1\delta' / (CS \,; ST)^{\nu'}\varrho_{\nu'}}$ is a c.p.t., so $v_1\delta' \in (CS \,; ST)^{\nu'}\varrho_{\nu'} @ u_1\delta'$.
As $G\delta' = G\delta$, this is the same as $v_1\delta \in (CS \,; ST)^{\nu'}\varrho_{\nu'} @ u_1\delta$ and $E_0 \vDash \psi_1\delta$, so $\sigma_{vars(G)}\rho$ is a solution of G.

(b) $G = u_1 \to v_1 / CS(\bar{t})^{\mu}\varrho_{\mu} \,;\, ST^{\mu}\varrho_{\mu} \,(\wedge\, \Delta) \mid \psi_1 \mid V, \mu$, where $CS(\bar{t})^{\mu}\varrho_{\mu} = CS^{\mu}(\bar{t}\mu\varrho_{\mu})$, $G \leadsto_{[c1]} u_1 \to v_1 / ST_2\gamma \,;\, ST : (\wedge\, \Delta) \mid \psi_1 \mid V, \mu = G'$, and $G' \leadsto^{+}_{\sigma} nil \mid \psi \mid V, \nu$, where $\nu = (\mu\sigma)_V$, we call $\varrho_{\nu} = (\varrho_{\mu}\sigma)_{\setminus V}$, `sd` $CS(\bar{x}) := ST_1 \in Call^{\mu}_{\mathcal{R}}$, $\gamma = \{\bar{x} \mapsto \bar{t}\mu\varrho_{\mu}\}$, ST_2 is a fresh version of ST_1, with some renaming γ', where $dom(\gamma') = V_{ST_1} \setminus (\hat{x} \cup V^{\mu})$, so $ST_2 = ST_1\gamma'$, hence $\sigma_{V_G}|\psi$ is a computed answer for G and $\sigma_{V_{G'}}|\psi$ is a computed answer for G'. As $V_{CS^\mu(\bar{t}\mu\varrho_\mu)} = \mathit{ran}(\gamma)$ and $\hat{x} \subseteq V_{ST_2}$ then $V_G \subseteq V_{G'}$, so $\mathit{ran}(\sigma_{V_G}) \subseteq \mathit{ran}(\sigma_{V_{G'}})$. $ST_2 = ST_2[\bar{x}']_{\bar{p}}$, for proper $\bar{x}'$ and $\bar{p}$, where $\hat{x}' = \hat{x}$ and $V_{ST_2[]_{\bar{p}}} \cap \bar{x} = \emptyset$, so $ST_2\gamma = ST_2[\bar{x}']_{\bar{p}}\gamma = ST_2[\bar{x}'\gamma]_{\bar{p}}$. Call $\gamma_0 = \{\bar{x} \mapsto \bar{t}\}$, so $\bar{x}'\gamma = \bar{x}'\gamma_0\mu\varrho_\mu$. Then:

i. as $\mathit{dom}(\varrho_\mu) \cap V^\mu = \emptyset$, invariant for admissible goals, and $ST_2[]_{\bar{p}}$ has only new variables except for V^μ, then $ST_2[]_{\bar{p}} = ST_2[]_{\bar{p}}\varrho_\mu = ST_2\varrho_\mu[]_{\bar{p}} = ST_1\gamma'\varrho_\mu[]_{\bar{p}}$, and

ii. as `sd` $CS(\bar{x}) := ST_1 \in \mathit{Call}^\mu_{\mathcal{R}}$ then there is a definition `sd` $CS(\bar{x}) := ST_0$ in $\mathit{Call}_{\mathcal{R}}$ such that $ST_0^\mu = ST_1$. Then, we get $ST_2\gamma = ST_2[\bar{x}'\gamma]_{\bar{p}} = ST_1\gamma'\varrho_\mu[\bar{x}'\gamma]_{\bar{p}} = ST_1\gamma'\varrho_\mu[\bar{x}'\gamma]_{\bar{p}} = ST_0^\mu\gamma'\varrho_\mu[\bar{x}'\gamma_0\mu\varrho_\mu]_{\bar{p}} = (ST_0^\mu\gamma'[\bar{x}'\gamma_0\mu]_{\bar{p}})\varrho_\mu = ((ST_0\gamma')^\mu[\bar{x}'\gamma_0\mu]_{\bar{p}})\varrho_\mu = (ST_0\gamma'[\bar{x}'\gamma_0]_{\bar{p}})^\mu\varrho_\mu$, since $\mathit{dom}(\gamma') \cap V^\mu = \emptyset$ and $\mathit{ran}(\gamma') \cap \mathit{dom}(\mu) = \emptyset$.

Let $\rho : V_{G\sigma} \to \mathcal{T}_\Sigma$ such that $\psi\rho$ is satisfiable, call $\delta = \sigma_{V_G}\rho$, so $\delta : V_G \to \mathcal{T}_\Sigma$, call $\nu' = (\nu\rho)_V$, where $\mathit{dom}(\nu') = V$ and $\mathit{ran}(\nu') = \emptyset$, and call $\varrho_{\nu'} = (\varrho_\nu\rho)_{\setminus V}$. Let $\rho_1 : V_{G'\sigma} \setminus V_{G\sigma} \to \mathcal{T}_\Sigma$, so $\mathit{dom}(\rho) \cap \mathit{dom}(\rho_1) = \emptyset$ and $\mathit{dom}(\rho) \cup \mathit{dom}(\rho_1) = V_{G'\sigma}$, such that $\psi(\rho \uplus \rho_1)$ is satisfiable. Call $\rho' = \rho \uplus \rho_1$, so $\rho' : V_{G'\sigma} \to \mathcal{T}_\Sigma$, and call $\delta' = \sigma_{V_{G'}}\rho'$, so $\delta' : V_{G'} \to \mathcal{T}_\Sigma$. As $\mathit{dom}(\nu') = V$ and $\mathit{ran}(\nu') = \emptyset$ then $(\nu\rho')_V = (\nu\rho)_V = \nu'$. Also, as $V_G \subseteq V_{G'}$ and $\mathit{dom}(\rho_1) = V_{G'\sigma} \setminus V_{G\sigma}$, then $G\delta' = G\sigma_{V_{G'}}\rho' = G\sigma_{V_G}\rho' = G\sigma\rho' = G\sigma(\rho \uplus \rho_1) = G\sigma\rho = G\sigma_{V_G}\rho = G\delta$.

By I.H, $E_0 \vDash \psi_1\delta'$, so $\dfrac{\dfrac{F_1}{u_1\delta' \to w/(ST_0\gamma'[\bar{x}'\gamma_0]_{\bar{p}})^{\nu'}\varrho_{\nu'}} \quad \dfrac{F_2}{w \to v_1\delta'/ST^{\nu'}\varrho_{\nu'}}}{u_1\delta' \to v_1\delta'/(ST_0\gamma'[\bar{x}'\gamma_0]_{\bar{p}}\,;ST))^{\nu'}\varrho_{\nu'}}$ is a c.p.t., for some term $w \in \mathcal{H}_\Sigma$, with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},\mathit{Call}_{\mathcal{R}}}$. As ϱ_ν is idempotent and ρ is ground then $\varrho_{\nu'}$ is also idempotent. Then, as ν' is ground, $\mathit{dom}(\nu') = V$, and $\mathit{dom}(\varrho_{\nu'}) \cap V = \emptyset$, we can write $\dfrac{F_1}{u_1\delta' \to w/(ST_0\gamma'[\bar{x}'\gamma_0]_{\bar{p}})^{\nu'}\varrho_{\nu'}}$ as $\dfrac{F_1}{u_1\delta' \to w/(ST_0\gamma'[\bar{x}'\gamma_0\varrho_{\nu'}]_{\bar{p}})^{\nu'}\varrho_{\nu'}}$. Let α be a renaming such that $\mathit{dom}(\alpha) = V_{\varrho_{\nu'}}$ and $\mathit{ran}(\alpha)$ is away from all known variables. By Lemma 5 there is a c.p.t. of the form $\dfrac{F_3}{u_1\delta' \to w/(ST_0\gamma'[\bar{x}'\gamma_0\varrho_{\nu'}]_{\bar{p}})^{\nu'}(\varrho_{\nu'}\alpha)}$. Now, we can apply Lemma 6, so there is also a closed proof tree of the form $\dfrac{F_4}{u_1\delta' \to w/(ST_0\gamma'[\bar{x}'\gamma_0\varrho_{\nu'}]_{\bar{p}})^{\nu'}}$. This c.p.t. shows that partial generalization of $\mathit{dom}(\varrho_{\nu'})$ is also valid.

As $(ST_0\gamma'[\bar{x}'\gamma_0\varrho_{\nu'}]_{\bar{p}})^{\nu'} = ST_0^{\nu'}\gamma'[\bar{x}'\gamma_0\varrho_{\nu'}\nu']_{\bar{p}} = ST_0^{\nu'}\gamma'[\bar{x}'\gamma_0\nu'\varrho_{\nu'}]_{\bar{p}} = ST_0^{\nu'}(\gamma' \cup \gamma'')$, where $\gamma'' = \{\bar{x} \mapsto \bar{t}\nu'\varrho_{\nu'}\}$, since $(\mathit{dom}(\gamma') \cup \mathit{ran}(\gamma')) \cap V = \emptyset$, ν' is ground, $\mathit{dom}(\nu') = V$, and $\mathit{dom}(\varrho_{\nu'}) \cap V = \emptyset$, and also $CS(\bar{t})^{\nu'}\varrho_{\nu'} = CS(\bar{t}\nu'\varrho_{\nu'})$, then $\mathcal{D}^{\nu'}_{\mathcal{R},\mathit{Call}_{\mathcal{R}}}$ has derivation rules $\dfrac{u_1\delta' \to w/ST_0^{\nu'}(\gamma' \cup \gamma'')}{u_1\delta' \to w/CS(\bar{t}\nu'\varrho_{\nu'})}$, i.e., $\dfrac{u_1\delta' \to w/(ST_0\gamma'[\bar{x}'\gamma_0\varrho_{\nu'}]_{\bar{p}})^{\nu'}}{u_1\delta' \to w/CS(\bar{t}\nu'\varrho_{\nu'})}$, and $\dfrac{u_1\delta' \to w/CS(\bar{t}\nu'\varrho_{\nu'}) \quad w \to v_1\delta'/ST^{\nu'}\varrho_{\nu'}}{u_1\delta' \to v_1\delta'/(CS(\bar{t})\,;ST)^{\nu'}\varrho_{\nu'}}$. Then $\dfrac{\dfrac{\dfrac{F_4}{u_1\delta' \to w/(ST_0\gamma'[\bar{x}'\gamma_0\varrho_{\nu'}]_{\bar{p}})^{\nu'}}}{u_1\delta' \to w/CS(\bar{t}\nu'\varrho_{\nu'})} \quad \dfrac{F_2}{w \to v_1\delta'/ST^{\nu'}\varrho_{\nu'}}}{u_1\delta' \to v_1\delta'/(CS(\bar{t})\,;ST)^{\nu'}\varrho_{\nu'}}$ is a c.p.t., so $[v_1\delta']_E \in (CS(\bar{t})\,;ST)^{\nu'}\varrho_{\nu'}@[u_1\delta']_E$. As $G\delta' = G\delta$, this is the same as $[v_1\delta]_E \in (CS(\bar{t})\,;ST)^{\nu'}\varrho_{\nu'}@[u_1\delta]_E$ and $E_0 \vDash \psi_1\delta'$ is the same as $E_0 \vDash \psi_1\delta$, so $\sigma_{\mathit{vars}(G)}\rho$ is a solution of G.

15. Rule $[c2]$ (call strategy):
$G = u_1 \to v_1/CS(\bar{t})^\mu\varrho_\mu\,;ST^\mu\varrho_\mu\ (\wedge\,\Delta) \mid \psi_1 \mid V,\mu$, where $CS(\bar{t})^\mu\varrho_\mu = CS^\mu(\bar{t}\mu\varrho_\mu)$, $G \rightsquigarrow_{[c2]} \bigwedge_{j=1}^m (l_j\gamma'\gamma \to r_j\gamma'\gamma/\texttt{idle}) \wedge u_1 \to v_1/ST_2\gamma\,;ST^\mu\varrho_\mu\ (\wedge\,\Delta) \mid \psi_2 \mid V,\mu = G'$, $G' \rightsquigarrow^*_{\sigma'} u_1\sigma' \to v_1\sigma'/(ST_2\gamma\,;ST^\mu\varrho_\mu)\sigma'\ (\wedge\,\Delta\sigma') \mid \psi_3 \mid V,(\mu\sigma')_V = G''$, and $G'' \rightsquigarrow^+_{\sigma''} \mathit{nil} \mid \psi \mid V,\nu$, call $\sigma = \sigma'\sigma''$, where $\nu = (\mu\sigma)_V$, `csd` $CS(\bar{x}) := ST_1\ \mathit{if}\ C \in \mathit{Call}^\mu_{\mathcal{R}}$, $C = \bigwedge_{j=1}^m l_j \to r_j \wedge \phi$, $\gamma = \{\bar{x} \mapsto \bar{t}\mu\varrho_\mu\}$, call $\overline{C} = \bar{l},\bar{r},\phi$, $ST_2\ \mathit{if}\ C\gamma'$ is a fresh version of $ST_1\ \mathit{if}\ C$, with some renaming γ', $\mathit{dom}(\gamma') = V_{ST_1,C} \setminus (\hat{x} \cup V^\mu)$, so $ST_2 = ST_1\gamma'$, $\psi_2 = \psi_1 \wedge \phi\gamma'\gamma$, and $\psi_3 = \psi_2\sigma' \wedge \psi_4 = \psi_1\sigma' \wedge \phi\gamma'\gamma\sigma' \wedge \psi_4$, for proper ψ_4, hence $V_{C\gamma'\gamma\sigma'} \subseteq V_{G'\sigma'} \subseteq V_{G''}$, call $\psi_5 = \phi\gamma'\gamma\sigma' \wedge \psi_4$, $\sigma_{V_G}|\psi$ is a computed answer for G,

and $\sigma_{V_{G'}}|\psi$ is a computed answer for G', where $\psi = \psi_3\sigma'' \wedge \psi_6$, for proper ψ_6. We call $\varrho_\nu = (\varrho_\mu\sigma)_{V_{G'}\setminus V}$.
By invariant 11, G has the form $G_0^\mu\varrho_\mu$, so $(u_1, v_1, \psi_1) = (u_0, v_0, \psi_0)\mu\varrho_\mu$, for proper u_0, v_0, and ψ_0, and there exists Δ_0 such that $\Delta = \Delta_0^\mu\varrho_\mu$. As $V_{CS^\mu(\bar{t}_\mu\varrho_\mu)} = ran(\gamma)$ and $\hat{x} \subseteq V_{\overline{C}\gamma', ST_2}$ then $V_G \subseteq V_{G'}$, so $\sigma_{V_G} = (\sigma_{V_{G'}})_{V_G}$ and G' has the form $G_1^\mu\varrho_\mu$, where $\varrho_\mu = (\varrho_\mu)_{V_{G_1}\setminus V}$, by invariant 11. Also by invariant 11, G'' has the form $G_2^{\mu'}\varrho_{\mu'}$, where $\mu' = (\mu\sigma')_V$ and $\varrho_{\mu'} = (\varrho_\mu\sigma')_{V_{G_2}\setminus V}$. Then, $\psi_3 = \psi_1\sigma' \wedge \psi_5 = (\psi_0\mu\varrho_\mu)\sigma' \wedge \psi_5 = \psi_0\mu'\varrho_{\mu'} \wedge \psi_5$ has the form $(\psi_0 \wedge \phi_0)\mu'\varrho_{\mu'}$, for proper ϕ_0, and $(u_1, v_1)\sigma' = (u_0, v_0)\mu\varrho_\mu\sigma' = (u_0, v_0)\mu'\varrho_{\mu'}$, so $V_{G_0} \subseteq V_{G_2}$, hence $V_G = V_{G_0^\mu\varrho_\mu} \subseteq V_{G_2^\mu\varrho_\mu}$, $V_{G\sigma'} \subseteq V_{G_2^\mu\varrho_\mu\sigma'} = V_{G_2^{\mu'}\varrho_{\mu'}} = V_{G''}$, and $V_{G\sigma} \subseteq V_{G_2^\mu\varrho_\mu\sigma} = V_{G_2^\mu\varrho_\mu\sigma'\sigma''} = V_{G_2^{\mu'}\varrho_{\mu'}\sigma''} = V_{G''\sigma''}$.
$(ST_2, \overline{C}) = (ST_2[\bar{x}']_{\bar{p}}, \overline{C}[\bar{x}'']_{\bar{q}})$, for proper $(\bar{x}'', \bar{x}', \bar{q}, \bar{p})$, where $V_{(ST_2[]_{\bar{p}}), \overline{C}[]_{\bar{q}}} \cap \hat{x} = \emptyset$ and $\hat{x}' \cup \hat{x}'' = \hat{x}$, and $\overline{C}\gamma' = \overline{C}[\bar{x}'']_{\bar{q}}\gamma' = \overline{C}\gamma'[\bar{x}''\gamma']_{\bar{q}} = \overline{C}\gamma'[\bar{x}'']_{\bar{q}}$, since $dom(\gamma') \cap \hat{x} = \emptyset$.
Call $\gamma_0 = \{\bar{x} \mapsto \bar{t}\}$, so $(ST_2, \overline{C}\gamma')\gamma = (ST_2[\bar{x}']_{\bar{p}}, \overline{C}\gamma'[\bar{x}'']_{\bar{q}})\gamma = (ST_2[\bar{x}'\gamma]_{\bar{p}}, \overline{C}\gamma'[\bar{x}''\gamma]_{\bar{q}})$. Then:

(a) as $dom(\varrho_\mu) \cap V^\mu = \emptyset$, invariant for admissible goals, and $(ST_2[]_{\bar{p}}, \overline{C}\gamma'[]_{\bar{q}})$ has only new variables except for V^μ, then $(ST_2[]_{\bar{p}}, \overline{C}\gamma'[]_{\bar{q}}) = (ST_2[]_{\bar{p}}, \overline{C}\gamma'[]_{\bar{q}})\varrho_\mu = (ST_2\varrho_\mu[]_{\bar{p}}, \overline{C}\gamma'\varrho_\mu[]_{\bar{q}}) = (ST_1\gamma'\varrho_\mu[]_{\bar{p}}, \overline{C}\gamma'\varrho_\mu[]_{\bar{q}})$, and

(b) as `sd` $CS(\bar{x})$:= ST_1 *if* $C \in Call_{\mathcal{R}}^\mu$ then there is a call strategy definition `sd` $CS(\bar{x})$:= ST_0 *if* $C' \in Call_{\mathcal{R}}$, $C' = \bigwedge_{j=1}^m l_j' \to r_j' \wedge \phi'$, call $\overline{C}' = \bar{l}', \bar{r}', \phi'$, such that $(ST_0, \overline{C}')^\mu = (ST_1, \overline{C})$, so $\overline{C}'\mu = \overline{C} = \overline{C}[\bar{x}'']_{\bar{q}}$, hence $\overline{C}'\mu = \overline{C}'\mu[\bar{x}'']_{\bar{q}}$ and $\overline{C}' = \overline{C}'[\bar{x}'']_{\bar{q}}$, since $dom(\mu) \cap \hat{x} = \emptyset$. Then, since $dom(\gamma') \cap V^\mu = \emptyset$ and $ran(\gamma') \cap dom(\mu) = \emptyset$:
   - $ST_2\gamma = ST_2[\bar{x}'\gamma]_{\bar{p}} = ST_1\gamma'\varrho_\mu[\bar{x}'\gamma]_{\bar{p}} = ST_0^\mu\gamma'[\bar{x}'\gamma_0\mu]_{\bar{p}}\varrho_\mu = ST_0^\mu\gamma'[\bar{x}'\gamma_0\mu]_{\bar{p}}\varrho_\mu = (ST_0\gamma'[\bar{x}'\gamma_0]_{\bar{p}})^\mu\varrho_\mu$, call $ST_0' = ST_0\gamma'[\bar{x}'\gamma_0]_{\bar{p}}$, and
   - $\overline{C}\gamma' = \overline{C}\gamma'[\bar{x}''\gamma]_{\bar{q}} = \overline{C}\gamma'\varrho_\mu[\bar{x}''\gamma]_{\bar{q}} = \overline{C}'^\mu\gamma'\varrho_\mu[\bar{x}''\gamma]_{\bar{q}} = \overline{C}'^\mu\gamma'\varrho_\mu[\bar{x}''\gamma_0\mu\varrho_\mu]_{\bar{q}} = \overline{C}'^\mu\gamma'[\bar{x}''\gamma_0\mu]_{\bar{q}}\varrho_\mu = (\overline{C}'\gamma'[\bar{x}''\gamma_0]_{\bar{q}})^\mu\varrho_\mu = (\overline{C}'\gamma'[\bar{x}'']_{\bar{q}}\gamma_0)^\mu\varrho_\mu = (\overline{C}'[\bar{x}'']_{\bar{q}}\gamma'\gamma_0)^\mu\varrho_\mu = (\overline{C}'\gamma'\gamma_0)^\mu\varrho_\mu$.

As $G'' = G_2^{\mu'}\varrho_{\mu'}$, then $G_2 = u_0 \to v_0/ST_0' \,; ST\ (\wedge\,\Delta_0) \mid \psi_0 \wedge \phi_0 \mid V, none$, hence $(ST_0' \,; ST)^{\mu'}\varrho_{\mu'}$ is a strategy in G''.
Let $\rho : V_{G\sigma} \to \mathcal{T}_\Sigma$ such that $\psi\rho$ is satisfiable, call $\delta = \sigma_{V_G}\rho$, so $\delta : V_G \to \mathcal{T}_\Sigma$, call $\nu' = (\nu\rho)_V = (\mu\sigma\rho)_V$, where $dom(\nu') = V$ and $ran(\nu') = \emptyset$, and call $\varrho_{\nu'} = (\varrho_\nu\rho)_{V_{G_1}\setminus V} = (\varrho_\mu\sigma\rho)_{V_{G_1}\setminus V}$. As $dom(\rho) = V_{G\sigma}$ and $V_G \subseteq V_{G_2^\mu\varrho_\mu}$, so $V_{G\sigma} \subseteq V_{G_2^\mu\varrho_\mu\sigma} = V_{G_2^\mu\varrho_\mu\sigma'\sigma''} = V_{G''\sigma''}$, then $dom(\rho) \subseteq V_{G''}\sigma''$. Let $\rho_1 : V_{G''\sigma''} \setminus V_{G\sigma} \to \mathcal{T}_\Sigma$, so $dom(\rho) \cap dom(\rho_1) = \emptyset$ and $dom(\rho) \cup dom(\rho_1) = V_{G''\sigma''}$, such that $\psi(\rho \uplus \rho_1)$ is satisfiable. Call $\rho' = \rho \uplus \rho_1$, so $\rho' : V_{G''\sigma''} \to \mathcal{T}_\Sigma$ and $\rho'_{V_{G\sigma}} = \rho$.
By I.H., as $\rho' : V_{G''\sigma''} \to \mathcal{T}_\Sigma$ and $\psi\rho'$ is satisfiable, $\sigma''_{V_{G''}}\rho'$ is a solution for G'', call $\delta' = \sigma''_{V_{G''}}\rho'$, so $\delta' : V_{G''} \to \mathcal{T}_\Sigma$ and $\psi_1\sigma'\delta'$ is ground. As $dom(\nu') = V$ and $ran(\nu') = \emptyset$ then $(\nu\rho')_V = (\nu\rho)_V = \nu'$. Also, as $V_{G\sigma'} \subseteq V_{G''}$ and $dom(\rho_1) = V_{G''\sigma''} \setminus V_{G\sigma}$, then $G\sigma'\delta' = G\sigma'\sigma''_{V_{G''}}\rho' = G\sigma'\sigma''_{V_{G\sigma'}}\rho' = G\sigma'\sigma''\rho' = G\sigma\rho' = G\sigma(\rho \uplus \rho_1) = G\sigma\rho = G\sigma_{V_G}\rho = G\delta$. Also, as $V_{\psi_1} \subseteq V_G$, so $\psi_1\sigma'\delta' = \psi_1\delta$, and $\psi_1\sigma'\delta'$ is a subformula of $\psi\rho'$, so $\psi_1\sigma'\delta'$ is ground and satisfiable, then $E_0 \vDash \psi_1\delta$.
As δ' is a solution for $G'' = G_2^{\mu'}\varrho_{\mu'}$ and $G_2 = u_0 \to v_0/ST_0' \,; ST\ (\wedge\,\Delta_0) \mid \psi_0 \wedge \phi_0 \mid V, none$, then $[v_0\mu'\varrho_{\mu'}\delta']_E \in (ST_0'; ST)^{\mu'}\varrho_{\mu'}\delta'@[u_0\mu'\varrho_{\mu'}\delta']_E$ (†). Now, as $(u_0, v_0)\mu\varrho_\mu = (u_1, v_1)$ and $\delta' = \sigma''_{V_{G''}}\rho'$, then we can write (†) as $[v_1\delta']_E \in (ST_0'; ST)^{\mu'}\varrho_{\mu'}\sigma''_{V_{G''}}\rho'@[u_1\delta']_E$ (††).
As $(ST_0'; ST)^{\mu'}\varrho_{\mu'}$ is a strategy in G'', then $(ST_0'; ST)^{\mu'}\varrho_{\mu'}\sigma''_{V_{G''}}\rho' = (ST_0'; ST)^{\mu'}\varrho_{\mu'}\sigma''\rho' = (ST_0' \,; ST)^\mu\varrho_\mu\sigma'\sigma''\rho' = (ST_0' \,; ST)^\mu\varrho_\mu\sigma\rho' = (ST_0' \,; ST)^\nu\varrho_\nu\rho' = (ST_0' \,; ST)^\nu\varrho_\nu\rho = (ST_0' \,; ST)^{\nu'}\varrho_{\nu'}$, because $G\sigma\rho' = G\sigma\rho$ and $(ST_0' \,; ST)^\nu\varrho_\nu$ is a strategy in $G\sigma$, so we

can write (††) as $[v_1\delta']_E \in (ST'_0\,; ST)^{\nu'}\varrho_{\nu'}@[u_1\delta']_E$, hence there is a closed proof tree $\dfrac{\dfrac{F_1}{u_1\delta' \to w/(ST'_0)^{\nu'}\varrho_{\nu'}} \quad \dfrac{F_2}{w \to v_1\delta'/ST^{\nu'}\varrho_{\nu'}}}{u_1\delta' \to v_1\delta'/(ST'_0\,; ST)^{\nu'}\varrho_{\nu'}}$, for some term $w \in \mathcal{H}_\Sigma$, with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$.

As ϱ_ν is idempotent and ρ is ground then $\varrho_{\nu'}$ is also idempotent. Then, as ν' is ground, $dom(\nu') = V$, and $dom(\varrho_{\nu'}) \cap V = \emptyset$, we can write $\dfrac{F_1}{u_1\delta' \to w/(ST'_0)^{\nu'}\varrho_{\nu'}}$ as $\dfrac{F_1}{u_1\delta' \to w/(ST_0\gamma'[\bar{x}'\gamma_0\varrho_{\nu'}]_{\bar{p}})^{\nu'}\varrho_{\nu'}}$. Let α be a renaming such that $dom(\alpha) = V_{\varrho_{\nu'}}$ and $ran(\alpha)$ is away from all known variables. By Lemma 5 there is a c.p.t. of the form $\dfrac{F_3}{u_1\delta' \to w/(ST_0\gamma'[\bar{x}'\gamma_0\varrho_{\nu'}]_{\bar{p}})^{\nu'}(\varrho_{\nu'}\alpha)}$. Now, we can apply Lemma 6, so there is also a c.p.t. of the form $\dfrac{F_4}{u_1\delta' \to w/(ST_0\gamma'[\bar{x}'\gamma_0\varrho_{\nu'}]_{\bar{p}})^{\nu'}}$.

As $G' \rightsquigarrow^*_{\sigma'} G''$ and all the calculus rules apply always to the leftmost open goal of any goal, then also $G''' = \bigwedge_{j=1}^m (l_j\gamma'\gamma \to r_j\gamma'\gamma/\texttt{idle}) \mid \psi_2 \mid V, \mu \rightsquigarrow^*_{\sigma'} nil \mid \psi_2\sigma' \wedge \psi_4 \mid V, (\mu\sigma')_V$. Then, by I.H., for every substitution $\theta : V_{G'''\sigma'} \to \mathcal{T}_\Sigma$ such that $(\psi_2\sigma' \wedge \psi_4)\theta$ is satisfiable, $\sigma'_{V_{G'''}}\theta$ is a solution of G''', so $\bar{l}\gamma'\gamma\sigma'\theta =_E \bar{r}\gamma'\gamma\sigma'\theta$ (†).

Call $\gamma'' = \{\bar{x} \mapsto \bar{t}(\nu' \uplus \varrho_{\nu'})\}$, so $CS^{\nu'}(\bar{t}\nu'\varrho_{\nu'}) = ST_0^{\nu'}\gamma''$ *if* $(C')^{\nu'}\gamma''$, call $C'' = (C')^{\nu'}(\gamma' \cup \gamma'') = C'\nu'(\gamma' \cup \gamma'')$, and call $\delta'' = \sigma'\delta'$. Then:

- $C''\delta''_{V_{C''}} = C''\delta'' = C'\nu'(\gamma' \cup \gamma'')\delta'' = C'\nu'(\gamma' \cup \gamma'')\delta'' = C'\nu'[\bar{x}'']_{\bar{q}}(\gamma' \cup \gamma'')\delta'' = C'\gamma'\nu'[\bar{x}'']_{\bar{q}}\gamma''\delta'' = C'\gamma'\nu'[\bar{t}(\nu' \uplus \varrho_{\nu'})]_{\bar{q}}\delta'' = C'\gamma'[\bar{t}\varrho_{\nu'}]_{\bar{q}}\nu'\delta'' = C'\gamma'[\bar{t}\varrho_{\nu'}]_{\bar{q}}\nu'\sigma'\delta' = C'\gamma'[\bar{t}\varrho_{\nu'}]_{\bar{q}}\nu'\sigma'\sigma''_{V_{G''}}\rho'$ since $(dom(\gamma') \cup ran(\gamma')) \cap V = \emptyset$, ν' is ground, $dom(\nu') = V$, and $dom(\varrho_{\nu'}) \cap V = \emptyset$,
- $C\gamma'\gamma = C'\mu\gamma'\gamma = C'\mu[\bar{x}'']_{\bar{q}}\gamma'\gamma = C'\gamma'\mu[\bar{x}'']_{\bar{q}}\gamma = C'\gamma'\mu[\bar{t}\mu\varrho_\mu]_{\bar{q}} = C'\gamma'[\bar{t}\varrho_\mu]_{\bar{q}}\mu = C'\gamma'[\bar{t}(\varrho_\mu)_{V_{G_1}\setminus V}]_{\bar{q}}\mu$, because γ' is a renaming such that $(dom(\gamma') \cup ran(\gamma')) \cap (dom(\mu) \cup ran(\mu) \cup \hat{x}) = \emptyset$ and $V_{\bar{t}} \subseteq V_{G_1} \setminus V$,
- as $C\gamma'\gamma = C'\gamma'[\bar{t}(\varrho_\mu)_{V_{G_1}\setminus V}]_{\bar{q}}\mu$, $V_{C\gamma'\gamma\sigma'} \subseteq V_{G''}$, $\sigma = \sigma'\sigma''$ is idempotent, and $\sigma'\sigma''_{V_{G''}}$ is a restriction of σ, hence also idempotent, then $C\gamma'\gamma\sigma'\delta' = C\gamma'\gamma\sigma'\sigma''_{V_{G''}}\rho' = C'\gamma'[\bar{t}(\varrho_\mu)_{V_{G_1}\setminus V}]_{\bar{q}}\mu\sigma'\sigma''_{V_{G''}}\rho' = C'\gamma'[\bar{t}(\varrho_\mu\sigma'\sigma''_{V_{G''}})_{V_{G_1}\setminus V}]_{\bar{q}}(\mu\sigma'\sigma''_{V_{G''}})_V\sigma'\sigma''_{V_{G''}}\rho' = C'\gamma'[\bar{t}(\varrho_\mu\sigma'\sigma'')_{V_{G_1}\setminus V}]_{\bar{q}}(\mu\sigma'\sigma'')_V\sigma'\sigma''_{V_{G''}}\rho' = C'\gamma'[\bar{t}(\varrho_\mu\sigma)_{V_{G_1}\setminus V}]_{\bar{q}}(\mu\sigma)_V\sigma'\sigma''_{V_{G''}}\rho'$, and
- as $\rho' = \rho \uplus \rho_1$, and ρ is ground, then $C'\gamma'[\bar{t}(\varrho_\mu\sigma)_{V_{G_1}\setminus V}]_{\bar{q}}(\mu\sigma)_V\sigma'\sigma''_{V_{G''}}\rho' = C'\gamma'[\bar{t}(\varrho_\mu\sigma\rho)_{V_{G_1}\setminus V}]_{\bar{q}}(\mu\sigma\rho)_V\sigma'\sigma''_{V_{G''}}\rho' = C'\gamma'[\bar{t}\varrho_{\nu'}]_{\bar{q}}\nu'\sigma'\sigma''_{V_{G''}}\rho'$,

so $C''\delta''_{V_{C''}} = C\gamma'\gamma\sigma'\delta'$. As $C\gamma'\gamma\sigma'\delta'$ is ground, then $\delta''_{V_{C''}}$ is ground, i.e., $\delta''_{V_{C''}} : V_{C''} \to \mathcal{T}_\Sigma$, and $\phi\gamma'\gamma\sigma'\delta'$ is ground.

As $\psi\rho'$ is satisfiable and $\psi\rho' = \psi_3\sigma''\rho' \wedge \psi_6\rho'$, then also $\psi_3\sigma''\rho' = (\psi_2\sigma' \wedge \psi_4)\sigma''\rho' = (\psi_2\sigma' \wedge \psi_4)\sigma''_{V_{G''}}\rho' = (\psi_2\sigma' \wedge \psi_4)\delta' = ((\psi_1 \wedge \phi\gamma'\gamma)\sigma' \wedge \psi_4)\delta'$ is satisfiable, so $\phi\gamma'\gamma\sigma'\delta'$, i.e., $\phi\gamma'\gamma\delta''$, is satisfiable. As $\phi\gamma'\gamma\sigma'\delta'$ is also ground, then $E_0 \vDash \phi\gamma'\gamma\delta''$.

By (†), as $(\psi_2\sigma' \wedge \psi_4)\delta'$ is satisfiable, $\bar{l}\gamma'\gamma\sigma'\delta' =_E \bar{r}\gamma'\gamma\sigma'\delta'$, i.e., $\bar{l}\gamma'\gamma\delta'' =_E \bar{r}\gamma'\gamma\delta''$. As also $CS(\bar{t})^{\nu'}\varrho_{\nu'} = CS(\bar{t}\nu'\varrho_{\nu'})$, then there are derivation rules $\dfrac{u_1\delta' \to w/ST_0^{\nu'}(\gamma' \cup \gamma'')\delta''}{u_1\delta' \to w/CS(\bar{t}\nu'\varrho_{\nu'})}$, i.e., $\dfrac{u_1\delta' \to w/(ST_0\gamma'[\bar{x}'\gamma_0\varrho_{\nu'}]_{\bar{p}})^{\nu'}}{u_1\delta' \to w/CS(\bar{t}\nu'\varrho_{\nu'})}$, and $\dfrac{u_1\delta' \to w/CS(\bar{t}\nu'\varrho_{\nu'}) \quad w \to v_1\delta'/ST^{\nu'}\varrho_{\nu'}}{u_1\delta' \to v_1\delta'/(CS(\bar{t})\,; ST)^{\nu'}\varrho_{\nu'}}$ in $\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$, so there is a c.p.t. $\dfrac{\dfrac{\dfrac{F_4}{u_1\delta' \to w/(ST_0\gamma'[\bar{x}'\gamma_0\varrho_{\nu'}]_{\bar{p}})^{\nu'}}}{u_1\delta' \to w/CS(\bar{t}\nu'\varrho_{\nu'})} \quad \dfrac{F_2}{w \to v_1\delta'/ST^{\nu'}\varrho_{\nu'}}}{u_1\delta' \to v_1\delta'/(CS(\bar{t})\,; ST)^{\nu'}\varrho_{\nu'}}$, and $v_1\delta' \in (CS(\bar{t}); ST)^{\nu'}\varrho_{\nu'}@u_1\delta'$. As $G\delta' = G\delta$, this is the same as $v_1\delta \in (CS(\bar{t})\,; ST)^{\nu'}\varrho_{\nu'}@u_1\delta$ so, as $E_0 \vDash \psi_1\delta$, $\sigma_{vars(G)}\rho$ is a solution of G.

16. Rule $[m]$ (match):
    $G = u_1 \to v_1$ / $(\texttt{match}\ t_1\ \texttt{s.t.}\ \bigwedge_{j=1}^m (l'_j = r'_j) \wedge \phi_1\,; ST)^\mu\varrho_\mu\ (\wedge\,\Delta) \mid \psi_1 \mid V, \mu \rightsquigarrow_{[m],\sigma_1} (\bigwedge_{j=1}^m (l'_j \to r'_j/\texttt{idle})^\mu\varrho_\mu \wedge u_1 \to v_1/ST^\mu\varrho_\mu\ (\wedge\,\Delta) \mid \psi_2 \mid V, \mu)\sigma_1 = G'\sigma_1$, call $t = t_1^\mu\varrho_\mu$, $\phi = \phi_1^\mu\varrho_\mu$, $\bar{l} = (\bar{l}')^\mu\varrho_\mu$, and $\bar{r} = (\bar{r}')^\mu\varrho_\mu$, where $abstract_{\Sigma_1}(t) = \langle \lambda\bar{x}.t^\circ; \sigma^\circ; \phi^\circ \rangle$,

$t^\circ = t[\bar{x}]_{\bar{q}}$, with $\bar{x} = x_1, \ldots, x_l$ and $\bar{q} = q_1, \ldots, q_l$, $\phi^\circ = (\bigwedge_{i=1}^{l} x_i = t|_{q_i})$, hence $V_{t^\circ} \cup V_{\phi^\circ} = V_t \cup \hat{x}$, $\sigma_1 \in CSU_B(u_1 = t^\circ)$, $\psi_2 = \psi_1 \wedge \phi \wedge \phi^\circ$, so $V_G \subseteq V_{G'}$, $\psi_2\sigma_1$ is satisfiable, and $G'\sigma_1 \rightsquigarrow^{+}_{\sigma'} nil \mid \psi \mid V, \nu$, call $\sigma = \sigma_1\sigma'$, where $\nu = (\mu\sigma)_V = (\mu\sigma_1\sigma')_V$, so $\sigma_{V_G} \mid \psi$ is a computed answer for G and $\sigma'_{V_{G'\sigma_1}} \mid \psi$ is a computed answer for $G'\sigma_1$.

Let $\rho : V_{G\sigma} \to \mathcal{T}_\Sigma$ be a substitution such that $\psi\rho$ is satisfiable, call $\delta = \sigma_{V_G}\rho$, so $\delta : V_G \to \mathcal{T}_\Sigma$, $\rho_1 = \rho_{V_{G'\sigma}}$, so also $\psi\rho_1$ is satisfiable, $\nu' = (\nu\rho)_V$, where $dom(\nu') = V$ and $ran(\nu') = \emptyset$, and $\varrho_{\nu'} = (\varrho_\mu\delta)_{\setminus V}$. As $dom(\rho) = V_{G\sigma}$ then $dom(\rho_1) = V_{G\sigma} \cap V_{G'\sigma}$. Let $\rho_2 = \rho_{V_{G\sigma} \setminus V_{G'\sigma}}$, so $\rho = \rho_1 \uplus \rho_2$, and let $\rho'_1 : V_{G'\sigma} \setminus V_{G\sigma} \to \mathcal{T}_\Sigma$, so $dom(\rho_1) \cap dom(\rho'_1) = \emptyset$ and $dom(\rho_1) \cup dom(\rho'_1) = V_{G'\sigma}$, such that $\psi(\rho_1 \uplus \rho'_1)$ is satisfiable, and call $\rho' = \rho_1 \uplus \rho'_1$, so $\rho' : V_{G'\sigma} \to \mathcal{T}_\Sigma$. By definition of ν and ρ_1, $ran(\nu) \cup (V \setminus dom(\nu)) \subseteq dom(\rho_1)$ so, as $dom(\nu') = V$ and $ran(\nu') = \emptyset$, $\nu' = (\nu\rho)_V = (\nu\rho_1)_V = (\nu\rho')_V$.

By I.H., as $\rho' : V_{G'\sigma_1\sigma'} \to \mathcal{T}_\Sigma$ and $\psi\rho'$ is satisfiable, $\sigma'_{V_{G'\sigma_1}}\rho'$ is a solution for $G'\sigma_1$, call $\delta' = \sigma_1\sigma'_{V_{G'\sigma_1}}\rho'$, $\varrho' = (\varrho_\mu\delta')_{\setminus V}$, and $\rho'' = \delta'_{V_{t,\phi,\bar{l},\bar{r}} \setminus V_G}$.

As in rule $[i1]$, if then else, and using the fact that $V_{\bar{l},\bar{r}} \subseteq V_{G'}$, we have the following intermediate results:

- $(\mu\delta)_V = (\mu\delta')_V$,
- $V_{(t,\phi,\bar{l},\bar{r})\sigma} \subseteq V_{G'\sigma}$,
- $V_{(t_1,\phi_1,\bar{l}',\bar{r}')^{\nu'}} \subseteq V_{(t_1,\phi_1,\bar{l}',\bar{r}')^{\mu}}$,
- $V_{(t_1,\phi_1,\bar{l}',\bar{r}')^{\mu}} \setminus V_{(t_1,\phi_1,\bar{l}',\bar{r}')^{\nu'}} \subseteq V^\mu$, and
- $(t, \phi)\sigma\rho' = (t_1, \phi_1)^{\nu'}\varrho_{\nu'}\rho''$.

Using the proof for the last result we also get $(\bar{l}, \bar{r})\sigma\rho' = (\bar{l}', \bar{r}')^{\nu'}\varrho_{\nu'}\rho''$.

As $\sigma'_{V_{G'\sigma_1}}\rho'$ is a solution for $G'\sigma_1$ then, by I.H.:

(a) $E_0 \vDash \psi_2\delta'$, i.e., $E_0 \vDash (\psi_1 \wedge \phi \wedge \phi^\circ)\delta'$,

(b) there are closed proof trees for each open goal in $\Delta\delta'$, with respect to $\mathcal{D}^{(\mu\delta')_V}_{\mathcal{R},Call_\mathcal{R}}$ ($=\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$, we use ν' instead of $(\mu\delta')_V$ in (c) and (d)),

(c) $[v_1\delta']_E \in ST^{\nu'}\varrho'@[u_1\delta']_E$, and

(d) $[r_j\delta']_E \in \texttt{idle}@[l_j\delta']_E$, for $1 \leq j \leq m$, i.e., $\bar{l}\delta' =_E \bar{r}\delta'$,

so:

(a)
   i. $V_{\psi_2} \subseteq V_{G'}$ implies $\psi_2\sigma_1\sigma'_{V_{G'\sigma_1}} = \psi_2\sigma_1\sigma' = \psi_2\sigma$, so $E_0 \vDash \psi_2\sigma\rho'$, where $\psi_2\sigma\rho'$ is ground, because $V_{\psi_2\sigma} \subseteq V_{G'\sigma}$ and $\rho' : V_{G'\sigma} \to \mathcal{T}_\Sigma$, hence $E_0 \vDash \psi_1\sigma\rho'$, $E_0 \vDash \phi^\circ\sigma\rho'$, and $E_0 \vDash \phi\sigma\rho'$, all ground expressions.
   ii. $V_{\psi_1\sigma} \subseteq V_{G\sigma}$ and $dom(\rho) = V_{G\sigma}$ implies $\psi_1\sigma\rho \in \mathcal{T}_\Sigma$ so, as $\rho' = \rho \uplus \rho'_1$, $\psi_1\sigma\rho' = \psi_1\sigma(\rho \uplus \rho'_1) = \psi_1\sigma\rho = \psi_1\delta$, hence $E_0 \vDash \psi_1\delta$ (†).

(b) As in subcase (a)-ii, $V_\Delta \subseteq V_G$ implies $\Delta\delta' = \Delta\delta$, and the same closed proof trees are valid for each open goal in $\Delta\delta$ with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$ (††).

(c) Again, $V_{v_1,u_1} \subseteq V_G$ implies that $v_1\delta' = v_1\delta$ and $u_1\delta' = u_1\delta$. Then there is a c.p.t. of the form $\frac{F}{u_1\delta \to v_1\delta / ST^{\nu'}\varrho'}$, with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$.

(d) As $(\bar{l}, \bar{r})\delta' = (\bar{l}, \bar{r})\sigma_1\sigma'_{V_{G'\sigma_1}}\rho' = (\bar{l}, \bar{r})\sigma_1\sigma'\rho' = (\bar{l}, \bar{r})\sigma\rho' = (\bar{l}', \bar{r}')^{\nu'}\varrho_{\nu'}\rho''$, then $(\bar{l}')^{\nu'}\varrho_{\nu'}\rho'' =_E (\bar{r}')^{\nu'}\varrho_{\nu'}\rho''$.

We prove (a) $ST^{\nu'}\varrho_{\nu'}\rho'' = ST^{\nu'}\varrho'$ and (b) $\rho'' : V_{t,\phi,\bar{l},\bar{r}} \setminus V_G \to \mathcal{T}_\Sigma$:

(a) As $\varrho_{\nu'} = (\varrho_\mu\delta)_{\setminus V}$, $\delta = \sigma_{V_G}\rho$, $\sigma = \sigma_1\sigma'$, $\delta' = \sigma_1\sigma'_{V_{G'\sigma_1}}\rho'$, $\varrho' = (\varrho_\mu\delta')_{\setminus V}$, and $V_{ST^{\nu'}} \cap V = \emptyset$ this is the same as $ST^{\nu'}\varrho_\mu(\sigma_1\sigma')_{V_G}\rho\rho'' = ST^{\nu'}\varrho_\mu\sigma_1\sigma'_{V_{G'\sigma_1}}\rho'$.
   Let $y \in V_{ST^{\nu'}\varrho_\mu}$, so $y \notin V$. There are two options:

i. $y \in V_G$. Then $V_{y\sigma_1} \subseteq V_{G\sigma_1} \subseteq V_{G'\sigma_1}$, so $y(\sigma_1\sigma')_{V_G} = y\sigma_1\sigma' = y\sigma_1\sigma'_{V_{G'\sigma_1}}$. Also $y(\sigma_1\sigma')_{V_G} = y\sigma$, hence $V_{y(\sigma_1\sigma')_{V_G}} \subseteq V_{G\sigma}$. Then, as $\rho : V_{G\sigma} \to \mathcal{T}_\Sigma$, $y(\sigma_1\sigma')_{V_G}\rho$ is ground, so $y(\sigma_1\sigma')_{V_G}\rho\rho'' = y(\sigma_1\sigma')_{V_G}\rho = y\sigma_1\sigma'_{V_{G'\sigma_1}}\rho = y\sigma_1\sigma'_{V_{G'\sigma_1}}(\rho\cup\rho'_1) = y\sigma_1\sigma'_{V_{G'\sigma_1}}\rho'$;

ii. $y \notin V_G$, so $y(\sigma_1\sigma')_{V_G} = y$. As $ran(\sigma) \cap V_{ST^\mu\varrho_\mu} = \emptyset$ and $V_{ST^{\nu'}\varrho_\mu} \subseteq V_{ST^\mu\varrho_\mu}$ then $ran(\sigma)\cap V_{ST^{\nu'}\varrho_\mu} = \emptyset$ so $y \notin V_{G\sigma}$ and, as $dom(\rho) = V_{G\sigma}$, $y(\sigma_1\sigma')_{V_G}\rho = y$. Then:

A. if $y \in V_{t,\phi,\bar{l},\bar{r}}$ then $y(\sigma_1\sigma')_{V_G}\rho\rho'' = y\rho'' = y\delta'_{V_{t,\phi,\bar{l},\bar{r}}\setminus V_G} = y\delta' = y\sigma_1\sigma'_{V_{G'\sigma_1}}\rho'$, ground term because $V_{y\sigma_1} \subseteq V_{(t,\phi,\bar{l},\bar{r})\sigma_1} \subseteq V_{G'\sigma_1}$ and $\rho' : V_{G'\sigma_1\sigma'} \to \mathcal{T}_\Sigma$;

B. if $y \notin V_{t,\phi,\bar{l},\bar{r}}$ then $y(\sigma_1\sigma')_{V_G}\rho\rho'' = y\rho'' = y\delta'_{V_{t,\phi,\bar{l},\bar{r}}\setminus V_G} = y$. As $dom(\sigma_1) \subseteq (V_{u_1} \cup V_{t^\circ}) \subseteq (V_G \cup V_{t,\phi} \cup \bar{x}) \subseteq (V_G \cup V_{t,\phi,\bar{l},\bar{r}} \cup \bar{x})$ and $y \notin (V_G \cup V_{t,\phi,\bar{l},\bar{r}})$ then $y\sigma_1 = y$ so, as $ran(\sigma_1) \cap V_{ST^{\nu'}\varrho_\mu} = \emptyset$, $y\sigma_1 \notin V_{G\sigma_1}$, and $y\sigma_1\sigma'_{V_{G'\sigma_1}} = y \notin V_{G\sigma_1\sigma'_{V_{G'\sigma_1}}}$ so, as $\rho' : V_{G'\sigma_1\sigma'} \to \mathcal{T}_\Sigma$, $y\sigma_1\sigma'_{V_{G'\sigma_1}}\rho' = y = y(\sigma_1\sigma')_{V_G}\rho\rho''$.

(b) As $dom(\rho'') \subseteq (V_{t,\phi,\bar{l},\bar{r}} \setminus V_G)$ and, from (a.ii.A), $y \in (V_{t,\phi,\bar{l},\bar{r}} \setminus V_G) \implies V_{y\rho''} = \emptyset$ then $dom(\rho'') = V_{t,\phi,\bar{l},\bar{r}} \setminus V_G$, hence $\rho'' : V_{t,\phi,\bar{l},\bar{r}} \setminus V_G \to \mathcal{T}_\Sigma$.

Now, we prove (a) $dom(\rho'') = V_{(t_1,\phi_1,\bar{l}',\bar{r}')^{\nu'}\varrho_{\nu'}}$, (b) $E_0 \vDash \phi_1^{\nu'}\varrho_{\nu'}\rho''$, and (c) $u_1\delta =_E t_1^{\nu'}\varrho_{\nu'}\rho''$:

(a) As $dom(\rho'') = V_{t,\phi,\bar{l},\bar{r}}\setminus V_G$, $V_{(t_1,\phi_1,\bar{l}',\bar{r}')^{\nu'}} \subseteq V_{(t_1,\phi_1,\bar{l}',\bar{r}')^{\mu}}$, $V_{(t_1,\phi_1,\bar{l}',\bar{r}')^{\mu}}\setminus V_{(t_1,\phi_1,\bar{l}',\bar{r}')^{\nu'}} \subseteq V^\mu \subseteq V_G$, and $dom(\varrho_\mu)\cap V^\mu = \emptyset$, then $dom(\rho'') = V_{(t,\phi,\bar{l},\bar{r})} \setminus V_G = V_{(t_1,\phi_1,\bar{l}',\bar{r}')^{\mu}\varrho_\mu} \setminus V_G = V_{(t_1,\phi_1,\bar{l}',\bar{r}')^{\nu'}\varrho_\mu} \setminus V_G$. Also, as $\varrho_{\nu'} = (\varrho_\mu\delta)_{\setminus V}$ and $V_{(t_1,\phi_1,\bar{l}',\bar{r}')^{\nu'}} \cap V = \emptyset$, then $V_{(t_1,\phi_1,\bar{l}',\bar{r}')^{\nu'}\varrho_{\nu'}} = V_{(t_1,\phi_1,\bar{l}',\bar{r}')^{\nu'}(\varrho_\mu\delta)_{\setminus V}} = V_{(t_1,\phi_1,\bar{l}',\bar{r}')^{\nu'}\varrho_\mu\delta}$, so we prove $V_{(t_1,\phi_1,\bar{l}',\bar{r}')^{\nu'}\varrho_\mu\delta} = V_{(t_1,\phi_1,\bar{l}',\bar{r}')^{\nu'}\varrho_\mu} \setminus V_G$, which is trivial, since $\delta : V_G \to \mathcal{T}_\Sigma$.

(b) Immediate, since $E_0 \vDash \phi\sigma\rho'$ and $\phi\sigma\rho' = \phi_1^{\nu'}\varrho_{\nu'}\rho''$.

(c) $u_1\sigma_1 =_B t^\circ\sigma_1$ and $\sigma = \sigma_1\sigma'$ imply $u_1\sigma =_B t^\circ\sigma$ so, as $V_{u_1} \subseteq V_G$, $u_1\sigma_{vars(G)} = u_1\sigma =_B t^\circ\sigma$. As $\rho : V_{G\sigma} \to \mathcal{T}_\Sigma$, so $u_1\sigma_{V_G}\rho$ is a ground term, and $\rho' = \rho \uplus \rho'_1$ then $u_1\delta = u_1\sigma_{V_G}\rho = u_1\sigma_{V_G}\rho' =_B t^\circ\sigma\rho' = t[\bar{x}]_{\bar{q}}\sigma\rho' = t\sigma\rho'[\bar{x}\sigma\rho']_{\bar{q}}$.
As $E_0 \vDash \phi^\circ\sigma\rho'$ then $t\sigma\rho'[\bar{x}\sigma\rho']_{\bar{q}} =_{E_0} t\sigma\rho'[t|_{q_1}\sigma\rho', \ldots, t|_{q_l}\sigma\rho']_{\bar{q}} = t\sigma\rho'[t\sigma\rho'|_{\bar{q}}]_{\bar{q}} = t\sigma\rho' = t_1^{\nu'}\varrho_{\nu'}\rho''$, because $t\sigma\rho' = t_1^{\nu'}\varrho_{\nu'}\rho''$, so $u_1\delta =_B t^\circ\sigma\rho' =_{E_0} t_1^{\nu'}\varrho_{\nu'}\rho''$, i.e., $u_1\delta =_E t_1^{\nu'}\varrho_{\nu'}\rho''$.

Then, as $\rho'' : V_{(t_1,\phi_1,\bar{l}',\bar{r}')^{\nu'}\varrho_{\nu'}} \to \mathcal{T}_\Sigma$, $E_0 \vDash \phi_1^{\nu'}\varrho_{\nu'}\rho''$, and $(\bar{l}')^{\nu'}\varrho_{\nu'}\rho'' =_E (\bar{r}')^{\nu'}\varrho_{\nu'}\rho''$, there is a derivation rule $\frac{}{w \to w/\texttt{match}\ t_1^{\nu'}\varrho_{\nu'}\ \texttt{s.t.}\ \phi_1^{\nu'}\varrho_{\nu'}} \in \mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$, for some term w such that $t_1^{\nu'}\varrho_{\nu'}\rho'' =_E w$. As $u_1\delta =_E t_1^{\nu'}\varrho_{\nu'}\rho''$, then

$$\frac{\dfrac{}{u_1\delta \to u_1\delta/\texttt{match}\ t_1^{\nu'}\varrho_{\nu'}\ \texttt{s.t.}\ \phi_1^{\nu'}\varrho_{\nu'}} \quad \dfrac{F}{u_1\delta \to v_1\delta/ST^{\nu'}\varrho_{\nu'}}}{u_1\delta \to v_1\delta/\texttt{match}\ t_1^{\nu'}\varrho_{\nu'}\ \texttt{s.t.}\ \phi_1^{\nu'}\varrho_{\nu'} ; ST^{\nu'}\varrho_{\nu'}}$$

is a c.p.t., $\rho : vars(G\sigma) \to \mathcal{T}_\Sigma$, $\psi\rho$ is satisfiable, $E_0 \vDash \psi_1\delta$ (†), and there are closed proof trees for each open goal in $\Delta\delta$ with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$ (††), hence $\sigma_{vars(G)}\rho$ is a solution of G.

17. Rule $[w]$ (matchrew):
$MS = \texttt{matchrew}\ t_1\ \texttt{s.t.}\ C_1\ \texttt{by}\ z_1\ \texttt{using}\ ST_1, \ldots, z_n\ \texttt{using}\ ST_n$, call $\bar{z} = \{z_1, \ldots, z_n\}$, where $t_1 = t_1[\bar{z}]_{\bar{p}}$, for proper $\bar{p} = \{p_1, \ldots, p_n\}$. $G = u_1 \to v_1/(MS\ ; ST)^\mu\varrho_\mu\ (\wedge\ \Delta) \mid \psi_1 \mid V, \mu$, where $C_1 = \bigwedge_{j=1}^m (l'_j = r'_j) \wedge \phi_1$, call $t = t_1^\mu\varrho_\mu$, $\phi = \phi_1^\mu\varrho_\mu$, $\bar{l} = (\bar{l}')^\mu\varrho_\mu$, and $\bar{r} = (\bar{r}')^\mu\varrho_\mu$.

Now, $G \rightsquigarrow_{[w],\sigma_1} (\bigwedge_{j=1}^{m}(l_j\gamma \to r_j\gamma/\mathtt{idle}) \wedge \bigwedge_{i=1}^{n}(x_i \to y_i/ST_i^{\mu}\varrho_\mu\gamma;\mathtt{idle}) \wedge t[\bar{y}]_{\bar{p}} \to v_1/\ ST^{\mu}\varrho_\mu\ (\wedge\ \Delta) \mid \psi_2 \mid V,\mu)\sigma_1 = G'\sigma_1$, where $\bar{x}$ and $\bar{y}$ are fresh versions of $\bar{z}$, γ is a renaming from $\bar{z}$ to $\bar{x}$, $abstract_{\Sigma_1}(t[\bar{x}]_{\bar{p}}) = \langle\lambda\bar{z}.t^{\circ};\sigma^{\circ};\phi^{\circ}\rangle$, $t^{\circ} = t[z_1,\ldots,z_l]_{q_1\ldots q_l}$, $\phi^{\circ} = (\bigwedge_{i=1}^{l} z_i = t|_{q_i})$, $\sigma_1 \in CSU_B(u_1 = t^{\circ})$, $\psi_2 = \psi_1 \wedge \phi \wedge \phi^{\circ}$, so $V_G \subseteq V_{G'}$, $\psi_2\sigma_1$ is satisfiable, $G'\sigma_1 \rightsquigarrow_{\sigma_2}^{*} \bigwedge_{i=1}^{n}(x_i \to y_i/ST_i^{\mu}\varrho_\mu\gamma;\mathtt{idle})\sigma_1\sigma_2 \wedge (t[\bar{y}]_{\bar{p}} \to v_1/ST^{\mu}\varrho_\mu)\sigma_1\sigma_2(\wedge\Delta\sigma_1\sigma_2) \mid \psi_3 \mid V,(\mu\sigma_1\sigma_2)_V = G''$, and $G'' \rightsquigarrow_{\sigma''}^{+} nil \mid \psi \mid V,\nu$, call $\sigma' = \sigma_2\sigma''$ and $\sigma = \sigma_1\sigma'$, where $\nu = (\mu\sigma)_V$, so $\sigma_{V_G} \mid \psi$ is a computed answer for G and $\sigma'_{V_{G'\sigma_1}} \mid \psi$ is a computed answer for $G'\sigma_1$.

Let $\rho : V_{G\sigma} \to \mathcal{T}_\Sigma$ be a substitution such that $\psi\rho$ is satisfiable, call $\delta = \sigma_{V_G}\rho$, so $\delta : V_G \to \mathcal{T}_\Sigma$, $\rho_1 = \rho_{V_{G'\sigma}}$, so also $\psi\rho_1$ is satisfiable, and $\nu' = (\nu\rho)_V$, where $dom(\nu') = V$ and $ran(\nu') = \emptyset$. As $dom(\rho) = V_{G\sigma}$ then $dom(\rho_1) = V_{G\sigma} \cap V_{G'\sigma}$. Let $\rho_2 = \rho_{V_{G\sigma}\setminus V_{G'\sigma}}$, so $\rho = \rho_1 \uplus \rho_2$, and let $\rho'_1 : V_{G'\sigma}\setminus V_{G\sigma} \to \mathcal{T}_\Sigma$, so $dom(\rho_1) \cap dom(\rho'_1) = \emptyset$ and $dom(\rho_1) \cup dom(\rho'_1) = V_{G'\sigma}$, such that $\psi(\rho_1 \uplus \rho'_1)$ is satisfiable, and call $\rho' = \rho_1 \uplus \rho'_1$, so $\rho' : V_{G'\sigma} \to \mathcal{T}_\Sigma$, call $\delta' = \sigma_{V_{G'}}\rho'$, $\delta'_{\bar{x}} = \delta'_{V_{\bar{x}}}$, and $\delta'_{\bar{y}} = \delta'_{V_{\bar{y}}}$. By definition of ν and ρ_1, $ran(\nu) \cup (V \setminus dom(\nu)) \subseteq dom(\rho_1)$ so, as $dom(\nu') = V$ and $ran(\nu') = \emptyset$, $\nu' = (\nu\rho)_V = (\nu\rho_1)_V = (\nu\rho')_V$.

By I.H., as $\rho' : V_{G'\sigma_1\sigma'} \to \mathcal{T}_\Sigma$ and $\psi\rho'$ is satisfiable, $\sigma'_{V_{G'\sigma_1}}\rho'$ is a solution for $G'\sigma_1$, call $\delta' = \sigma_1\sigma'_{V_{G'\sigma_1}}\rho'$, $\varrho' = (\varrho_\mu\delta')_{\setminus V}$, and $\rho'' = \delta'_{V_{t,\phi,\bar{l},\bar{r}}\setminus V_G}$.

As in rule $[m]$, match, we have the following intermediate results:

- $(\mu\delta)_V = (\mu\delta')_V$,
- $V_{(t,\phi,\bar{l},\bar{r})\sigma} \subseteq V_{G'\sigma}$,
- $V_{(t_1,\phi_1,\bar{l}',\bar{r}')^{\nu'}} \subseteq V_{(t_1,\phi_1,\bar{l}',\bar{r}')^{\mu}}$,
- $V_{(t_1,\phi_1,\bar{l}',\bar{r}')^{\mu}} \setminus V_{(t_1,\phi_1,\bar{l}',\bar{r}')^{\nu'}} \subseteq V^{\mu}$, and
- $(t,\phi,\bar{l},\bar{r})\sigma\rho' = (t_1,\phi_1,\bar{l}',\bar{r}')^{\nu'}\varrho_{\nu'}\rho''$.

As $\sigma'_{V_{G'\sigma_1}}\rho'$ is a solution for $G'\sigma_1$ then, by I.H.:

(a) $E_0 \vDash \psi_2\delta'$, i.e., $E_0 \vDash (\psi_1 \wedge \phi \wedge \phi^{\circ})\delta'$,

(b) there are closed proof trees for each open goal in $\Delta\delta'$, with respect to $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}^{(\mu\delta')_V}$ ($=\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}^{\nu'}$, we use ν' instead of $(\mu\delta')_V$ in (c)-(e)),

(c) $[v_1\delta']_E \in ST^{\nu'}\varrho'@[t[\bar{y}]_{\bar{p}}\delta']_E$,

(d) $[r_j\delta']_E \in \mathtt{idle}@[l_j\delta']_E$, for $1 \le j \le m$, i.e., $\bar{l}\delta' =_E \bar{r}\delta'$, and

(e) $[y_i\delta']_E \in ST_i^{\nu'}\varrho'@[x_i\delta']_E$, for $1 \le i \le n$,

so:

(a)
 i. $V_{\psi_2} \subseteq V_{G'}$ implies $\psi_2\sigma_1\sigma'_{V_{G'\sigma_1}} = \psi_2\sigma_1\sigma' = \psi_2\sigma$, so $E_0 \vDash \psi_2\sigma\rho'$, where $\psi_2\sigma\rho'$ is ground, because $V_{\psi_2\sigma} \subseteq V_{G'\sigma}$ and $\rho' : V_{G'\sigma} \to \mathcal{T}_\Sigma$, hence $E_0 \vDash \psi_1\sigma\rho'$, $E_0 \vDash \phi^{\circ}\sigma\rho'$, and $E_0 \vDash \phi\sigma\rho'$, so also $E_0 \vDash \phi_1^{\nu'}\varrho_{\nu'}\rho''$ (†), all ground expressions.
 ii. $V_{\psi_1\sigma} \subseteq V_{G\sigma}$ and $dom(\rho) = V_{G\sigma}$ implies $\psi_1\sigma\rho \in \mathcal{T}_\Sigma$ so, as $\rho' = \rho \uplus \rho'_1$, $\psi_1\sigma\rho' = \psi_1\sigma(\rho \uplus \rho'_1) = \psi_1\sigma\rho = \psi_1\delta$, hence $E_0 \vDash \psi_1\delta$ (††).

(b) As in subcase (a)-ii, $V_\Delta \subseteq V_G$ implies $\Delta\delta' = \Delta\delta$, and the same closed proof trees are valid for each open goal in $\Delta\delta$ with respect to $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}^{\nu'}$ (†††).

(c) Again, $V_{v_1} \subseteq V_G$ implies that $v_1\delta' = v_1\delta$. Then there is a c.p.t. of the form $\frac{F}{t[\bar{y}]_{\bar{p}}\delta' \to v_1\delta/ST^{\nu'}\varrho'}$, with respect to $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}^{\nu'}$.

(d) As $(\bar{l},\bar{r})\delta' = (\bar{l},\bar{r})\sigma_1\sigma'_{V_{G'\sigma_1}}\rho' = (\bar{l},\bar{r})\sigma_1\sigma'\rho' = (\bar{l},\bar{r})\sigma\rho' = (\bar{l}',\bar{r}')^{\nu'}\varrho_{\nu'}\rho''$, then $(\bar{l}')^{\nu'}\varrho_{\nu'}\rho'' =_E (\bar{r}')^{\nu'}\varrho_{\nu'}\rho''$.

(e) As in the previous subcase, $(\bar{x},\bar{y})\delta' = (\bar{x}',\bar{y}')^{\nu'}\varrho_{\nu'}\rho''$, so there are closed proof trees of the form $\frac{F_i}{x_i^{\nu'}\varrho_{\nu'}\rho'' \to y_i^{\nu'}\varrho_{\nu'}\rho''/ST_i^{\nu'}\varrho'}$, for $1 \le i \le n$, with respect to $\mathcal{D}_{\mathcal{R},Call_{\mathcal{R}}}^{\nu'}$.

Using the same proofs shown in rule $[m]$, match, we get $ST^{\nu'}\varrho_{\nu'}\rho'' = ST^{\nu'}\varrho'$ and $\rho'' : V_{t,\phi,\bar{l},\bar{r}} \setminus V_G \to \mathcal{T}_\Sigma$.

Also using these proofs, we get: (a) $dom(\rho'') = V_{(t_1,\phi_1,\bar{l}',\bar{r}')^{\nu'}\varrho_{\nu'}}$, (b) $E_0 \vDash \phi_1^{\nu'}\varrho_{\nu'}\rho''$, and (c) $u_1\delta =_E t_1^{\nu'}\varrho_{\nu'}\rho''$.

As $V_{(t_1,\phi_1,\bar{l}',\bar{r}')^{\nu'}\varrho_{\nu'}} \subseteq V_{(t_1,\phi_1,\bar{l}',\bar{r}')^{\mu}\varrho_\mu} \subseteq V_{MS^\mu\varrho_\mu}$, then $\rho''_{V_{MS^\mu\varrho_\mu}} = \rho''$, so $ran(\rho''_{V_{MS^\mu\varrho_\mu}}) \subseteq \mathcal{T}_\Sigma \subseteq \mathcal{T}_\Sigma(\mathcal{X})$ and, as $t_1 = t_1[\bar{x}]_{\bar{p}}$, $(\bar{l}')^{\nu'}\varrho_{\nu'}\rho'' =_E (\bar{r}')^{\nu'}\varrho_{\nu'}\rho''$ and $E_0 \vDash \phi_1^{\nu'}\varrho_{\nu'}\rho''$ (†), there is a derivation rule $\frac{x_1^{\nu'}\varrho_{\nu'}\rho'' \to y_1^{\nu'}\varrho_{\nu'}\rho''/ST_1^{\nu'}\varrho' \cdots x_n^{\nu'}\varrho_{\nu'}\rho'' \to y_n^{\nu'}\varrho_{\nu'}\rho''/ST_n^{\nu'}\varrho'}{t_1^{\nu'}\varrho_{\nu'}\rho'' \to t_1[\bar{y}]_{\bar{p}}^{\nu'}\varrho_{\nu'}\rho''/MS^{\nu'}\varrho'}$ in $\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$.

Also $\frac{u_1\delta \to t_1[\bar{y}]_{\bar{p}}^{\nu'}\varrho_{\nu'}\rho''/MS^{\nu'}\varrho_{\nu'} \quad t_1[\bar{y}]_{\bar{p}}^{\nu'}\varrho_{\nu'}\rho'' \to v_1\delta/ST^{\nu'}\varrho_{\nu'}}{u_1\delta \to v_1\delta/(MS\,;ST)^{\nu'}\varrho_{\nu'}}$ is a derivation rule in $\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$.

As $u_1\delta =_E t_1^{\nu'}\varrho_{\nu'}\rho''$, then

$$\frac{\dfrac{\dfrac{F_1}{x_1^{\nu'}\varrho_{\nu'}\rho'' \to y_1^{\nu'}\varrho_{\nu'}\rho''/ST_1^{\nu'}\varrho'} \cdots \dfrac{F_n}{x_n^{\nu'}\varrho_{\nu'}\rho'' \to y_n^{\nu'}\varrho_{\nu'}\rho''/ST_n^{\nu'}\varrho'}}{u_1\delta \to t_1[\bar{y}]_{\bar{p}}^{\nu'}\varrho_{\nu'}\rho''/MS^{\nu'}\varrho_{\nu'}} \qquad \dfrac{F}{t_1[\bar{y}]_{\bar{p}}^{\nu'}\varrho_{\nu'}\rho'' \to v_1\delta/ST^{\nu'}\varrho_{\nu'}}}{u_1\delta \to v_1\delta/(MS\,;ST)^{\nu'}\varrho_{\nu'}}$$

is a c.p.t. with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$. As $\rho : vars(G\sigma) \to \mathcal{T}_\Sigma$, $\psi\rho$ is satisfiable, $E_0 \vDash \psi_1\delta$ (††), and there are closed proof trees for each open goal in $\Delta\delta$ with respect to $\mathcal{D}^{\nu'}_{\mathcal{R},Call_\mathcal{R}}$ (†††), then $\sigma_{vars(G)}\rho$ is a solution of G.

□

Lemma 7. Given $\mathcal{R}_B = (\Sigma, E_0 \cup B, R_B)$, an associated rewrite theory of $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ closed under B-extensions, and a goal $G = \bigwedge_{j=1}^m (l_j \to r_j/\mathtt{idle}) \wedge \Delta^\mu\varrho_\mu \mid \psi \mid V,\mu$, if α is a ground substitution such that $V_G \subseteq dom(\alpha)$, $E_0 \vDash \psi\alpha$, and $\bar{l}\alpha =_E \bar{r}\alpha$, then there exist a ground substitution α°, substitutions $\beta_1,\ldots,\beta_m$ from CSUs, let $\beta_i^k = \beta_i\beta_{i+1}\cdots\beta_k$, and abstractions $abstract_{\Sigma_1}((l_j\beta_1^{j-1}, r_j\beta_1^{j-1})) = \langle\lambda(\bar{x}_j,\bar{y}_j).(l_j^\circ, r_j^\circ);(\theta_{l_j}^\circ,\theta_{r_j}^\circ);(\phi_{l_j}^\circ,\phi_{r_j}^\circ)\rangle$, for $1 \le j \le m$, where $\beta_1^0 = none$, let $\beta = \beta_1^m$, such that $dom(\alpha^\circ) = dom(\alpha) \cup V_{\hat{x},\hat{y}}$, $\alpha =_{E_0} \alpha^\circ_{dom(\alpha)}$, $\bar{l}^\circ\alpha^\circ =_E \bar{r}^\circ\alpha^\circ$, $\alpha^\circ \ll_E \beta_{dom(\alpha^\circ)}$, $G \rightsquigarrow^m_{[d1]} \Delta^\nu\varrho_\nu \mid \psi\beta \wedge \bigwedge_{j=1}^m(\phi_{l_j}^\circ \wedge \phi_{r_j}^\circ)\beta_j^m \mid V,\nu$, and for every pair of substitutions ρ and γ such that $ran(\rho)$ is away from all known variables, $\alpha^\circ \ll_E (\beta\rho)_{dom(\alpha^\circ)}$, and $\alpha^\circ =_E (\beta\rho)_{dom(\alpha^\circ)}\cdot\gamma$, it holds that $E_0 \vDash (\psi\beta \wedge \bigwedge_{j=1}^m(\phi_{l_j}^\circ \wedge \phi_{r_j}^\circ)\beta_j^m)\rho\gamma$ and $\Delta^\mu\varrho_\mu\alpha =_E \Delta^\mu\varrho_\mu\beta\rho\gamma$.

Proof. The proof is by induction over m, the number of equational conditions. We also prove that $dom(\beta) \subseteq dom(\alpha^\circ) \cup \bigcup_{j=1}^{m-1} ran(\beta_j)$ (∗).

1. Base case, $m = 1$:

   $\mathbf{G} = l \to r/\mathtt{idle} \wedge \Delta^\mu\varrho_\mu \mid \psi \mid V,\mu$, α is a ground substitution, $V_G \subseteq dom(\alpha)$, $E_0 \vDash \psi\alpha$, $l\alpha =_E r\alpha$, and $abstract_{\Sigma_1}((l\beta_1^0, r\beta_1^0)) = abstract_{\Sigma_1}((l,r)) = \langle\lambda(\bar{x},\bar{y}).(l^\circ,r^\circ);(\theta_l^\circ,\theta_r^\circ);(\phi_l^\circ,\phi_r^\circ)\rangle$. , where $l^\circ = l[\bar{x}]_{\bar{p}}$, $r^\circ = r[\bar{y}]_{\bar{q}}$, $\phi_l^\circ = \bigwedge_{i=1}^{i_x} x_i = l|_{p_i}$, and $\phi_r^\circ = \bigwedge_{i=1}^{i_y} y_i = r|_{q_i}$ for proper $\bar{p}$, $\bar{q}$, i_x, and i_y, so $V_{l^\circ,r^\circ,\phi_l^\circ,\phi_r^\circ} = V_{l,r} \cup \hat{x} \cup \hat{y} \subseteq V_G \cup \hat{x} \cup \hat{y} \subseteq dom(\alpha) \cup \hat{x} \cup \hat{y} = dom(\alpha^\circ)$, hence $V_{\phi_l^\circ,\phi_r^\circ} \subseteq dom(\alpha^\circ)$. As $V_{\phi_l^\circ,\phi_r^\circ} \subset \mathcal{X}_0$ then also $V_{\phi_l^\circ,\phi_r^\circ} \subseteq dom(\alpha^\circ) \cap \mathcal{X}_0$. Then:

   - by Lemma 4, there exists a ground substitution α° such that $l^\circ\alpha^\circ =_B r^\circ\alpha^\circ$, $E_0 \vDash (\phi_l^\circ \wedge \phi_r^\circ)\alpha^\circ$, $dom(\alpha^\circ) = dom(\alpha) \cup \hat{x} \cup \hat{y}$, so $V_{(l^\circ,r^\circ,\phi_l^\circ,\phi_r^\circ)\sigma^\circ} = \emptyset$, and $\alpha =_{E_0} \alpha^\circ_{dom(\alpha)}$, hence there also exists a substitution $\beta_1 \in CSU_B(l^\circ = r^\circ)$, where in this base case $\beta = \beta_1^1 = \beta_1$, such that $dom(\beta) \subseteq dom(\alpha^\circ) = dom(\alpha) \cup \hat{x} \cup \hat{y}$ (∗) and $\alpha^\circ \ll_B \beta$. As $\beta \ll \beta_{dom(\alpha^\circ)}$ then $\alpha^\circ \ll_B \beta_{dom(\alpha^\circ)}$, hence $\alpha^\circ \ll_E \beta_{dom(\alpha^\circ)}$;
   - as $E_0 \vDash (\phi_l^\circ \wedge \phi_r^\circ)\alpha^\circ$, $\psi\alpha$ is satisfiable, $dom(\alpha^\circ) = dom(\alpha) \cup \hat{x} \cup \hat{y}$, $V_\psi \cap (\hat{x} \cup \hat{y}) = \emptyset$, so $\psi\alpha =_{E_0} \psi\alpha^\circ$ hence $\psi\alpha^\circ$ is satisfiable, and $\alpha^\circ \ll_B \beta$, so $\alpha^\circ_{\mathcal{X}_0} \ll \beta_{\mathcal{X}_0}$, then $(\psi \wedge \phi_l^\circ \wedge \phi_r^\circ)\beta$ is satisfiable, and $\mathbf{G} \rightsquigarrow^1_{[d1]} \Delta^\mu\varrho_\mu\beta \mid (\psi \wedge \phi_l^\circ \wedge \phi_r^\circ))\beta \mid V,(\mu\beta)_V$;

- let ρ such that $\alpha^\circ \ll_E (\beta\rho)_{dom(\alpha^\circ)}$ and let γ such that $\alpha^\circ =_E (\beta\rho)_{dom(\alpha^\circ)} \cdot \gamma$. Then:
  (a) as $V_G \subseteq dom(\alpha)$, $V_\psi \subset \mathcal{X}_0$, and $dom(\alpha^\circ) = dom(\alpha) \cup \hat{x} \cup \hat{y}$ then $\psi\alpha = \psi\alpha^\circ =_{E_0} \psi(\beta\rho)_{dom(\alpha^\circ)}\gamma = \psi\beta\rho\gamma$ so, as $E_0 \vDash \psi\alpha$, also $E_0 \vDash \psi\beta\rho\gamma$;
  (b) as $V_{\phi_l^\circ,\phi_r^\circ} \subseteq dom(\alpha^\circ) \cap \mathcal{X}_0$, then $(\phi_l^\circ \wedge \phi_r^\circ)\alpha^\circ =_{E_0} (\phi_l^\circ \wedge \phi_r^\circ)(\beta\rho)_{dom(\alpha^\circ)}\gamma = (\phi_l^\circ \wedge \phi_r^\circ)\beta\rho\gamma$ so, as $E_0 \vDash (\phi_l^\circ \wedge \phi_r^\circ)\alpha^\circ$, also $E_0 \vDash (\phi_l^\circ \wedge \phi_r^\circ)\beta\rho\gamma$.

  From (a) and (b) we get $E_0 \vDash (\psi \wedge \phi_l^\circ \wedge \phi_r^\circ)\beta\rho\gamma$.
- As $V_G \subseteq dom(\alpha) \subseteq dom(\alpha^\circ)$, $dom(\beta) \subseteq dom(\alpha^\circ)$, $\alpha^\circ =_E (\beta\rho)_{dom(\alpha^\circ)} \cdot \gamma$, and $ran(\rho)$ is away from all known variables, then $\Delta^\mu \varrho_\mu \alpha =_{E_0} \Delta^\mu \varrho_\mu \alpha^\circ =_E \Delta^\mu \varrho_\mu (\beta\rho)_{dom(\alpha^\circ)} \cdot \gamma = \Delta^\mu \varrho_\mu \beta\rho\gamma$.

2. Induction step, $m > 1$:

   $\mathbf{G} = l_1 \to r_1/\texttt{idle} \wedge \bigwedge_{j=2}^m (l_j \to r_j/\texttt{idle}) \wedge \Delta^\mu \varrho_\mu \mid \psi \mid V, \mu$, let $\Delta_2^m = \bigwedge_{j=2}^m (l_j \to r_j/\texttt{idle})$. As in the base case, there exist a ground substitution δ° and a substitution $\beta_1 \in CSU_B(l_1^\circ = r_1^\circ)$, so $ran(\beta_1) \cap (V_G \cup V_{\hat{x},\hat{y}} \cup V_{l_1^\circ,r_1^\circ}) = \emptyset$, such that $\alpha =_{E_0} \delta^\circ_{dom(\alpha)}$, $dom(\beta_1) \subseteq dom(\delta^\circ) = dom(\alpha) \cup \hat{x}_1 \cup \hat{y}_1$, $\delta^\circ \ll_B \beta_1 \ll (\beta_1)_{dom(\delta^\circ)}$, $(\psi \wedge \phi_l^\circ \wedge \phi_r^\circ)\beta_1$ is satisfiable, so $\mathbf{G} \rightsquigarrow^1_{[d1]} (\Delta_2^m \wedge \Delta^\mu \varrho_\mu)\beta_1 \mid (\psi \wedge \phi_{l_1}^\circ \wedge \phi_{r_1}^\circ))\beta_1 \mid V, (\mu\beta_1)_V = \mathbf{G_1}$, and for every pair of substitutions ρ and γ such that $\delta^\circ \ll_E (\beta_1\rho)_{dom(\delta^\circ)}$ and $\delta^\circ =_E (\beta_1\rho)_{dom(\delta^\circ)} \cdot \gamma$ it holds that $E_0 \vDash (\psi \wedge \phi_l^\circ \wedge \phi_r^\circ)\beta_1\rho\gamma$ and $(\Delta_2^m \wedge \Delta^\mu \varrho_\mu)\alpha =_E (\Delta_2^m \wedge \Delta^\mu \varrho_\mu)\beta_1\rho\gamma$.

   As $\delta^\circ \ll_B \beta_1$ and δ° is ground, then there exists a ground substitution δ_1, such that $dom(\delta_1) = ran(\beta_1) \cup (dom(\delta^\circ) \setminus dom(\beta_1))$, where $ran(\beta_1) \cap V_G = \emptyset$, and $\delta^\circ =_B \beta_1 \cdot \delta_1$, so $dom(\beta_1\delta_1) = ran(\beta_1) \cup dom(\delta^\circ) = ran(\beta_1) \cup dom(\alpha) \cup \hat{x}_1 \cup \hat{y}_1$. Then:

   - as $\delta^\circ =_B (\beta_1\delta_1)_{\setminus ran(\beta_1)}$, so $\delta^\circ_{\mathcal{X}_0} = (\beta_1\delta_1)_{\mathcal{X}_0 \setminus ran(\beta_1)}$, $dom(\delta^\circ) = dom(\alpha) \cup \hat{x}_1 \cup \hat{y}_1$, and $\alpha =_{E_0} \delta^\circ_{dom(\alpha)} = \delta^\circ_{\setminus(\hat{x}_1 \cup \hat{y}_1)}$, then $\alpha =_{E_0} \delta^\circ_{\setminus(\hat{x}_1 \cup \hat{y}_1)} =_B (\beta_1\delta_1)_{\setminus(ran(\beta_1) \cup \hat{x}_1 \cup \hat{y}_1)} = (\beta_1\delta_1)_{dom(\alpha)}$, i.e., $\alpha =_E (\beta_1\delta_1)_{dom(\alpha)}$;
   - $V_{\Delta_2^m} \cap (\hat{x}_1 \cup \hat{y}_1) = \emptyset$ implies $\Delta_2^m\beta_1\delta_1 = \Delta_2^m(\beta_1\delta_1)_{\setminus(\hat{x}_1 \cup \hat{y}_1)} =_E \Delta_2^m\alpha$. Then, since $\bigwedge_{j=2}^m (l_j\alpha =_E r_j\alpha)$, $\bigwedge_{j=2}^m (l_j\beta_1\delta_1 =_E r_j\beta_1\delta_1)$ (†);
   - as $E_0 \vDash \psi\alpha$, $V_\psi \subset \mathcal{X}_0$, $\delta^\circ_{\mathcal{X}_0} = (\beta_1\delta_1)_{\mathcal{X}_0 \setminus ran(\beta_1)}$, and $V_\psi \cap (ran(\beta_1) \cup \hat{x}_1 \cup \hat{y}_1) = \emptyset$, then $\psi\beta_1\delta_1 = \psi(\beta_1\delta_1)_{\setminus(\hat{x}_1 \cup \hat{y}_1)} =_{E_0} \psi\alpha$, so $E_0 \vDash \psi\beta_1\delta_1$;
   - as $\delta^\circ_{\mathcal{X}_0} = (\beta_1\delta_1)_{\mathcal{X}_0 \setminus ran(\beta_1)}$ and $V_{\phi_{l_1}^\circ,\phi_{r_1}^\circ} \cap ran(\beta_1) = \emptyset$, then $(\phi_{l_1}^\circ \wedge \phi_{r_1}^\circ)\beta_1\delta_1 = (\phi_{l_1}^\circ \wedge \phi_{r_1}^\circ)\delta^\circ$ so, as $E_0 \vDash (\phi_{l_1}^\circ \wedge \phi_{r_1}^\circ)\delta^\circ$, also $E_0 \vDash (\phi_{l_1}^\circ \wedge \phi_{r_1}^\circ)\beta_1\delta_1$; and
   - as $E_0 \vDash (\phi_{l_1}^\circ \wedge \phi_{r_1}^\circ)\beta_1\delta_1$ and $E_0 \vDash \psi\beta_1\delta_1$, then $E_0 \vDash (\psi \wedge \phi_{l_1}^\circ \wedge \phi_{r_1}^\circ)\beta_1\delta_1$(††).

   Then, by (†) and (††), we can apply the I.H. and there exists a ground substitution δ_1°, substitutions $\beta_2, \ldots, \beta_m$ from CSUs, and abstractions $abstract_{\Sigma_1}((l_j\beta_1\beta_2^{j-1}, r_j\beta_1\beta_2^{j-1})) = \langle\lambda(\bar{x}_j, \bar{y}_j).(l_j^\circ, r_j^\circ);\ (\theta_{l_j}^\circ, \theta_{r_j}^\circ); (\phi_{l_j}^\circ, \phi_{r_j}^\circ)\rangle$, for $2 \leq j \leq m$, where $\beta_2^1 = none$, such that $dom(\beta_2^m) \subseteq dom(\delta_1^\circ) \cup \bigcup_{j=2}^{m-1} ran(\beta_j)$, $dom(\delta_1^\circ) = dom(\beta_1\delta_1) \cup (V_{\hat{x},\hat{y}} \setminus (\hat{x}_1 \cup \hat{y}_1))$, $\beta_1\delta_1 =_{E_0} (\delta_1^\circ)_{dom(\beta_1\delta_1)}$, $l_j^\circ\delta_1^\circ =_E r_j^\circ\delta_1^\circ$, for $2 \leq j \leq m$, $\delta_1^\circ \ll_E (\beta_2^m)_{dom(\delta_1^\circ)}$, $\mathbf{G_1} \rightsquigarrow^{m-1}_{[d1]} \Delta^\mu\varrho_\mu\beta_1\beta_2^m \mid (\psi \wedge \phi_{l_1}^\circ \wedge \phi_{r_1}^\circ)\beta_1\beta_2^m \wedge \bigwedge_{j=2}^m (\phi_{l_j}^\circ \wedge \phi_{r_j}^\circ)\beta_j^m \mid V, (\mu\beta_1\beta_2^m)_V$, and for every pair of substitutions ρ and γ such that $\delta_1^\circ \ll_E (\beta_2^m\rho)_{dom(\delta_1^\circ)}$ and $\delta_1^\circ =_E (\beta_2^m\rho)_{dom(\delta_1^\circ)} \cdot \gamma$ it holds that $E_0 \vDash ((\psi \wedge \phi_{l_1}^\circ \wedge \phi_{r_1}^\circ)\beta_1\beta_2^m \wedge \bigwedge_{j=2}^m (\phi_{l_j}^\circ \wedge \phi_{r_j}^\circ)\beta_j^m)\rho\gamma$ and $(\Delta_2^m \wedge \Delta^\mu\varrho_\mu)\beta_1\delta_1 =_E (\Delta_2^m \wedge \Delta^\mu\varrho_\mu)\beta_1\beta_2^m\rho\gamma$.

   As $\beta_1\beta_2^m = \beta_1^m = \beta$, this is the same as $\mathbf{G_1} \rightsquigarrow^{m-1}_{[d1]} \Delta^\mu\varrho_\mu\beta \mid \psi\beta \wedge \bigwedge_{j=1}^m (\phi_{l_j}^\circ \wedge \phi_{r_j}^\circ)\beta_j^m \mid V, (\mu\beta)_V$, $E_0 \vDash (\psi\beta \wedge \bigwedge_{j=1}^m (\phi_{l_j}^\circ \wedge \phi_{r_j}^\circ)\beta_j^m)\rho\gamma$, and $(\Delta_2^m \wedge \Delta^\mu\varrho_\mu)\beta_1\delta_1 =_E (\Delta_2^m \wedge \Delta^\mu\varrho_\mu)\beta\rho\gamma$ (†††).

As $dom(\beta_1) \subseteq dom(\delta^\circ) = dom(\alpha) \cup \hat{x}_1 \cup \hat{y}_1$, $dom(\alpha^\circ) = dom(\alpha) \cup V_{\hat{\hat{x}},\hat{y}}$, so $dom(\beta_1) \cup dom(\delta^\circ) \subseteq dom(\alpha^\circ)$, $dom(\delta_1^\circ) = dom(\beta_1\delta_1) \cup (V_{\hat{\hat{x}},\hat{y}} \setminus (\hat{x}_1 \cup \hat{y}_1))$, $dom(\delta_1) = ran(\beta_1) \cup (dom(\delta^\circ) \setminus dom(\beta_1))$, and $dom(\beta_2^m) \subseteq dom(\delta_1^\circ) \cup \bigcup_{j=2}^{m-1} ran(\beta_j)$, then:

$dom(\beta) = dom(\beta_1\beta_2^m) = dom(\beta_1) \cup dom(\beta_2^m) \subseteq dom(\alpha) \cup \hat{x}_1 \cup \hat{y}_1 \cup dom(\delta_1^\circ) \cup \bigcup_{j=2}^{m-1} ran(\beta_j) = dom(\alpha) \cup \hat{x}_1 \cup \hat{y}_1 \cup dom(\beta_1\delta_1) \cup (V_{\hat{\hat{x}},\hat{y}} \setminus (\hat{x}_1 \cup \hat{y}_1)) \cup \bigcup_{j=2}^{m-1} ran(\beta_j) = dom(\alpha) \cup V_{\hat{\hat{x}},\hat{y}} \cup dom(\beta_1\delta_1) \cup \bigcup_{j=2}^{m-1} ran(\beta_j) = dom(\alpha^\circ) \cup dom(\beta_1\delta_1) \cup \bigcup_{j=2}^{m-1} ran(\beta_j) \subseteq (dom(\alpha^\circ) \cup dom(\beta_1) \cup dom(\delta^\circ)) \cup (ran(\beta_1) \cup \bigcup_{j=2}^{m-1} ran(\beta_j)) = dom(\alpha^\circ) \cup \bigcup_{j=1}^{m-1} ran(\beta_j)$ $(*)$.

Let $\alpha^\circ = \delta^\circ(\delta_1^\circ)_{V_{\hat{\hat{x}},\hat{y}} \setminus (\hat{x}_1 \cup \hat{y}_1)} = \delta^\circ \cup (\delta_1^\circ)_{V_{\hat{\hat{x}},\hat{y}} \setminus (\hat{x}_1 \cup \hat{y}_1)}$, since δ° is ground and $dom(\delta^\circ) = dom(\alpha) \cup \hat{x}_1 \cup \hat{y}_1$. Then:

(a) as $\mathbf{G} \rightsquigarrow^1_{[d1]} \mathbf{G_1}$, then:
$\mathbf{G} \rightsquigarrow^m_{[d1]} \Delta^\mu \varrho_\mu \beta \mid \psi\beta \wedge \bigwedge_{j=1}^m (\phi^\circ_{l_j} \wedge \phi^\circ_{r_j})\beta_j^m \mid V, (\mu\beta)_V$.

(b) as $dom(\delta^\circ) = dom(\alpha) \cup \hat{x}_1 \cup \hat{y}_1$, then:
$dom(\alpha^\circ) = dom(\delta^\circ \cup (\delta_1^\circ)_{V_{\hat{\hat{x}},\hat{y}} \setminus (\hat{x}_1 \cup \hat{y}_1)}) = dom(\delta^\circ) \cup (V_{\hat{\hat{x}},\hat{y}} \setminus (\hat{x}_1 \cup \hat{y}_1)) = dom(\alpha) \cup \hat{x}_1 \cup \hat{y}_1 \cup (V_{\hat{\hat{x}},\hat{y}} \setminus (\hat{x}_1 \cup \hat{y}_1)) = dom(\alpha) \cup V_{\hat{\hat{x}},\hat{y}}$, i.e., $dom(\alpha^\circ) = dom(\alpha) \cup V_{\hat{\hat{x}},\hat{y}}$;

(c) as $\alpha =_{E_0} \delta^\circ_{dom(\alpha)}$, $dom(\alpha) \cap V_{\hat{\hat{x}},\hat{y}} = \emptyset$ and δ° is ground, then:
$\alpha^\circ_{dom(\alpha)} = (\delta^\circ \cup (\delta_1^\circ)_{V_{\hat{\hat{x}},\hat{y}} \setminus (\hat{x}_1 \cup \hat{y}_1)})_{dom(\alpha)} = \delta^\circ_{dom(\alpha)} =_{E_0} \alpha$, i.e., $\alpha =_{E_0} \alpha^\circ_{dom(\alpha)}$;

(d) as $\delta^\circ =_B (\beta_1\delta_1)_{\setminus ran(\beta_1)}$, $\beta_1\delta_1 =_{E_0} (\delta_1^\circ)_{dom(\beta_1\delta_1)}$, and $dom(\delta_1^\circ) = dom(\beta_1\delta_1) \cup (V_{\hat{\hat{x}},\hat{y}} \setminus (\hat{x}_1 \cup \hat{y}_1))$, then:
$\alpha^\circ = \delta^\circ \cup (\delta_1^\circ)_{V_{\hat{\hat{x}},\hat{y}} \setminus (\hat{x}_1 \cup \hat{y}_1)} =_B (\beta_1\delta_1)_{\setminus ran(\beta_1)} \cup (\delta_1^\circ)_{V_{\hat{\hat{x}},\hat{y}} \setminus (\hat{x}_1 \cup \hat{y}_1)} =_{E_0} (\delta_1^\circ)_{dom(\beta_1\delta_1) \setminus ran(\beta_1)} \cup (\delta_1^\circ)_{V_{\hat{\hat{x}},\hat{y}} \setminus (\hat{x}_1 \cup \hat{y}_1)} = (\delta_1^\circ)_{\setminus ran(\beta_1)}$, i.e., $\alpha^\circ =_E (\delta_1^\circ)_{\setminus ran(\beta_1)}$, so:
- as $ran(\beta_1) \cap V_G = \emptyset$ and $l_j^\circ \delta_1^\circ =_E r_j^\circ \delta_1^\circ$, for $2 \leq j \leq m$ then $l_j^\circ \alpha^\circ =_E r_j^\circ \alpha^\circ$, for $2 \leq j \leq m$
- as $ran(\beta_1) \cap V_{l_1^\circ, r_1^\circ} = \emptyset$ and $l_1^\circ \beta_1 =_B r_1^\circ \beta_1$, then:
  - $l_1^\circ (\beta_1)_{\setminus ran(\beta_1)} = l_1^\circ \beta_1 =_B r_1^\circ \beta_1 = r_1^\circ (\beta_1)_{\setminus ran(\beta_1)}$,
  - $l_1^\circ (\beta_1\delta_1)_{\setminus ran(\beta_1)} =_B r_1^\circ (\beta_1\delta_1)_{\setminus ran(\beta_1)}$,
  - $l_1^\circ (\delta_1^\circ)_{dom(\beta_1\delta_1) \setminus ran(\beta_1)} =_E r_1^\circ (\delta_1^\circ)_{dom(\beta_1\delta_1) \setminus ran(\beta_1)}$, and
  - $l_1^\circ (\delta_1^\circ)_{\setminus ran(\beta_1)} =_E r_1^\circ (\delta_1^\circ)_{\setminus ran(\beta_1)}$, i.e., $l_1^\circ \alpha^\circ =_E r_1^\circ \alpha^\circ$.

In conclusion, $\bar{l}^\circ \alpha^\circ =_E \bar{r}^\circ \alpha^\circ$;

(e)
- as $dom(\beta_1\delta_1) = ran(\beta_1) \cup dom(\alpha) \cup \hat{x}_1 \cup \hat{y}_1$, then $(\beta_1\delta_1)_{\setminus ran(\beta_1)} = (\beta_1\delta_1)_{dom(\alpha) \cup \hat{x}_1 \cup \hat{y}_1}$;
- as $\beta_1\delta_1 =_{E_0} (\delta_1^\circ)_{dom(\beta_1\delta_1)} = (\delta_1^\circ)_{dom(\beta_1)} \cup (\delta_1^\circ)_{dom(\delta_1)}$ then $\delta_1 =_{E_0} (\delta_1^\circ)_{dom(\delta_1)}$;
- as $dom(\beta_1\delta_1) = ran(\beta_1) \cup dom(\alpha) \cup \hat{x}_1 \cup \hat{y}_1$ then $dom(\delta_1) \subseteq ran(\beta_1) \cup dom(\alpha) \cup \hat{x}_1 \cup \hat{y}_1$;
- then, as $\delta^\circ =_B (\beta_1\delta_1)_{\setminus ran(\beta_1)}$ and $dom(\delta_1^\circ) = dom(\alpha^\circ) \cup ran(\beta_1)$:
  $\alpha^\circ = \delta^\circ(\delta_1^\circ)_{V_{\hat{\hat{x}},\hat{y}} \setminus (\hat{x}_1 \cup \hat{y}_1)} = (\delta^\circ(\delta_1^\circ)_{V_{\hat{\hat{x}},\hat{y}} \setminus (\hat{x}_1 \cup \hat{y}_1)})_{dom(\alpha^\circ)} =_B$
  $((\beta_1\delta_1)_{\setminus ran(\beta_1)}(\delta_1^\circ)_{V_{\hat{\hat{x}},\hat{y}} \setminus (\hat{x}_1 \cup \hat{y}_1)})_{dom(\alpha^\circ)} =$
  $((\beta_1\delta_1)_{dom(\alpha) \cup \hat{x}_1 \cup \hat{y}_1}(\delta_1^\circ)_{V_{\hat{\hat{x}},\hat{y}} \setminus (\hat{x}_1 \cup \hat{y}_1)})_{dom(\alpha^\circ)} =_{E_0}$
  $((\beta_1(\delta_1^\circ)_{dom(\delta_1)})_{dom(\alpha) \cup \hat{x}_1 \cup \hat{y}_1}(\delta_1^\circ)_{V_{\hat{\hat{x}},\hat{y}} \setminus (\hat{x}_1 \cup \hat{y}_1)})_{dom(\alpha^\circ)} =$
  $((\beta_1\delta_1^\circ)_{dom(\alpha) \cup \hat{x}_1 \cup \hat{y}_1}(\delta_1^\circ)_{V_{\hat{\hat{x}},\hat{y}} \setminus (\hat{x}_1 \cup \hat{y}_1)})_{dom(\alpha^\circ)} = ((\beta_1\delta_1^\circ)_{dom(\alpha) \cup V_{\hat{\hat{x}},\hat{y}}})_{dom(\alpha^\circ)} =$
  $(\beta_1\delta_1^\circ)_{dom(\alpha^\circ)} \ll_E (\beta_1(\beta_2^m)_{dom(\delta_1^\circ)})_{dom(\alpha^\circ)} = (\beta_1\beta_2^m)_{dom(\alpha^\circ)} = (\beta_1^m)_{dom(\alpha^\circ)}$.

In conclusion, $\alpha^\circ \ll_E (\beta_1^m)_{dom(\alpha^\circ)}$;

(f) let ρ and γ such that $\alpha^\circ \ll_E (\beta\rho)_{dom(\alpha^\circ)}$ and $\alpha^\circ =_E (\beta\rho)_{dom(\alpha^\circ)} \cdot \gamma$. Then:
- as $\delta_1 =_{E_0} (\delta_1^\circ)_{dom(\delta_1)}$ then $(\beta_1\delta_1)_{dom(\beta_1\delta_1)} =_{E_0} (\beta_1\delta_1^\circ)_{dom(\beta_1\delta_1)}$, hence $(\beta_1\delta_1)_{dom(\alpha) \cup \hat{x}_1 \cup \hat{y}_1} =_{E_0} (\beta_1\delta_1^\circ)_{dom(\alpha) \cup \hat{x}_1 \cup \hat{y}_1}$;

- then, as $\delta^\circ =_B (\beta_1\delta_1)_{\setminus ran(\beta_1)}$ and $dom(\beta_1\delta_1) = ran(\beta_1) \cup dom(\alpha) \cup \hat{x}_1 \cup \hat{y}_1$: $\alpha^\circ = \delta^\circ(\delta_1^\circ)_{V_{\hat{x},\hat{y}}\setminus(\hat{x}_1\cup\hat{y}_1)} =_B (\beta_1\delta_1)_{\setminus ran(\beta_1)}(\delta_1^\circ)_{V_{\hat{x},\hat{y}}\setminus(\hat{x}_1\cup\hat{y}_1)} =$ $(\beta_1\delta_1)_{dom(\alpha)\cup\hat{x}_1\cup\hat{y}_1}(\delta_1^\circ)_{V_{\hat{x},\hat{y}}\setminus(\hat{x}_1\cup\hat{y}_1)} =_{E_0} (\beta_1\delta_1^\circ)_{dom(\alpha)\cup\hat{x}_1\cup\hat{y}_1}(\delta_1^\circ)_{V_{\hat{x},\hat{y}}\setminus(\hat{x}_1\cup\hat{y}_1)} =$ $(\beta_1\delta_1^\circ)_{dom(\alpha^\circ)}$, i.e., $(\beta_1\delta_1^\circ)_{dom(\alpha^\circ)} =_E \alpha^\circ$;
- as $dom(\delta_1^\circ) = dom(\alpha^\circ) \cup ran(\beta_1)$ and $(\beta_1\delta_1^\circ)_{dom(\alpha^\circ)} =_E \alpha^\circ \ll_E (\beta\rho)_{dom(\alpha^\circ)} =$ $(\beta_1\beta_2^m\rho)_{dom(\alpha^\circ)}$, then $(\delta_1^\circ)_{dom(\alpha^\circ)\cup ran(\beta_1)} \ll_E (\beta_2^m\rho)_{dom(\alpha^\circ)\cup ran(\beta_1)}$, i.e., $\delta_1^\circ \ll_E (\beta_2^m\rho)_{dom(\delta_1^\circ)}$;
- as $dom(\delta_1^\circ) = dom(\alpha^\circ)\cup ran(\beta_1)$ and $(\beta_1\beta_2^m\rho)_{dom(\alpha^\circ)}\gamma = (\beta\rho)_{dom(\alpha^\circ)}\gamma =_E \alpha^\circ =_E$ $(\beta_1\delta_1^\circ)_{dom(\alpha^\circ)}$, then $(\beta_2^m\rho)_{dom(\alpha^\circ)\cup ran(\beta_1)}\gamma =_E (\delta_1^\circ)_{dom(\alpha^\circ)\cup ran(\beta_1)}$, i.e., $\delta_1^\circ =_E (\beta_2^m\rho)_{dom(\delta_1^\circ)}\gamma$.

In conclusion, as $\delta_1^\circ \ll_E (\beta_2^m\rho)_{dom(\delta_1^\circ)}$ and $\delta_1^\circ =_E (\beta_2^m\rho)_{dom(\delta_1^\circ)}\gamma$ then, by (†††), $E_0 \vDash (\psi\beta \wedge \bigwedge_{j=1}^m(\phi_{l_j}^\circ \wedge \phi_{r_j}^\circ)\beta_j^m)\rho\gamma$;

- Also by (†††), $(\Delta_2^m \wedge \Delta^\mu\varrho_\mu)\beta_1\delta_1 =_E (\Delta_2^m \wedge \Delta^\mu\varrho_\mu)\beta\rho\gamma$, so $\Delta^\mu\varrho_\mu\beta_1\delta_1 =_E \Delta^\mu\varrho_\mu\beta\rho\gamma$;
- As $\alpha =_{E_0} \delta^\circ_{dom(\alpha)}$, $\delta^\circ =_B \beta_1 \cdot \delta_1$, β_1 is a CSU, so $V_\Delta^\mu\varrho_\mu \cap ran(\beta_1) = \emptyset$, and $V_\Delta^\mu\varrho_\mu \subseteq V_G \subseteq dom(\alpha)$, then: $\Delta^\mu\varrho_\mu\beta\rho\gamma =_E \Delta^\mu\varrho_\mu\beta_1\delta_1 =_B \Delta^\mu\varrho_\mu\delta^\circ = \Delta^\mu\varrho_\mu\delta^\circ_{dom(\alpha)} =_{E_0} \Delta^\mu\varrho_\mu\alpha$.

So also $\Delta^\mu\varrho_\mu\alpha =_E \Delta^\mu\varrho_\mu\beta\rho\gamma$.

□

Theorem 3. Given an associated rewrite theory $\mathcal{R} = (\Sigma, E_0 \cup B, R)$ closed under B-extensions and a reachability problem $P = \bigwedge_{i=1}^n u_i \to v_i/ST_i \mid \phi \mid V, \mu$, where μ is R/E-normalized, if $\sigma : V \to \mathcal{T}_\Sigma$ is a R/E-normalized solution for P then there exist a formula $\psi \in QF(\mathcal{X}_0)$ and two substitutions, say λ and ρ, call $\nu = (\mu\lambda)_V$, such that $\bigwedge_{i=1}^n u_i\mu \to v_i\mu/ST_i^\mu;\mathtt{idle} \mid \phi\mu \mid V, \mu \rightsquigarrow_\lambda^+ nil \mid \psi \mid V, \nu$, $\sigma =_E \nu{\cdot}\rho$, and $\psi\rho$ is satisfiable.

Proof. The proof is by induction over the sum $\mathbf{h}$ of the number of nodes in each c.p.t. for the solution σ. No simplification is applied to the reachability formulas that appear in the generated path.

In the following we will make use of the following two facts. For any term t and substitution α it holds that:

1. $pos_\Sigma(t) \subseteq pos_\Sigma(t\alpha)$ because, by definition, the variables of t that α instantiates are located at positions in $pos_\mathcal{X}(t)$, and
2. $top_{\Sigma_0}(t) \subseteq top_{\Sigma_0}(t\alpha)$, because α only may add new top_{Σ_0} positions for non-Σ_0 variables in its domain, but cannot remove any existing position in $top_{\Sigma_0}(t)$.

We will call $u = u_1\mu$ and $v = v_1\mu$. In all cases $\sigma = \mu\cdot\sigma'$, for proper σ' such that $dom(\sigma') = V^\mu$, $[v_1\sigma]_E \in ST_1^\sigma @[u_1\sigma]_E$, and $E_0 \vDash \phi\sigma$. As σ is ground and R/E-normalized, then σ' has to be also ground and, by Proposition 7, R/E-normalized.

(i) Base step: $\mathbf{h} = 1$.

Then P has the form $u_1 \to v_1/ST_1 \mid \phi \mid V, \mu$, with $V_P = V_{u_1,v_1,\phi} \subseteq V$ and the c.p.t. T for P_0 and σ has the form $\dfrac{\overline{u_1\sigma\to v_1\sigma/ST_1^\sigma} \quad \overline{v_1\sigma\to v_1\sigma/\mathtt{idle}}}{u_1\sigma\to v_1\sigma/ST_1^\sigma;\mathtt{idle}}$.

There are four strategies in the base case: `idle`, $c[\gamma]$, $\mathtt{top}(c[\gamma])$, and the `match` test.

1. $ST_1 = \mathtt{idle}$.

   $P = u_1 \to v_1/\mathtt{idle} \mid \phi \mid V, \mu$. As, by definition 33, $V_{u_1,v_1,\phi} \subseteq V$ then $V_{u,v,\phi\mu} \subseteq V^\mu =$ $dom(\sigma')$, and as $[v_1\sigma]_E \in \mathtt{idle}@[u_1\sigma]_E$ then, as shown in example 10, $u_1\sigma =_E v_1\sigma$, i.e., $u\sigma' =_E v\sigma'$, all ground terms.

Let $abstract_{\Sigma_1}((u,v)) = \langle\lambda(\bar{x},\bar{y}).(u^\circ,v^\circ);(\theta_u^\circ,\theta_v^\circ);(\phi_u^\circ,\phi_v^\circ)\rangle$. As $dom(\sigma') = V^\mu$ then, by Lemma 4, there exists a ground substitution σ° such that $u^\circ\sigma^\circ =_B v^\circ\sigma^\circ$, $E_0 \vDash (\phi_u^\circ \wedge \phi_v^\circ)\sigma^\circ$, $dom(\sigma^\circ) = V^\mu \cup \hat{x} \cup \hat{y}$, and $\sigma' =_{E_0} \sigma^\circ_{V^\mu}$.

As $u^\circ\sigma^\circ =_B v^\circ\sigma^\circ$, then there exist substitutions ν' and ρ' such that $\nu' \in CSU_B(u^\circ = v^\circ)$ and $\sigma^\circ =_B \nu'\cdot\rho'$, call $\nu = (\mu\nu')_V$ and $\rho = \rho'_{ran(\nu)\cup(V\setminus dom(\nu))}$. As $dom(\mu) \subseteq V$ and $\sigma' =_{E_0} \sigma^\circ_{V^\mu}$ then $\sigma = \mu\sigma' =_{E_0} \mu\sigma^\circ_{V^\mu} =_B \mu(\nu'\rho')_{V^\mu} = (\mu\nu'\rho')_V = (\mu\nu')_V\cdot\rho'_{ran((\mu\nu')_V)\cup(V\setminus dom((\mu\nu')_V))} = \nu\cdot\rho'_{ran(\nu)\cup(V\setminus dom(\nu))} = \nu\cdot\rho$, i.e., $\sigma =_E \nu\cdot\rho$.

As $E_0 \vDash \phi\sigma$, $V_{\phi\mu} \subseteq V^\mu$, and $\sigma' =_{E_0} \sigma^\circ_{V^\mu}$ then $\phi\mu\sigma^\circ =_{E_0} \phi\mu\sigma' = \phi\sigma$, so $E_0 \vDash \phi\mu\sigma^\circ$. Now, as $E_0 \vDash (\phi_u^\circ \wedge \phi_v^\circ)\sigma^\circ$, then $E_0 \vDash (\phi\mu \wedge \phi_u^\circ \wedge \phi_v^\circ)\sigma^\circ$, call $\psi^\circ = \phi\mu \wedge \phi_u^\circ \wedge \phi_v^\circ$ and let $\psi = \psi^\circ\nu'$. As $E_0 \vDash \psi^\circ\sigma^\circ$, $\sigma^\circ =_B \nu'\cdot\rho'$, and $V_{\psi^\circ} \cap ran(\nu') = \emptyset$, so $\psi^\circ\sigma^\circ = \psi^\circ\nu'\rho'$, and ρ is more general than ρ', then $\psi^\circ\nu'\rho$, i.e., $\psi\rho$, is satisfiable, hence ψ is also satisfiable.

As $u = u_1\mu$, $v = v_1\mu$, and $\nu' \in CSU_B(u^\circ = v^\circ)$, then $u \to v/\texttt{idle};\texttt{idle} \mid \phi\mu \mid V,\mu \rightsquigarrow_{[d2]} u \to v/\texttt{idle} \mid \phi\mu \mid V,\mu \rightsquigarrow_{[d1],\nu'} nil \mid \psi \mid V,\nu$, where ψ is satisfiable and $\sigma =_E \nu\rho$.

2. $ST_1 = c[\gamma]$.

   $P = u_1 \to v_1/c[\gamma] \mid \phi \mid V,\mu$, with $c : l \to r \text{ if } \chi \in R$, and $[v_1\sigma]_E \in c^\sigma[\gamma\sigma_{ran(\gamma)}]@[u_1\sigma]_E$. Then, by Lemma 5 point 3, $u_1\sigma \xrightarrow[c^\sigma\gamma\sigma_{ran(\gamma)}\ R^\sigma/E]{1} v_1\sigma$, so $E_0 \vDash \chi\sigma\gamma\sigma_{ran(\gamma)}$. Call $c' = c^\sigma\gamma\sigma_{ran(\gamma)}$ ($= c^{\gamma\sigma}$ because σ is ground and, by definition, $dom(\gamma) \cap dom(\sigma) = \emptyset$, hence $E_0 \vDash \chi\gamma\sigma$), $\mathcal{R}(c') = (\Sigma, E_0 \cup B, \{c'\})$, and $\mathcal{R}_B(c') = (\Sigma, E_0 \cup B, c'_B)$. Then also $u_1\sigma \xrightarrow[c'\ \{c'\}/E]{1} v_1\sigma$ so, by Theorem 1, $u_1\sigma \to^1_{\{c'\},B} v_1\sigma$.

   As $u_1\sigma \to^1_{\{c'\},B} v_1\sigma$ and $vars(B) \cap vars(c\gamma) = \emptyset$, then this rewrite step uses a rule $c'_1 \in c'_B$ where:

   - if $c'_1 = c'$ then c'_1 has the form $c'_1 : l\gamma\sigma \to r\gamma\sigma \text{ if } \chi\gamma\sigma$, call $l_0 = l$ and $r_0 = r$, and
   - if $c'_1 \neq c'$ then c'_1 has the form $c'_1 : w[l\gamma\sigma]_{p'} \to w[r\gamma\sigma]_{p'} \text{ if } \chi\gamma\sigma$, by Definition 18, for proper w and p'. As by Definition 18, $V_w \cap V_{c'} = \emptyset$, by Definition 33, $V_w \cap V = \emptyset$, and also $dom(\gamma) \subseteq V_{c'}$ and $dom(\sigma) \subseteq V$, this is the same as $c'_1 : w[l]_{p'}\gamma\sigma \to w[r]_{p'}\gamma\sigma \text{ if } \chi\gamma\sigma$, call $l_0 = w[l]_{p'}$ and $r_0 = w[r]_{p'}$.

   In either case, c'_1 has the form, $c'_1 : l_0\gamma\sigma \to r_0\gamma\sigma \text{ if } \chi\gamma\sigma$. Let $c_0 : l_0 \to r_0 \text{ if } \chi$. As $c'_1 \in c_B^{\gamma\sigma}$ and $c'_1 = c_0^{\gamma\sigma}$ then, by proposition 6, $c_0 \in c_B$. Since $\sigma = \mu\sigma'$, if we call $l_1 = l_0\gamma\mu$ and $r_1 = r_0\gamma\mu$ then c'_1 has also the form $c'_1 : l_1\sigma' \to r_1\sigma' \text{ if } \chi\gamma\sigma$.
   Let $c_2 : l_2 \to r_2 \text{ if } \chi_2$ be a fresh version of c_0^μ except for $dom(\gamma) \cup V^\mu$ ($= dom(\gamma) \cup dom(\sigma')$), and let τ be the renaming that verifies $c_2 = c_0^\mu\tau$, so $(l_2, r_2, \chi_2) = (l_0, r_0, \chi)(\mu \uplus \tau)$, where $(dom(\tau) \cup ran(\tau)) \cap (dom(\gamma) \cup V^\mu) = \emptyset$. Then $l_2(\gamma\mu)_{dom(\gamma)} = l_0(\mu \uplus \tau)(\gamma\mu)_{dom(\gamma)} = l_0((\gamma\mu)_{dom(\gamma)} \uplus \mu \uplus \tau) = l_0((\gamma\mu)_{dom(\gamma)} \uplus \mu)\tau = l_0\gamma\mu\tau = l_1\tau$, so also $r_2(\gamma\mu)_{dom(\gamma)} = r_1\tau$ and $\chi_2(\gamma\mu)_{dom(\gamma)} = \chi\gamma\mu\tau$. Call $l_c = l_2(\gamma\mu)_{dom(\gamma)}$ and $\sigma'' = \tau^{-1}\sigma'$. Then $l_c\sigma'' = l_1\tau\tau^{-1}\sigma' = l_1\sigma'$. Now:

   (a) $abstract_{\Sigma_1}(l_c) = \langle\lambda\bar{y}.l^\circ;\theta_l^\circ;\phi_l^\circ\rangle$, where $\bar{y} = y_1,\ldots,y_{i_y}$, $l^\circ = l_c[\bar{y}]_{\bar{p}}$, $\bar{p} = p_1,\ldots,p_{i_y}$, $\hat{p} = top_{\Sigma_0}(l_c)$, $\theta_l^\circ = \bigcup_{i=1}^{i_y}\{y_i \mapsto l_c|_{p_i}\}$, and $\phi_l^\circ = \bigwedge_{i=1}^{i_y} y_i = l_c|_{p_i}$;

   (b) since $l_1\sigma' = l_c\sigma''$ and $top_{\Sigma_0}(l_c) \subseteq top_{\Sigma_0}(l_c\sigma'')$ then $abstract_{\Sigma_1}(l_1\sigma') = abstract_{\Sigma_1}(l_c\sigma'') = \langle\lambda\bar{y}\bar{z}.l^\circ_{c\sigma''};\theta^\circ_{c\sigma''};\phi^\circ_{c\sigma''}\rangle$, where $\bar{z} = z_1,\ldots,z_{i_z}$, $l^\circ_{c\sigma''} = l_c\sigma''[\bar{y}]_{\bar{p}}[\bar{z}]_{\bar{q}}$, $\hat{q} = top_{\Sigma_0}(l_c\sigma'') \setminus top_{\Sigma_0}(l_c)$, $\theta^\circ_{c\sigma''} = \bigcup_{i=1}^{i_y}\{y_i \mapsto l_c|_{p_i}\sigma''\} \cup \bigcup_{j=1}^{i_z}\{z_j \mapsto l_c\sigma''|_{q_j}\}$, and $\phi^\circ_{c\sigma''} = (\bigwedge_{i=1}^{i_y} y_i = l_c|_{p_i}\sigma'' \wedge \bigwedge_{j=1}^{i_z} z_j = l_c\sigma''|_{q_j})$;

   (c) as $u_1\sigma \to^1_{\{c'\},B} v_1\sigma$ with c'_1, then there are a position p in $pos_{\Sigma_1}(u_1\sigma)$ and a substitution $\delta : \hat{y} \cup \hat{z} \cup V_{c'_1} \to \mathcal{T}_\Sigma$ such that $rep(u_1\sigma|_p) =_B l^\circ_{c\sigma''}\delta$, $v_1\sigma =_E u_1\sigma[r_0\gamma\sigma\delta]_p = u_1\sigma[r_0\gamma\mu\sigma'\delta]_p = u_1\sigma[r_1\sigma'\delta]_p$, and $E_0 \vDash (\chi\gamma\sigma \wedge \phi^\circ_{c\sigma''})\delta$, so $E_0 \vDash \chi\gamma\sigma\delta$, i.e., $E_0 \vDash \chi\gamma\mu\sigma'\delta$, $\bar{y}\delta =_{E_0} l_c|_{\bar{p}}\sigma''\delta$ and $\bar{z}\delta =_{E_0} l_c\sigma''|_{\bar{q}}\delta$;

(d) as $p \in pos_{\Sigma_1}(u_1\sigma)$ and σ is R/E-normalized, hence R, E-normalized by Theorem 1, then $p \in pos_{\Sigma_1}(u_1)$, so $u_1\sigma|_p = u_1|_p\sigma = u_1|_p\mu\sigma' = u_1\mu|_p\sigma' = u|_p\sigma'$; and

(e) as τ is a fresh renaming then $\emptyset = V_u \cap ran(\tau) = V_u \cap dom(\tau^{-1})$, so $u|_p\tau^{-1}\sigma' = u|_p\sigma' = u_1\sigma|_p =_{E_0} rep(u_1\sigma|_p) =_B l^\circ_{c\sigma''}\delta = l_c\sigma''[\bar{y}]_{\bar{p}}[\bar{z}]_{\bar{q}}\delta =_{E_0} l_c\sigma'' = l_c\tau^{-1}\sigma'$, i.e., $u|_p\tau^{-1}\sigma' =_E l_c\tau^{-1}\sigma'$;

Let $abstract_{\Sigma_1}(u|_p) = \langle \lambda\bar{x}.u^\circ; \theta^\circ_u; \phi^\circ_u\rangle$. As $dom(\tau^{-1}\sigma') = ran(\tau) \cup V^\mu$ then, by Lemma 4, there exists a ground substitution σ° such that $u^\circ\sigma^\circ =_B l^\circ\sigma^\circ$, $E_0 \vDash (\phi^\circ_u \wedge \phi^\circ_l)\sigma^\circ$, $dom(\sigma^\circ) = dom(\tau^{-1}\sigma') \cup \hat{x} \cup \hat{y} = ran(\tau) \cup V^\mu \cup \hat{x} \cup \hat{y}$, and $\tau^{-1}\sigma' =_{E_0} \sigma^\circ_{dom(\tau^{-1}\sigma')} = \sigma^\circ_{ran(\tau)\cup V^\mu}$, so $(\tau^{-1}\sigma')_{V^\mu} =_{E_0} \sigma^\circ_{V^\mu}$. As $(dom(\tau) \cup ran(\tau)) \cap V^\mu = \emptyset$ and $dom(\sigma') = V^\mu$ then $\sigma' = \sigma'_{V^\mu} = (\tau^{-1}\sigma')_{V^\mu} =_{E_0} \sigma^\circ_{V^\mu}$.

As $u^\circ\sigma^\circ =_B l^\circ\sigma^\circ$, then there exist substitutions ν' and ρ' such that $\nu' \in CSU_B(u^\circ = l^\circ)$ and $\sigma^\circ =_B \nu'\rho'$, call $\nu = (\mu\nu')_V$ and $\rho = \rho'_{ran(\nu)\cup(V\setminus dom(\nu))}$. As $dom(\mu) \subseteq V$ and $\sigma' =_{E_0} \sigma^\circ_{V^\mu}$ then $\sigma = \mu\sigma' =_{E_0} \mu\sigma^\circ_{V^\mu} =_B \mu(\nu'\rho')_{V^\mu} = (\mu\nu'\rho')_V = (\mu\nu')_V\rho'_{ran((\mu\nu')_V)\cup(V\setminus dom((\mu\nu')_V))} = \nu\rho'_{ran(\nu)\cup(V\setminus dom(\nu))} = \nu\rho$, i.e., $\sigma =_E \nu\rho$.

As $\chi_2(\gamma\mu)_{dom(\gamma)} = \chi\gamma\mu\tau$, $dom(\sigma^\circ) = ran(\tau) \cup V^\mu \cup \hat{x} \cup \hat{y}$, and $\tau^{-1}\sigma' =_{E_0} \sigma^\circ_{ran(\tau)\cup V^\mu}$, then $\chi_2(\gamma\mu)_{dom(\gamma)}\sigma^\circ\delta = \chi\gamma\mu\tau\sigma^\circ_{ran(\tau)\cup V^\mu}\delta =_{E_0} \chi\gamma\mu\tau\tau^{-1}\sigma'\delta = \chi\gamma\mu\sigma'\delta$ so, as $E_0 \vDash \chi\gamma\mu\sigma'\delta$, $E_0 \vDash \chi_2(\gamma\mu)_{dom(\gamma)}\sigma^\circ\delta$.

As $E_0 \vDash \phi\sigma$, $V_{\phi\mu} \subseteq V^\mu$, and $\sigma' =_{E_0} \sigma^\circ_{V^\mu}$ then $\phi\mu\sigma^\circ =_{E_0} \phi\mu\sigma' = \phi\sigma$, so $E_0 \vDash \phi\mu\sigma^\circ$. Now, as $E_0 \vDash (\phi^\circ_u \wedge \phi^\circ_l)\sigma^\circ$, then $E_0 \vDash (\phi\mu \wedge \phi^\circ_u \wedge \phi^\circ_l)\sigma^\circ$ ground formula, so $E_0 \vDash (\phi\mu \wedge \phi^\circ_u \wedge \phi^\circ_l)\sigma^\circ\delta$ and $E_0 \vDash (\phi\mu \wedge \phi^\circ_u \wedge \phi^\circ_l \wedge \chi_2(\gamma\mu)_{dom(\gamma)})\sigma^\circ\delta$. Call $\varphi^\circ = \phi\mu \wedge \phi^\circ_u \wedge \phi^\circ_l \wedge \chi_2(\gamma\mu)_{dom(\gamma)}$, and let $\varphi = \varphi^\circ\nu'$. As $\sigma^\circ =_B \nu'\rho'$, so $\varphi^\circ\sigma^\circ = \varphi^\circ\nu'\rho' = \varphi\rho'$, then $E_0 \vDash \varphi\rho'\delta$, call $\delta' = \rho'\delta$, hence φ is also satisfiable.

Now, $\mathbf{G_0} = u \to v/c^\mu[(\gamma\mu)_{dom(\gamma)}];\texttt{idle} \mid \phi\mu \mid V, \mu \rightsquigarrow_{[t]} u \to^1 x_0, x_0 \to v/c^\mu[(\gamma\mu)_{dom(\gamma)}];\texttt{idle} \mid \phi\mu \mid V, \mu \rightsquigarrow^*_{[c]} u|_p \to^1 x, u[x]_p \to v/c^\mu[(\gamma\mu)_{dom(\gamma)}];\texttt{idle} \mid \phi\mu \mid V, \mu = \mathbf{G_1}$, where $u|_p$ cannot be a variable, say x_u, because as $p \in pos_\Sigma(u_1)$ then, by (c), also $x_u\sigma' \to^1_{R,B} r_0\gamma\sigma\delta$, so σ would not be R/E-normalized. As $c_2 : l_2 \to r_2 \text{ if } \chi_2$, where $r_2(\gamma\mu)_{dom(\gamma)} = r_1\tau$, and $\nu' \in CSU_B(u^\circ = l^\circ)$ then $\mathbf{G_1} \rightsquigarrow_{[r],\nu'\cup\{x\mapsto r_1\tau\nu'\}} (u[r_1\tau]_p \to v\ /\texttt{idle})\nu' \mid \varphi \mid V, \nu = \mathbf{G_2}$.

We already know that $E_0 \vDash \varphi\delta'$. We prove that $u[r_1\tau]_p\nu'\delta' =_E v\nu'\delta'$:

- as $\tau^{-1}\sigma' =_{E_0} \sigma^\circ_{dom(\tau^{-1}\sigma')}$ and $dom(\sigma^\circ) = dom(\tau^{-1}\sigma') \cup \hat{x} \cup \hat{y}$, then $\tau^{-1}\sigma' \uplus \sigma^\circ_{\hat{x}\cup\hat{y}} =_{E_0} \sigma^\circ_{dom(\tau^{-1}\sigma')} \uplus \sigma^\circ_{\hat{x}\cup\hat{y}} = \sigma^\circ$, where $V_{G_2} \cap (\hat{x} \cup \hat{y}) = \emptyset$, $u = u\tau^{-1}$, and $v = v\tau^{-1}$;
- $u[r_1\tau]_p\nu'\delta' =_B u[r_1\tau]_p\sigma^\circ\delta =_{E_0} u[r_1\tau]_p(\tau^{-1}\sigma' \uplus \sigma^\circ_{\hat{x}\cup\hat{y}})\delta = u[r_1\tau]_p\tau^{-1}\sigma'\delta = u[r_1]_p\sigma'\delta$;
- $v\nu'\delta' =_B v\sigma^\circ\delta =_{E_0} v(\tau^{-1}\sigma' \uplus \sigma^\circ_{\hat{x}\cup\hat{y}})\delta = v\tau^{-1}\sigma'\delta = v\sigma'\delta$;
- by (c), $v_1\sigma =_E u_1\sigma[r_1\sigma'\delta]_p$, i.e., $v\sigma' =_E u\sigma'[r_1\sigma'\delta]_p$, ground expression so, as δ is ground, $v\sigma'\delta =_E u\sigma'\delta[r_1\sigma'\delta]_p = u[r_1]_p\sigma'\delta$, hence $u[r_1\tau]_p\nu'\delta' =_E v\nu'\delta'$.

Let $abstract_{\Sigma_1}((u[r_1\tau]_p, v\nu')) = \langle \lambda(\bar{x}', \bar{y}').(r^\circ, v^\circ); (\theta^\circ_r, \theta^\circ_v); (\phi^\circ_r, \phi^\circ_v)\rangle$. Then, by Lemma 4, there exists a ground substitution δ° such that $r^\circ\delta^\circ =_B v^\circ\delta^\circ$, $E_0 \vDash (\phi^\circ_r \wedge \phi^\circ_v)\delta^\circ$, $dom(\delta^\circ) = dom(\delta') \cup \hat{x}' \cup \hat{y}'$, and $\delta' =_{E_0} \delta^\circ_{dom(\delta')}$, so there exist substitutions ν'' and ρ'' such that $\nu'' \in CSU_B(r^\circ = v^\circ)$ and $\delta^\circ =_B \nu''\rho''$, call $\nu_1 = (\nu'\nu'')_V$ and $\rho_1 = \rho''_{ran(\nu_1)\cup(V\setminus dom(\nu_1))}$.

As $E_0 \vDash \varphi\delta'$, ground formula, and $\delta' =_{E_0} \delta^\circ_{dom(\delta')}$ then $E_0 \vDash \varphi\delta^\circ$ so $E_0 \vDash (\varphi \wedge \phi^\circ_r \wedge \phi^\circ_v)\delta^\circ$, call $\psi' = \varphi \wedge \phi^\circ_r \wedge \phi^\circ_v$. Now, as $\delta^\circ =_B \nu''\rho''$ implies $\psi'\delta^\circ = \psi'\nu''\rho''$, then also $E_0 \vDash \psi'\nu''\rho''$, call $\psi = \psi'\nu''$, so ψ and $\psi\rho_1$ are satisfiable.

As $\nu'' \in CSU_B(r^\circ = v^\circ)$ and ψ is satisfiable, then $\mathbf{G_2} \rightsquigarrow_{[d1],\nu''} nil \mid \psi \mid V, \nu_1$, where $\nu_1 = (\nu\nu'')_V$. Then, as $\psi\rho_1$ is satisfiable, all that is left to prove is $\sigma =_E \nu_1\rho_1$.

As $dom(\delta^\circ) = dom(\delta') \cup \hat{x}' \cup \hat{y}'$ and $\delta' =_{E_0} \delta^\circ_{dom(\delta')}$ then $dom(\delta^\circ) \cap V = dom(\delta')$ and $\delta^\circ_V =_{E_0} \delta'_V$, so, as $dom(\sigma) = V$ and σ $(=_B \mu(\nu'\rho')_{V^\mu})$ is ground, call $\sigma^B = \mu(\nu'\rho')_{V^\mu}$, then $\nu_1\rho_1 = (\nu\nu'')_V \rho''_{ran(\nu_1)\cup(V\setminus dom(\nu_1))} = (\nu\nu''\rho'')_V =_B (\nu\delta^\circ)_V =_{E_0} (\nu\delta')_V = (\mu\nu'\delta')_V = (\mu\nu'\rho'\delta)_V = \mu(\nu'\rho'\delta)_{V^\mu} = \sigma^B\delta_{ran(\sigma^B)\cup(V\setminus dom(\sigma^B))} =_B \sigma\delta_{ran(\sigma)\cup(V\setminus dom(\sigma))} = \sigma\delta_\emptyset = \sigma$, i.e., $\sigma =_E \nu_1\rho_1$.

3. $ST_1 = \mathtt{top}(c[\gamma])$.

   The proof is almost exactly the same as the previous one, particularized for the case $p = \epsilon$. The only difference is found in the initial narrowing steps, where instead of:

   - $\mathbf{G_0} = u \to v/c^\mu[(\gamma\mu)_{dom(\gamma)}];\mathtt{idle} \mid \phi\mu \mid V, \mu \rightsquigarrow_{[t]} u \to^1 x_0, x_0 \to v/c^\mu[(\gamma\mu)_{dom(\gamma)}];\mathtt{idle} \mid \phi\mu \mid V, \mu \rightsquigarrow^*_{[c]} u|_p \to^1 x, u[x]_p \to v/c^\mu[(\gamma\mu)_{dom(\gamma)}];\mathtt{idle} \mid \phi\mu \mid V, \mu = \mathbf{G_1}$ and

   - $\mathbf{G_1} \rightsquigarrow_{[r],\nu'\cup\{x_0\mapsto r_1\tau\nu'\}} (u[r_1\tau]_p \to v\ /\mathtt{idle})\nu' \mid \varphi \mid V, \nu = \mathbf{G_2}$,

   now we have:

   $\mathbf{G_0} = u \to v/\mathtt{top}(c^\mu[(\gamma\mu)_{dom(\gamma)}]);\mathtt{idle} \mid \phi\mu \mid V, \mu \rightsquigarrow_{[tp],\nu'} (r_1\tau \to v\ /\mathtt{idle})\nu' \mid \varphi \mid V, \nu = \mathbf{G_2}$.

4. $ST_1 = \mathtt{match}\, t\ \mathtt{s.t.}\ \bigwedge_{j=1}^m (l_j = r_j) \wedge \chi$.

   $P = u_1 \to v_1/ST_1 \mid \phi \mid V, \mu$, $vars(P) = vars(\bar{u}_1, \bar{v}_1, \phi) \subseteq V$, $ST_1^\sigma = \mathtt{match}\, t\sigma\ \mathtt{s.t.}\ \bigwedge_{j=1}^m (l_j\sigma = r_j\sigma) \wedge \chi\sigma$, and there exists a substitution $\delta : V_{ST_1^\sigma} \to \mathcal{T}_\Sigma$, such that $v_1\sigma =_E u_1\sigma =_E t\sigma\delta$, $l_j\sigma\delta =_E r_j\sigma\delta$, for $1 \le j \le m$, and $E_0 \vDash (\phi \wedge \chi)\mu\sigma'\delta$.

   Let $abstract_{\Sigma_1}((u, t\mu)) = \langle \lambda(\bar{x}, \bar{y}).(u^\circ, t^\circ); (\theta^\circ_u, \theta^\circ_t); (\phi^\circ_u, \phi^\circ_t)\rangle$. As $u_1\sigma$ is ground then $u\sigma'\delta = u_1\mu\sigma'\delta = u_1\sigma\delta = u_1\sigma =_E t\sigma\delta = t\mu\sigma'\delta$ so, by Lemma 4, there exists a ground substitution σ° such that $u^\circ\sigma^\circ =_B t^\circ\sigma^\circ$, $E_0 \vDash (\phi^\circ_u \wedge \phi^\circ_t)\sigma^\circ$, $dom(\sigma^\circ) = dom(\sigma'\delta) \cup \hat{x} \cup \hat{y}$, and $\sigma'\delta =_{E_0} \sigma^\circ_{dom(\sigma'\delta)}$.

   Call $\psi_1 = (\phi \wedge \chi)\mu \wedge \phi^\circ_u \wedge \phi^\circ_t$. As $E_0 \vDash (\phi \wedge \chi)\mu\sigma'\delta$, $V_{(\phi\wedge\chi)\mu\sigma'\delta} \cap (\hat{x} \cup \hat{y}) = \emptyset$, and $\sigma'\delta =_{E_0} \sigma^\circ_{dom(\sigma'\delta)} = \sigma^\circ_{\setminus(\hat{x}\cup\hat{y})}$, then $E_0 \vDash (\phi \wedge \chi)\mu\sigma^\circ$, so $E_0 \vDash \psi_1\sigma^\circ$.

   As $u^\circ\sigma^\circ =_B t^\circ\sigma^\circ$, then there exist substitutions ν and τ such that $\eta \in CSU_B(u^\circ = t^\circ)$ and $\sigma^\circ =_B \eta\cdot\tau$, so $\psi_1\sigma^\circ = \psi_1\eta\tau$, hence $E_0 \vDash \psi_1\eta\tau$ and $\psi_1\eta$ is satisfiable.

   Now, $\mathbf{G_0} = u \to v/ST_1^\mu;\mathtt{idle} \mid \phi\mu \mid V, \mu \rightsquigarrow_{[m],\eta} (\bigwedge_{j=1}^m (l_j \to r_j/\mathtt{idle}) \wedge u_1 \to v_1/\mathtt{idle})\mu\eta \mid \psi_1\eta \mid V, (\mu\eta)_V = \mathbf{G_1}$.

   As $\bar{l}\sigma\delta =_E \bar{r}\sigma\delta$, $\sigma'\delta =_{E_0} \sigma^\circ_{dom(\sigma'\delta)}$, $\sigma'\delta =_{E_0} \sigma^\circ_{\setminus(\hat{x}\cup\hat{y})}$, $\sigma^\circ =_B \eta\cdot\tau$, and $V_{\bar{l}\mu,\bar{r}\mu} \cap (\hat{x} \cup \hat{y} \cup ran(\eta)) = \emptyset$, then $(\bar{l},\bar{r})\mu\eta\tau = (\bar{l},\bar{r})\mu(\eta\cdot\tau) =_B (\bar{l},\bar{r})\mu\sigma^\circ = (\bar{l},\bar{r})\mu\sigma^\circ_{\setminus(\hat{x}\cup\hat{y})} =_{E_0} (\bar{l},\bar{r})\mu\sigma'\delta$, i.e., $(\bar{l},\bar{r})\mu\eta\tau =_E (\bar{l},\bar{r})\sigma\delta$, so $\bar{l}\mu\eta\tau =_E \bar{r}\mu\eta\tau$.

   By Lemma 7, as τ is a substitution such that $E_0 \vDash \psi_1\eta\tau$ and $\bar{l}\mu\eta\tau =_E \bar{r}\mu\eta\tau$, then there exist a ground substitution τ°, substitutions $\beta_1, \ldots, \beta_m$, let $\beta = \beta_1^m$, and abstractions $abstract_{\Sigma_1}((l_j\beta_1^{j-1}, r_j\beta_1^{j-1})) = \langle \lambda(\bar{x}_j, \bar{y}_j).(l^\circ_j, r^\circ_j); (\theta^\circ_{l_j}, \theta^\circ_{r_j}); (\phi^\circ_{l_j}, \phi^\circ_{r_j})\rangle$, for $1 \le j \le m$, such that $dom(\tau^\circ) = dom(\tau) \cup V_{\hat{\bar{x}},\hat{\bar{y}}}$, $\tau =_{E_0} \tau^\circ_{dom(\tau)}$, $\bar{l}^\circ\tau^\circ =_E \bar{r}^\circ\tau^\circ$, $\tau^\circ \ll_E \beta_{dom(\tau^\circ)}$, call $\psi_2 = \psi_1\eta\beta \wedge \bigwedge_{j=1}^m (\phi^\circ_{l_j} \wedge \phi^\circ_{r_j})\beta_j^m$, $\mathbf{G_1} \rightsquigarrow^m_{[d1]} (u_1 \to v_1\ /\mathtt{idle})\mu\eta\beta \mid \psi_2 \mid V, (\mu\eta\beta)_V = \mathbf{G_2}$, and for every pair of substitutions ρ and γ such that $\tau^\circ \ll_E (\beta\rho)_{dom(\tau^\circ)}$ and $\tau^\circ =_E (\beta\rho)_{dom(\tau^\circ)} \cdot \gamma$ it holds that $E_0 \vDash \psi_2\rho\gamma$ and $(u_1 \to v_1\ /\mathtt{idle})\mu\eta\tau =_E (u_1 \to v_1\ /\mathtt{idle})\mu\eta\beta\rho\gamma$ (†).

   Take $\rho = none$. As $\tau^\circ \ll_E \beta_{dom(\tau^\circ)}$, then there exists γ such that $\tau^\circ =_E \beta_{dom(\tau^\circ)} \cdot \gamma$ and $ran(\tau^\circ) = ran(\beta_{dom(\tau^\circ)} \cdot \gamma)$, so as τ° is ground then γ is ground. By (†), $E_0 \vDash \psi_2\gamma$ and $(u_1 \to v_1\ /\mathtt{idle})\mu\eta\tau =_E (u_1 \to v_1\ /\mathtt{idle})\mu\eta\beta\gamma$. Now, as $V_{u_1,v_1} \subseteq V = dom(\sigma)$ and σ is ground, then $V\sigma = V\sigma\delta = V\mu\sigma'\delta =_{E_0} V\mu\sigma^\circ =_B V\mu\eta\tau =_E V\mu\eta\beta\gamma$ so, as $u_1\sigma =_E v_1\sigma$, also $u_1\mu\eta\beta\gamma =_E v_1\mu\eta\beta\gamma$, ground Σ-equation, hence $V_{(u_1,v_1)\mu\eta\beta} \subseteq dom(\gamma)$.

   Let $abstract_{\Sigma_1}((u_1\mu\eta\beta, v_1\mu\eta\beta)) = \langle \lambda(\bar{x}', \bar{y}').(\mathfrak{u}^\circ, \mathfrak{v}^\circ); (\theta^\circ_{\mathfrak{u}}, \theta^\circ_{\mathfrak{v}}); (\phi^\circ_{\mathfrak{u}}, \phi^\circ_{\mathfrak{v}})\rangle$. As $u_1\mu\eta\beta\gamma =_E v_1\mu\eta\beta\gamma$, $V_{(u_1,v_1)\mu\eta\beta} \subseteq dom(\gamma)$, and γ is ground then, by Lemma 4, there exists a ground substitution γ° such that $\mathfrak{u}^\circ\gamma^\circ =_B \mathfrak{v}^\circ\gamma^\circ$, $E_0 \models (\phi^\circ_{\mathfrak{u}} \wedge \phi^\circ_{\mathfrak{v}})\gamma^\circ$, $dom(\gamma^\circ) = dom(\gamma) \cup \hat{x}' \cup \hat{y}'$, and $\gamma =_{E_0} \gamma^\circ_{dom(\gamma)}$.

As $\mathfrak{u}^\circ\gamma^\circ =_B \mathfrak{v}^\circ\gamma^\circ$, then there exist substitutions α and ε such that $\alpha \in CSU_B(u^\circ = v^\circ)$ and $\gamma^\circ =_B \alpha\cdot\varepsilon$. Now, as $\tau^\circ =_E \beta_{dom(\tau^\circ)}\cdot\gamma =_{E_0} \beta_{dom(\tau^\circ)}\cdot\gamma^\circ_{dom(\gamma)} =_B \beta_{dom(\tau^\circ)}\cdot(\alpha\cdot\varepsilon)_{dom(\gamma)}$ and τ° is ground, then $\tau^\circ =_E (\beta\alpha\varepsilon)_{dom(\tau^\circ)}$, so $\tau^\circ =_E (\beta\alpha)_{dom(\tau^\circ)}\cdot\varepsilon$, hence $\tau^\circ \ll_E (\beta\alpha)_{dom(\tau^\circ)}$ and, by (†), $E_0 \models \psi_2\alpha\varepsilon$.

Call $\psi = (\psi_2 \wedge \phi^\circ_{\mathfrak{u}} \wedge \phi^\circ_{\mathfrak{v}})\alpha$, ground formula, and $\nu = (\mu\eta\beta\alpha)_V$. As $E_0 \models (\phi^\circ_{\mathfrak{u}} \wedge \phi^\circ_{\mathfrak{v}})\gamma^\circ$ and $\gamma^\circ =_B \alpha\cdot\varepsilon$ then $E_0 \models (\phi^\circ_{\mathfrak{u}} \wedge \phi^\circ_{\mathfrak{v}})\alpha\cdot\varepsilon$, so also $E_0 \models (\phi^\circ_{\mathfrak{u}} \wedge \phi^\circ_{\mathfrak{v}})\alpha\varepsilon$, hence, as $E_0 \models \psi_2\alpha\varepsilon$, $E_0 \models \psi\varepsilon$. Finally:

- as $E_0 \models \psi\varepsilon$ then ψ is satisfiable, so $\mathbf{G_2} \rightsquigarrow_{[d1],\alpha} nil \mid \psi \mid V, \nu$, i.e., $\mathbf{G_0} \rightsquigarrow^+ nil \mid \psi \mid V, \nu$,
- as $E_0 \models \psi\varepsilon$ then $\psi\varepsilon$ is satisfiable, and
- as $V\sigma =_E V\mu\eta\beta\gamma =_{E_0} V\mu\eta\beta\gamma^\circ_{dom(\gamma)} =_B V\mu\eta\beta(\alpha\cdot\varepsilon)_{dom(\gamma)}$ then:
  $\sigma = \sigma_V =_E (\mu\eta\beta(\alpha\cdot\varepsilon)_{dom(\gamma)})_V = (\mu\eta\beta\alpha\varepsilon)_V = (\mu\eta\beta\alpha)_V\cdot\varepsilon = \nu\cdot\varepsilon$, i.e., $\sigma =_E \nu\cdot\varepsilon$.

(i) Induction step: $\mathbf{h} > 1$.

- First, we prove the induction step when P has several open goals and the first open goal is one the base cases: $P = u_1 \to v_1/ST_1 \wedge \Omega \mid \phi \mid V, \mu$, $\Omega = \bigwedge_{i=2}^n u_i \to v_i/ST_i$ and $n > 1$, let $\Delta = \bigwedge_{i=2}^n u_i \to v_i/ST_i;\texttt{idle}$.

  We have proved for all of these cases that there exist a formula ψ_1 and substitutions λ', ν', and ρ' such that $\mathbf{G} = u_1\mu \to v_1\mu/ST_1^\mu;\texttt{idle} \mid \phi\mu \mid V, \mu \rightsquigarrow^+_{\lambda'} nil \mid \psi_1 \mid V, \nu'$, $\sigma =_E \nu'\cdot\rho'$, and $\psi_1\rho'$ is satisfiable. Then, also $\mathbf{G_0} = u_1\mu \to v_1\mu/ST_1^\mu;\texttt{idle} \wedge \Delta\mu \mid \phi\mu \mid V, \mu \rightsquigarrow^+_{\lambda'} \Delta(\mu\lambda') \mid \psi_1 \mid V, \nu' = \mathbf{G_1}$, where $\sigma =_E \nu'\cdot\rho'$ and $\psi_1\rho'$ is satisfiable.

  Now, we prove that $\mathbf{G_1} \rightsquigarrow^+_{\lambda''} nil \mid \psi \mid V, \nu$, for proper ν, ψ, and λ'', and that there exist a substitution ρ such that $\sigma =_E \nu\cdot\rho$ and $\psi\rho$ is satisfiable, so the theorem holds. This generic proof is valid for many of the other cases of the induction step, so we prove it only once. We provide a specific proof for each case where this proof does not apply.

  All the variables in $dom(\lambda')$ are either variables in V^μ or fresh variables generated by the calculus rules, so $V_{\Delta\mu} \cap dom(\lambda') \subseteq V^\mu$, hence $\Delta(\mu\lambda') = \Delta(\mu\lambda')_V = \Delta\nu'$ and $\mathbf{G_1} = \Delta\nu' \mid \psi_1 \mid V, \nu'$. As any narrowing step will preserve ϕ, instantiated with the substitution used in that step, as part of a conjunction of formulas, and $V_\phi \subseteq V$ then $\psi_1 = \phi\nu' \wedge \psi_2$, for proper ψ_2.

  As $\psi_1\rho'$ is satisfiable and $\phi\sigma$ is ground, so $\phi\nu'\rho'$ is ground, then there exists a ground substitution α such that $dom(\alpha) = V_{\psi_2\rho'}$, where all the variables are either fresh or belong to $ran(\nu')$, so $dom(\alpha) \cap ran(\nu') = \emptyset$, and $E_0 \models (\phi\nu' \wedge \psi_2)\rho'\alpha$, where $\phi\nu'\rho'\alpha = \phi\nu'\rho'$. As $\nu'\cdot\rho'$ is ground, so ρ' is also ground, and $dom(\alpha) \cap ran(\nu') = \emptyset$, then: (i) $\nu'\cdot(\rho'\cdot\alpha) = (\nu'\cdot\rho')\cdot\alpha = (\nu'\cdot\rho')\alpha$ and (ii) $\rho'\cdot\alpha = \rho'\alpha$, so $E_0 \models \psi_1(\rho'\cdot\alpha)$. Call $V' = V^{\nu'} \cup V_{\psi_2}$.

  Consider the problem $P' = \Omega\nu' \mid \psi_1 \mid V', none$ in $\mathcal{R}^{\nu'}$ and $Call^{\nu'}_{\mathcal{R}}$, whose corresponding goal is $\mathbf{G_1'} = \Delta\nu' \mid \psi_1 \mid V', none$. As $\sigma =_E \nu'\cdot\rho'$, both ground substitutions, then $V_{\Omega\sigma} = V_{\Omega(\nu'\cdot\rho')} \subseteq V_\Omega$, so $V_{\Omega(\nu'\cdot\rho')} \cap dom(\alpha) = \emptyset$ and $\Omega(\nu'\cdot\rho')\alpha = \Omega(\nu'\cdot\rho') =_E \Omega\sigma$. As there is a c.p.t. for $[v_i\sigma]_E \in ST_i^\sigma[u_i\sigma]_E$, for $2 \le i \le n$, then, by Lemma 5, there are closed proof trees for all the open goals in $\Omega(\nu'\cdot\rho')$, i.e., $\Omega(\nu'\cdot\rho')\cdot\alpha$, each c.p.t. having the same depth and number of nodes as its correspondent c.p.t. for $\Omega\sigma$. As $E_0 \models \psi_1(\rho'\cdot\alpha)$ then $\rho'\cdot\alpha$ is a solution of P' with less nodes that those in the solution σ for P_0, since we have excluded the nodes in the c.p.t. for the first open goal, so we can apply the I.H. to P', and there exist a formula ψ and substitutions λ'' and ρ'', call $\lambda = \lambda'\lambda''$ and $\nu = (\mu\lambda)_V$, such that $\mathbf{G_1'} = \Delta\nu' \mid \psi_1 \mid V', none \rightsquigarrow^{+}_{\lambda''} nil \mid \psi \mid V', \lambda''_{V'}$, $\rho' \cdot \alpha =_E \lambda'' \cdot \rho''$, and $\psi\rho''$ is satisfiable, call $\rho = \rho''_{V \cup ran(\nu)}$. Then, also $\mathbf{G_1} = \Delta\nu' \mid \psi_1 \mid V, \nu' \rightsquigarrow^{+}_{\lambda''} nil \mid \psi \mid V, \nu$, so $\mathbf{G_0} \rightsquigarrow^{+}_{\lambda} nil \mid \psi \mid V, \nu$, and $\psi\rho$ is satisfiable. Finally, $\nu{\cdot}\rho = (\nu{\cdot}\rho'')_V = (\mu\lambda'\lambda''\rho'')_V =_E (\mu\lambda'\rho'\alpha)_V = (\nu'\rho'\alpha)_V =_E (\sigma\alpha)_V = \sigma$.

- In the rest of cases, the strategy in the first open goal of P may be a concatenation or not, so P has the form $u_1 \rightarrow v_1/ST_1(\,;ST) \wedge \Omega \mid \phi \mid V, \mu$, let $ST_0 = ST_1(\,;ST)$, with $\Omega = \bigwedge_{i=2}^{n} u_i \rightarrow v_i/ST_i$ and $n \geq 1$, let $\Delta = \bigwedge_{i=2}^{n} u_i \rightarrow v_i/ST_i; \texttt{idle}$, where ST is allowed to be a concatenation of strategies but ST_1 is not (in case of several concatenations), σ is a solution of the reachability problem $P' = \Omega \mid \phi \mid V, \mu$, so $[v_1\sigma]_E \in ST_0^{\sigma}@[u_1\sigma]_E$ and, for $2 \leq i \leq n$, $[v_i\sigma]_E \in ST_i^{\sigma}@[u_i\sigma]_E$, hence there is a c.p.t. for $[v_i\sigma]_E \in ST_i^{\sigma}[u_i\sigma]_E$ where the sum of the number of nodes in each c.p.t. for P' is lower than $\mathbf{h}$.

  1. $ST_1 = S_1 \mid S_2$.

     Then, one c.p.t., T, for P and σ has the form $\dfrac{\dfrac{\frac{F_1}{u_1\sigma \rightarrow w/S_i^{\sigma}}}{u_1\sigma \rightarrow w/ST_1^{\sigma}} \quad (\frac{F_2}{w \rightarrow v_1\sigma/ST^{\sigma}})}{u_1\sigma \rightarrow v_1\sigma/ST_0^{\sigma}}$, with respect to $\mathcal{D}^{\sigma}_{\mathcal{R},Call_{\mathcal{R}}}$, where w ($= v_1\sigma$ if $ST_0 = ST_1$) is a term in $\mathcal{T}_{\Sigma}$ and i in $\{1, 2\}$, let $S = S_i(;ST)$. Consider the problem $P' = u_1 \rightarrow v_1/S \mid \phi \mid V, \mu$ which for the same solution σ has a c.p.t. $T' = \dfrac{\frac{F_1}{u_1\sigma \rightarrow w/S_i^{\sigma}} \quad (\frac{F_2}{w \rightarrow v_1\sigma/ST^{\sigma}})}{u_1\sigma \rightarrow v_1\sigma/S^{\sigma}}$ with one less node than T.
     Then, by I.H., there exist a formula ψ_1 and two substitutions, λ' and ρ', let $\nu' = (\mu\lambda')_V$, such that $u \rightarrow v/S^{\mu}; \texttt{idle} \mid \phi\mu \mid V, \mu \rightsquigarrow^{+}_{\lambda'} nil \mid \psi_1 \mid V, \nu'$, $\sigma =_E \nu' \cdot \rho'$, and $\psi_1\rho'$ is satisfiable.
     But then, also:
     - if $n = 1$ then $\mathbf{G_0} = u \rightarrow v/ST_0^{\mu}; \texttt{idle} \mid \phi\mu \mid V, \mu \rightsquigarrow_{[o1\ or\ o2]} u \rightarrow v/S^{\mu}; \texttt{idle} \mid \phi\mu \mid V, \mu \rightsquigarrow^{+}_{\lambda'} nil \mid \psi_1 \mid V, \nu'$, $\sigma =_E \nu' \cdot \rho'$, and $\psi_1\rho'$ is satisfiable, so $\psi = \psi_1$, $\lambda = \lambda'$, $\nu = \nu'$, and $\rho = \rho'$;
     - else $\mathbf{G_0} = u \rightarrow v/ST_0^{\mu}; \texttt{idle} \wedge \Delta\mu \mid \phi\mu \mid V, \mu \rightsquigarrow_{[o1\ or\ o2]} u \rightarrow v/S^{\mu}; \texttt{idle} \wedge \Delta\mu \mid \phi\mu \mid V, \mu \rightsquigarrow^{+}_{\lambda'} \Delta(\mu\lambda') \mid \psi_1 \mid V, \nu' = \mathbf{G_1}$, $\sigma =_E \nu' \cdot \rho'$, and $\psi_1\rho'$ is satisfiable. The rest of the proof is the one given at the end of the induction step for the base cases.

  2. $ST_1 = S_1{+}$.

     Then there is a c.p.t. T, with respect to $\mathcal{D}^{\sigma}_{\mathcal{R},Call_{\mathcal{R}}}$, of the form $\dfrac{\frac{T_1}{u_1\sigma \rightarrow w/ST_1^{\sigma}} \quad (\frac{F}{w \rightarrow v_1\sigma/ST^{\sigma}})}{u_1\sigma \rightarrow v_1\sigma/ST_0^{\sigma}}$, where w ($= v_1\sigma$ if $ST_0 = ST_1$) is a term in $\mathcal{T}_{\Sigma}$ and either $head(T_1) = u_1\sigma \rightarrow w/S_1^{\sigma}$ or $head(T_1) = u_1\sigma \rightarrow w/S_1^{\sigma}\,;\,S_1^{\sigma}{+}$, let $S = S_1$ or $S = S_1\,;\,S_1{+}$, depending on the case, and $S_0 = S(;ST)$.
     Consider the problem $P' = u_1 \rightarrow v_1/S_0 \mid \phi \mid V, \mu$ which for the same solution σ has a c.p.t. $T' = \dfrac{T_1 \quad (\frac{F}{w \rightarrow v_1\sigma/ST^{\sigma}})}{u_1\sigma \rightarrow v_1\sigma/S_0^{\sigma}}$ with one less node than T.
     Then, by I.H., there exist a formula ψ_1 and two substitutions, λ' and ρ', let $\nu' = (\mu\lambda')_V$, such that $u \rightarrow v/S_0^{\mu}; \texttt{idle} \mid \phi\mu \mid V, \mu \rightsquigarrow^{+}_{\lambda'} nil \mid \psi_1 \mid V, \nu'$, $\sigma =_E \nu' \cdot \rho'$, and $\psi_1\rho'$ is satisfiable.
     But then, also:
     - if $n = 1$ then $\mathbf{G_0} = u \rightarrow v/ST_0^{\mu}; \texttt{idle} \mid \phi\mu \mid V, \mu \rightsquigarrow_{[p1\ or\ p2]}$ $u \rightarrow v/S_0^{\mu}; \texttt{idle} \mid \phi\mu \mid V, \mu \rightsquigarrow^{+}_{\lambda'} nil \mid \psi_1 \mid V, \nu'$, $\sigma =_E \nu' {\cdot} \rho'$, and $\psi_1\rho'$ is satisfiable, so $\psi = \psi_1$, $\lambda = \lambda'$, $\nu = \nu'$, and $\rho = \rho'$;

– else $\mathbf{G_0} = u \to v/ST_0^{\mu}; \texttt{idle} \wedge \Delta\mu \mid \phi\mu \mid V, \mu \leadsto_{[p1\ or\ p2]} u \to v/S_0^{\mu}; \texttt{idle} \wedge \Delta\mu \mid \phi\mu \mid V, \mu \leadsto^{+}_{\lambda'} \Delta(\mu\lambda') \mid \psi_1 \mid V, \nu' = \mathbf{G_1}$, $\sigma =_E \nu' \cdot \rho'$, and $\psi_1\rho'$ is satisfiable. The rest of the proof is the one given at the end of the induction step for the base cases.

3. $ST_1 = CS$, where $\texttt{sd}\ CS\ := \ S$, let $S_0 = S(;ST)$.

   Then there is a c.p.t. T, with respect to $\mathcal{D}^{\sigma}_{\mathcal{R},Call_{\mathcal{R}}}$, of the form $\frac{\frac{T_1}{u_1\sigma \to w/ST_1^{\sigma}} \quad (\frac{F}{w \to v_1\sigma/ST^{\sigma}})}{u_1\sigma \to v_1\sigma/ST_0^{\sigma}}$, where w ($= v_1\sigma$ if $ST_0 = ST_1$) is a term in $\mathcal{T}_{\Sigma}$ and $head(T_1) = u_1\sigma \to w/S^{\sigma}$.
   Consider the problem $P' = u_1 \to v_1/S_0 \mid \phi \mid V, \mu$ which for the same solution σ has a c.p.t. $T' = \frac{T_1 \quad (\frac{F}{w \to v_1\sigma/ST^{\sigma}})}{u_1\sigma \to v_1\sigma/S_0^{\sigma}}$ with one less node than T.
   Then, by I.H., there exist a formula ψ_1 and two substitutions, λ' and ρ', let $\nu' = (\mu\lambda')_V$, such that $u \to v/S_0^{\mu}; \texttt{idle} \mid \phi\mu \mid V, \mu \leadsto^{+}_{\lambda'} nil \mid \psi_1 \mid V, \nu'$, $\sigma =_E \nu' \cdot \rho'$, and $\psi_1\rho'$ is satisfiable.
   But then, also:
   – if $n = 1$ then $\mathbf{G_0} = u \to v/ST_0^{\mu}; \texttt{idle} \mid \phi\mu \mid V, \mu \leadsto_{[c1]}$ $u \to v/S_0^{\mu}; \texttt{idle} \mid \phi\mu \mid V, \mu \leadsto^{+}_{\lambda'} nil \mid \psi_1 \mid V, \nu'$, $\sigma =_E \nu' \cdot \rho'$, and $\psi_1\rho'$ is satisfiable, so $\psi = \psi_1$, $\lambda = \lambda'$, $\nu = \nu'$, and $\rho = \rho'$;
   – else $\mathbf{G_0} = u \to v/ST_0^{\mu}; \texttt{idle} \wedge \Delta\mu \mid \phi\mu \mid V, \mu \leadsto_{[c1]} u \to v/S_0^{\mu}; \texttt{idle} \wedge \Delta\mu \mid \phi\mu \mid V, \mu \leadsto^{+}_{\lambda'} \Delta(\mu\lambda') \mid \psi_1 \mid V, \nu' = \mathbf{G_1}$, $\sigma =_E \nu' \cdot \rho'$, and $\psi_1\rho'$ is satisfiable. The rest of the proof is the one given at the end of the induction step for the base cases.

4. $ST_1 = CS(\bar{t})$, where $\texttt{sd}\ CS(\bar{x})\ := \ S \in Call_{\mathcal{R}}$, let $\gamma = \{\bar{x} \mapsto \bar{t}\}$ and $S_0 = S\gamma(;ST)$.

   Then there is a c.p.t. T, with respect to $\mathcal{D}^{\sigma}_{\mathcal{R},Call_{\mathcal{R}}}$, of the form $\frac{\frac{T_1}{u_1\sigma \to w/ST_1^{\sigma}} \quad (\frac{F}{w \to v_1\sigma/ST^{\sigma}})}{u_1\sigma \to v_1\sigma/ST_0^{\sigma}}$, where w ($= v_1\sigma$ if $ST_0 = ST_1$) is a term in $\mathcal{T}_{\Sigma}$ and $head(T_1) = u_1\sigma \to w/(S\gamma)^{\sigma}$.
   Consider the problem $P' = u_1 \to v_1/S_0 \mid \phi \mid V, \mu$ which for the same solution σ has a c.p.t. $T' = \frac{T_1 \quad (\frac{F}{w \to v_1\sigma/ST^{\sigma}})}{u_1\sigma \to v_1\sigma/S_0^{\sigma}}$ with one less node than T.
   Then, by I.H., there exist a formula ψ_1 and two substitutions, λ' and ρ', let $\nu' = (\mu\lambda')_V$, such that $u \to v/S_0^{\mu}; \texttt{idle} \mid \phi\mu \mid V, \mu \leadsto^{+}_{\lambda'} nil \mid \psi_1 \mid V, \nu'$, $\sigma =_E \nu' \cdot \rho'$, and $\psi_1\rho'$ is satisfiable.
   But then, also:
   – if $n = 1$ then $\mathbf{G_0} = u \to v/ST_0^{\mu}; \texttt{idle} \mid \phi\mu \mid V, \mu \leadsto_{[c1]}$ $u \to v/S_0^{\mu}; \texttt{idle} \mid \phi\mu \mid V, \mu \leadsto^{+}_{\lambda'} nil \mid \psi_1 \mid V, \nu'$, $\sigma =_E \nu' \cdot \rho'$, and $\psi_1\rho'$ is satisfiable, so $\psi = \psi_1$, $\lambda = \lambda'$, $\nu = \nu'$, and $\rho = \rho'$;
   – else $\mathbf{G_0} = u \to v/ST_0^{\mu}; \texttt{idle} \wedge \Delta\mu \mid \phi\mu \mid V, \mu \leadsto_{[c1]} u \to v/S_0^{\mu}; \texttt{idle} \wedge \Delta\mu \mid \phi\mu \mid V, \mu \leadsto^{+}_{\lambda'} \Delta(\mu\lambda') \mid \psi_1 \mid V, \nu' = \mathbf{G_1}$, $\sigma =_E \nu' \cdot \rho'$, and $\psi_1\rho'$ is satisfiable. The rest of the proof is the one given at the end of the induction step for the base cases.

5. $ST_1 = CS(\bar{t})$, where $\texttt{csd}\ CS(\bar{x})\ := \ S\ if\ C \in Call_{\mathcal{R}}$, with C of the form $\bar{l} = \bar{r} \wedge \chi$, with $\mid \bar{l} \mid = \mid \bar{r} \mid = m$, let $\theta = \{\bar{x} \mapsto \bar{t}\}$ and let ϵ, with $dom(\epsilon) = V_{CS} \setminus (V \cup \hat{x})$, be a fresh renaming.

   Then there is a c.p.t. T, with respect to $\mathcal{D}^{\sigma}_{\mathcal{R},Call_{\mathcal{R}}}$, of the form $\frac{\frac{\frac{F_1}{u_1\sigma \to w/(S\epsilon\theta\delta)^{\sigma}}}{u_1\sigma \to w/ST_1^{\sigma}} \quad (\frac{F_2}{w \to v_1\sigma/ST^{\sigma}})}{u_1\sigma \to v_1\sigma/ST_0^{\sigma}}$, where w ($= v_1\sigma$ if $ST_0 = ST_1$) is a term in $\mathcal{T}_{\Sigma}$, $\delta : vars(C\epsilon\theta\sigma) \to \mathcal{T}_{\Sigma}$ is a substitution such that $\bar{l}\epsilon\theta\sigma\delta =_E \bar{r}\epsilon\theta\sigma\delta$, $E_0 \vDash \chi\epsilon\theta\sigma\delta$, and σ and δ ground and $dom(\sigma) \cap dom(\delta) = \emptyset$ implies $(S\epsilon\theta)^{\sigma}\delta = (S\epsilon\theta\delta)^{\sigma}$. Let $S_0 = S\epsilon\theta(;ST)$, $\tau = \sigma_1\delta$, $\mathbf{\Theta} = u_1 \to v_1/S_0; \texttt{idle}(\wedge\Delta)$, $\mathbf{\Theta'} = u_1 \to v_1/S_0(\wedge\Omega)$, and $\psi_1 = (\phi \wedge \chi\epsilon\theta)\mu$.

   Then $\mathbf{G_0} = u \to v/ST_0^{\mu}; \texttt{idle}(\wedge\Delta\mu) \mid \phi\mu \mid V, \mu \leadsto_{[c2]} \bigwedge_{j=1}^{m}(l_j\eta \to r_j\eta/\texttt{idle}) \wedge \mathbf{\Theta}\mu \mid$

$\psi_1 \mid V, \mu = \mathbf{G_1}$. As $E_0 \vDash \phi\mu\sigma_1$, ground formula, because $V_\phi \subseteq V$, then $\phi\mu\sigma_1 = \phi\mu\tau$ and $E_0 \vDash \psi_1\tau$.

By Lemma 7, as τ is a substitution such that $E_0 \vDash \psi_1\tau$ and $\bar{l}\eta\tau =_E \bar{r}\eta\tau$, then there exist a ground substitution τ°, substitutions $\beta_1, \ldots, \beta_m$, let $\beta = \beta_1^m$, and abstractions $\mathit{abstract}_{\Sigma_1}((l_j\eta\beta_1^{j-1}, r_j\eta\beta_1^{j-1})) = \langle\lambda(\bar{w}_j, \bar{w}'_j).(l_j^\circ, r_j^\circ); (\theta_{l_j}^\circ, \theta_{r_j}^\circ); (\phi_{l_j}^\circ, \phi_{r_j}^\circ)\rangle$, for $1 \le j \le m$, such that $\mathit{dom}(\tau^\circ) = \mathit{dom}(\tau) \cup V_{\hat{w},\hat{w}'}$, $\tau =_{E_0} \tau^\circ_{\mathit{dom}(\tau)}$, $\bar{l}^\circ\tau^\circ =_E \bar{r}^\circ\tau^\circ$, $\tau^\circ \ll_E \beta_{\mathit{dom}(\tau^\circ)}$, call $\psi_2 = \psi_1\beta \wedge \bigwedge_{j=1}^m(\phi_{l_j}^\circ \wedge \phi_{r_j}^\circ)\beta_j^m$, $\mathbf{G_1} \rightsquigarrow^m_{[d1],\beta} \mathbf{\Theta}\mu\beta \mid \psi_2 \mid V, (\mu\beta)_V = \mathbf{G_2}$, call $\xi = \mu\beta$, and for every pair of substitutions ρ and γ such that $\tau^\circ \ll_E (\beta\rho)_{\mathit{dom}(\tau^\circ)}$ and $\tau^\circ =_E (\beta\rho)_{\mathit{dom}(\tau^\circ)} \cdot \gamma$ it holds that $E_0 \vDash \psi_2\rho\gamma$ and $\mathbf{\Theta}\mu\tau =_E \mathbf{\Theta}\xi\rho\gamma$ (†).

Consider the problem $P' = \mathbf{\Theta}'\xi \mid \psi_2 \mid V^\xi, \mathit{none}$ in $\mathcal{R}^{\xi_V}$ and $\mathit{Call}_{\mathcal{R}}^{\xi_V}$, whose corresponding goal is $\mathbf{G}' = \mathbf{\Theta}\xi \mid \psi_2 \mid V^\xi, \mathit{none}$, and take $\rho = \mathit{none}$. As $\tau^\circ \ll_E \beta_{\mathit{dom}(\tau^\circ)}$, then there exists γ' such that $\tau^\circ =_E \beta_{\mathit{dom}(\tau^\circ)} \cdot \gamma'$ and $\mathit{ran}(\tau^\circ) = \mathit{ran}(\beta_{\mathit{dom}(\tau^\circ)} \cdot \gamma')$, so as τ° is ground then γ' is ground. By (†), $E_0 \vDash \psi_2\gamma'$ and $\mathbf{\Theta}\mu\tau =_E \mathbf{\Theta}\xi\gamma'$, so also $\mathbf{\Theta}'\mu\tau =_E \mathbf{\Theta}'\xi\gamma'$, where all the terms and formulas are ground.

Now, $\mathbf{\Theta}'\xi\gamma' =_E \mathbf{\Theta}'\mu\tau = (u_1 \to v_1/S_0(\wedge\Omega))\mu\tau = (u_1 \to v_1/S_0(\wedge\Omega))\mu\sigma_1\delta = (u_1 \to v_1/S_0(\wedge\Omega))\sigma\delta = (u_1\sigma \to v_1\sigma/S_0^\sigma\delta(\wedge\Omega^\sigma)) = \mathbf{\Theta}''$. For the first open goal of $\mathbf{\Theta}''$ there is a c.p.t. $T' = \dfrac{\frac{F_1}{u_1\sigma \to w/(S\epsilon\theta\delta)^\sigma} \quad (\frac{F_2}{w \to v_1\sigma/ST^\sigma})}{u_1\sigma \to v_1\sigma/S_0^\sigma\delta}$ with one less node than T, since $S_0^\sigma\delta = (S\epsilon\theta\delta)^\sigma(;ST^\sigma)$. As we have closed proof trees for all the other open goals in $\mathbf{\Theta}''$ then, by Lemma 5, there are closed proof trees for all the open goals in $\mathbf{\Theta}'\xi\gamma'$, each c.p.t. having the same depth and number of nodes as its correspondent c.p.t. in $\mathbf{\Theta}''$. As $E_0 \vDash \psi_2\gamma'$, then γ' is a solution for P', so we can apply the I.H. to $\mathbf{\Theta}''$, and there exist a formula ψ and substitutions ν' and ρ', such that $\mathbf{\Theta}\xi \mid \psi_2 \mid V^\xi, \mathit{none} \rightsquigarrow^+_{\nu'} \mathit{nil} \mid \psi \mid V^\xi, \nu'$, $\gamma' =_E \nu' \cdot \rho'$, and $\psi\rho'$ is satisfiable, where $\mathit{dom}(\nu') \subseteq V^\xi \subseteq \mathit{ran}(\xi)$. But then, call $\lambda = \beta\nu'$, $\nu = (\xi\nu')_V$, and $\rho = \rho'_{V \cup \mathit{ran}(\nu)}$, also $\mathbf{G_0} \rightsquigarrow^+_\beta \mathbf{\Theta}\xi \mid \psi_2 \mid V, \xi_V \rightsquigarrow^+_{\nu'} \mathit{nil} \mid \psi \mid V, \nu$, i.e., $\rho = \rho'_{V \cup \mathit{ran}(\nu)}$, so $\mathbf{G_0} \rightsquigarrow^+_\lambda \mathit{nil} \mid \psi \mid V, \nu$, and $\psi\rho$ is satisfiable.

As $\mathit{dom}(\nu) \subseteq V$, then $\nu \cdot \rho = (\nu\rho)_{\setminus \mathit{ran}(\nu)} = (\nu\rho'_{V \cup \mathit{ran}(\nu)})_{\setminus \mathit{ran}(\nu)} = (\rho'_V \cup \nu\rho'_{\mathit{ran}(\nu)})_{\setminus \mathit{ran}(\nu)} = \rho'_{V \setminus \mathit{ran}(\nu)} \cup (\nu\rho'_{\mathit{ran}(\nu)})_{\setminus \mathit{ran}(\nu)} = \rho'_V \cup (\nu\rho')_{V \setminus \mathit{dom}(\rho')} = (\nu\rho')_V = (\xi\nu'\rho')_V = (\mu\beta\nu'\rho')_V =_E (\mu\beta\gamma')_V =_E (\mu\tau^\circ)_V =_{E_0} (\mu\tau')_V =_B (\mu\sigma^\circ)_V =_{E_0} (\mu\sigma'\delta)_V = (\sigma\delta)_V = \sigma$, i.e., $\sigma =_E \nu \cdot \rho$.

Finally, as $\psi\rho'$ is satisfiable and ρ is more general than ρ' then $\psi\rho$ is also satisfiable.

6. $ST_1 = \texttt{match}\, t\, \texttt{s.t.}\, \chi\ ?\ S_1 \ :\ S_2$ and there exists a substitution $\delta : V_{ST_1^\sigma} \to \mathcal{T}_\Sigma$, such that $u_1\sigma =_E t\sigma\delta$ and $E_0 \vDash (\phi \wedge \chi)\sigma\delta$ (the proof with S_2 instead of S_1, when $E_0 \vDash (\phi \wedge \neg\chi)\sigma\delta$, is exactly the same).

   Then there is a c.p.t. T, with respect to $\mathcal{D}^\sigma_{\mathcal{R},\mathit{Call}_{\mathcal{R}}}$, of the form $\dfrac{\frac{\frac{F_1}{u_1\sigma \to w/(S_1\delta)^\sigma}}{u_1\sigma \to w/ST_1^\sigma} \quad (\frac{F_2}{w \to v_1\sigma/ST^\sigma})}{u_1\sigma \to v_1\sigma/ST_0^\sigma}$, where w ($= v_1\sigma$ if $ST_0 = ST_1$) is a term in $\mathcal{T}_\Sigma$, and σ and δ ground and $\mathit{dom}(\sigma) \cap \mathit{dom}(\delta) = \emptyset$ implies $(S_1)^\sigma\delta = (S_1\delta)^\sigma$. Let $S_0 = S_1(;ST)$, $\tau = \sigma_1\delta$, $\mathbf{\Theta} = u_1 \to v_1/S_0; \texttt{idle}(\wedge\Delta)$, and $\mathbf{\Theta}' = u_1 \to v_1/S_0(\wedge\Omega)$.

   Let $\mathit{abstract}_{\Sigma_1}((u, t\mu)) = \langle\lambda(\bar{x}, \bar{y}).(u^\circ, t^\circ); (\theta_u^\circ, \theta_t^\circ); (\phi_u^\circ, \phi_t^\circ)\rangle$. As $u_1\sigma$ is ground then $u\sigma'\delta = u_1\mu\sigma'\delta = u_1\sigma\delta = u_1\sigma =_E t\sigma\delta = t\mu\sigma'\delta$ so, by Lemma 4, there exists a ground substitution σ° such that $u^\circ\sigma^\circ =_B t^\circ\sigma^\circ$, $E_0 \vDash (\phi_u^\circ \wedge \phi_t^\circ)\sigma^\circ$, $\mathit{dom}(\sigma^\circ) = \mathit{dom}(\sigma'\delta) \cup \hat{x} \cup \hat{y}$, and $\sigma'\delta =_{E_0} \sigma^\circ_{\mathit{dom}(\sigma'\delta)}$.

   Call $\psi_1 = (\phi \wedge \chi)\mu \wedge \phi_u^\circ \wedge \phi_t^\circ$. As $E_0 \vDash (\phi \wedge \chi)\mu\sigma'\delta$, $V_{(\phi\wedge\chi)\mu\sigma'\delta} \cap (\hat{x} \cup \hat{y}) = \emptyset$, and $\sigma'\delta =_{E_0} \sigma^\circ_{\mathit{dom}(\sigma'\delta)} = \sigma^\circ_{\setminus(\hat{x}\cup\hat{y})}$, then $E_0 \vDash (\phi \wedge \chi)\mu\sigma^\circ$, so $E_0 \vDash \psi_1\sigma^\circ$.

As $u^\circ\sigma^\circ =_B t^\circ\sigma^\circ$, then there exist substitutions ν and τ such that $\eta \in CSU_B(u^\circ = t^\circ)$ and $\sigma^\circ =_B \eta{\cdot}\tau$, so $\psi_1\sigma^\circ = \psi_1\eta\tau$, hence $E_0 \vDash \psi_1\eta\tau$ and $\psi_1\eta$ is satisfiable. Call $\xi = \mu\eta$. Now, $\mathbf{G_0} = (u_1 \to v_1/ST_1; \mathtt{idle}(\wedge\Delta))\mu \mid \phi\mu \mid V, \mu \leadsto_{[i1],\eta} (u_1 \to v_1/S_0; \mathtt{idle}(\wedge\Delta))\xi \mid \psi_1\eta \mid V, \xi_V = \mathbf{G_1}$.

Consider the problem $P' = \mathbf{\Theta}'\xi \mid \psi_1\eta \mid V^\xi, none$ in $\mathcal{R}^{\xi_V}$ and $Call_{\mathcal{R}}^{\xi_V}$, whose corresponding goal is $\mathbf{G}' = \mathbf{\Theta}\xi \mid \psi_1\eta \mid V^\xi, none$, and take $\rho = none$.

Now, $\mathbf{\Theta}'\xi\tau = \mathbf{\Theta}'\mu\eta\tau =_B \mathbf{\Theta}'\mu\sigma^\circ = (u_1 \to v_1/S_0(\wedge\Omega))\mu\sigma^\circ = (u_1 \to v_1/S_0(\wedge\Omega))\mu\sigma^\circ =_{E_0} (u_1 \to v_1/S_0(\wedge\Omega))\mu\sigma_1\delta = (u_1 \to v_1/S_0(\wedge\Omega))\sigma\delta = (u_1\sigma \to v_1\sigma/S_0^\sigma\delta(\wedge\Omega^\sigma)) = \mathbf{\Theta}''$.

For the first open goal of $\mathbf{\Theta}''$ there is a c.p.t. $T' = \dfrac{\dfrac{F_1}{u_1\sigma \to w/(S_1\delta)^\sigma} \quad (\dfrac{F_2}{w \to v_1\sigma/ST^\sigma})}{u_1\sigma \to v_1\sigma/S_0^\sigma\delta}$ with one less node than T, since $S_0^\sigma\delta = (S_1\delta)^\sigma(;ST^\sigma)$. As we have closed proof trees for all the other open goals in $\mathbf{\Theta}''$ then, by Lemma 5, there are closed proof trees for all the open goals in $\mathbf{\Theta}'\xi\tau$, each c.p.t. having the same depth and number of nodes as its correspondent c.p.t. in $\mathbf{\Theta}''$. As $E_0 \vDash \psi_1\eta\tau$, then τ is a solution for P', so we can apply the I.H. to $\mathbf{\Theta}''$, and there exist a formula ψ_2 and substitutions ν'' and ρ'', such that $\mathbf{\Theta}\xi \mid \psi_1\eta \mid V^\xi, none \leadsto^+_{\nu''} nil \mid \psi_2 \mid V^\xi, \nu''$, $\tau =_E \nu''{\cdot}\rho''$, and $\psi_2\rho''$ is satisfiable, where $dom(\nu'') \subseteq V^\xi \subseteq ran(\xi)$. Call $\lambda' = \eta\nu''$, $\nu' = (\xi\nu'')_V$, and $\rho' = \rho''_{V \cup ran(\nu')}$. As ρ' is more general than ρ'' and $\psi_2\rho''$ is satisfiable then $\psi_2\rho'$ is satisfiable. Also, $\nu' \cdot \rho' = (\xi\nu'')_V \cdot \rho''_{V \cup ran(\nu')} = (\xi\nu''\rho'')_V = (\mu\eta\nu''\rho'')_V =_E (\mu\eta\tau)_V =_B (\mu\sigma^\circ)_V =_{E_0} (\mu\sigma'\delta)_V = (\sigma\delta)_V = \sigma_V = \sigma$, i.e., $\sigma =_E \nu' \cdot \rho'$. Now:

- if $n = 1$ then $\mathbf{G_0} \leadsto_{[i1],\eta} (u_1 \to v_1\ /S_0; \mathtt{idle})\xi \mid \psi_1\eta \mid V, \xi_V \leadsto^+_{\nu''} nil \mid \psi_2 \mid V, \nu'$, i.e., $\mathbf{G_0} \leadsto^+_{\lambda'} nil \mid \psi_2 \mid V, \nu'$, $\sigma =_E \nu' \cdot \rho'$, and $\psi_2\rho'$ is satisfiable, so $\psi = \psi_2$, $\lambda = \lambda'$, $\nu = \nu'$, and $\rho = \rho'$;
- else $\mathbf{G_0} \leadsto_{[i1],\eta} (u_1 \to v_1\ /S_0; \mathtt{idle} \wedge \Delta)\mu\eta \mid \psi_1\eta \mid V, \xi_V \leadsto^+_{\nu''} \Delta(\mu\lambda') \mid \psi_2 \mid V, \nu'$, i.e., $\mathbf{G_0} \leadsto^+_{\lambda'} \Delta(\mu\lambda') \mid \psi_2 \mid V, \nu'$, $\sigma =_E \nu' \cdot \rho'$, and $\psi_2\rho'$ is satisfiable. The rest of the proof is the one given at the end of the induction step for the base cases.

7. $ST_1 = \mathtt{matchrew}\ t\ \mathtt{s.t.}\ \bar{l} = \bar{r} \wedge \chi\ \mathtt{by}\ \bar{z}\ \mathtt{using}\ \bar{S}$, where $|\bar{z}| = k$, $|\bar{l}| = |\bar{r}| = m$, and $t = t[\bar{z}]_{\bar{p}}$.

   By definition, $V \cap \hat{z} = \emptyset$ and $\hat{z} \subset \mathcal{X}_1$. As $[v_1\sigma]_E \in ST_0^\sigma @ [u_1\sigma]_E$ then there is a c.p.t., with respect to $\mathcal{D}^\sigma_{\mathcal{R},Call_{\mathcal{R}}}$, of the form $\dfrac{\dfrac{\dfrac{F_1}{z_1\delta \to t_1/S_1^\sigma\delta} \cdots \dfrac{F_k}{z_k\delta \to t_k/S_k^\sigma\delta}}{u_1\sigma \to t\sigma\delta[\bar{t}]_{\bar{p}}/ST_1^\sigma} \quad (\dfrac{F}{t\sigma\delta[\bar{t}]_{\bar{p}} \to v_1\sigma/ST^\sigma})}{u_1\sigma \to v_1\sigma/ST_0^\sigma}$, where $\hat{z} \subseteq dom(\delta)$, ground substitution, $u_1\sigma =_E t\sigma\delta$, $\bar{l}\sigma\delta =_E \bar{r}\sigma\delta$, and $E_0 \vDash \chi\sigma\delta$, with all these terms and the formula ground. Also, if $ST_0 = ST_1$ then $t\sigma\delta[\bar{t}]_{\bar{p}} =_E v_1\sigma$.

   Let $abstract_{\Sigma_1}((u, t\mu)) = \langle\lambda(\bar{w}, \bar{w}').(u^\circ, t^\circ); (\theta^\circ_u, \theta^\circ_t); (\phi^\circ_u, \phi^\circ_t)\rangle$. As $u_1\sigma$ is ground, then $u\sigma'\delta = u_1\mu\sigma'\delta = u_1\sigma\delta = u_1\sigma =_E t\sigma\delta = t\mu\sigma'\delta$ so, by Lemma 4, there exists a ground substitution σ° such that $u^\circ\sigma^\circ =_B t^\circ\sigma^\circ$, $E_0 \vDash (\phi^\circ_u \wedge \phi^\circ_t)\sigma^\circ$, $dom(\sigma^\circ) = dom(\sigma'\delta) \cup \hat{w} \cup \hat{w}'$, and $\sigma'\delta =_{E_0} \sigma^\circ_{dom(\sigma'\delta)}$.

   Call $\psi_1 = (\phi \wedge \chi)\mu \wedge \phi^\circ_u \wedge \phi^\circ_t$. As $E_0 \vDash (\phi \wedge \chi)\mu\sigma'\delta$, $V_{(\phi\wedge\chi)\mu} \cap (\hat{w} \cup \hat{w}') = \emptyset$, and $\sigma'\delta =_{E_0} \sigma^\circ_{dom(\sigma'\delta)} = \sigma^\circ_{\setminus(\hat{w}\cup\hat{w}')}$, then $E_0 \vDash (\phi \wedge \chi)\mu\sigma^\circ$, so $E_0 \vDash \psi_1\sigma^\circ$.

   As $u^\circ\sigma^\circ =_B t^\circ\sigma^\circ$, then there exist substitutions ν and τ such that $\eta \in CSU_B(u^\circ = t^\circ)$ and $\sigma^\circ =_B \eta{\cdot}\tau$, so $\psi_1\sigma^\circ = \psi_1\eta\tau$, hence $E_0 \vDash \psi_1\eta\tau$, ground formula, and $\psi_1\eta$ is satisfiable. Call $\mathbf{\Theta} = \bigwedge_{j=1}^k (x_j \to y_j/S_j; \mathtt{idle}) \wedge t[\bar{y}]_{\bar{p}} \to v_1/ST; \mathtt{idle} \wedge \Delta$, where $\bar{x}$ and $\bar{y}$ are fresh versions of $\bar{z}$, and let λ be the renaming from $\bar{x}$ to $\bar{z}$, i.e., $\bar{x}\lambda = \bar{z}$, and let $\mathbf{\Theta}' = \bigwedge_{j=1}^k (x_j \to y_j/S_j) \wedge t[\bar{y}]_{\bar{p}} \to v_1/ST \wedge \Omega$.

   Now, $\mathbf{G_0} = (u_1 \to v_1/ST_1; ST; \mathtt{idle} \wedge \Delta)\mu \mid \phi\mu \mid V, \mu \leadsto_{[m],\eta} (\bigwedge_{j=1}^m (l_j \to r_j/\mathtt{idle}) \wedge \mathbf{\Theta})\mu\eta \mid \psi_1\eta \mid V, (\mu\eta)_V = \mathbf{G_1}$, all ground terms.

   As $\bar{l}\sigma\delta =_E \bar{r}\sigma\delta$, $\sigma'\delta =_{E_0} \sigma^\circ_{dom(\sigma'\delta)}$, $\sigma'\delta =_{E_0} \sigma^\circ_{\setminus(\hat{w}\cup\hat{w}')}$, $\sigma^\circ =_B \eta{\cdot}\tau$, and $V_{\bar{l}\mu,\bar{r}\mu} \cap (\hat{w} \cup \hat{w}' \cup ran(\eta)) = \emptyset$, then $(\bar{l}, \bar{r})\mu\eta\tau = (\bar{l}, \bar{r})\mu(\eta \cdot \tau) =_B (\bar{l}, \bar{r})\mu\sigma^\circ = (\bar{l}, \bar{r})\mu\sigma^\circ_{\setminus(\hat{w}\cup\hat{w}')} =_{E_0} (\bar{l}, \bar{r})\mu\sigma'\delta$, i.e., $(\bar{l}, \bar{r})\mu\eta\tau =_E (\bar{l}, \bar{r})\sigma\delta$, so $\bar{l}\mu\eta\tau =_E \bar{r}\mu\eta\tau$, since $\bar{l}\sigma\delta =_E \bar{r}\sigma\delta$.

In the same way, as $u_1\sigma =_E t\sigma\delta$, ground terms, and $v_1\sigma$ is also ground, then $V_{\theta\mu\eta\tau} = \hat{x} \cup \hat{y}$. Let $\tau' = \tau \cup \lambda \cdot \delta_{\hat{z}} \cup \{\bar{y} \mapsto \bar{t}\}$, so $V_{G_1\tau'} = \emptyset$. As $dom(\mu) \subseteq V$ then $\mathbf{\Theta}\mu = \mathbf{\Theta}^\mu$ so, by Lemma 7, as τ' is a ground substitution such that $V_{G_1} \subseteq dom(\tau')$, $E_0 \vDash \psi_1\eta\tau'$, and $\bar{l}\mu\eta\tau' =_E \bar{r}\mu\eta\tau'$, there exist a ground substitution τ°, substitutions $\beta_1, \ldots, \beta_m$, let $\beta = \beta_1^m$, and abstractions $abstract_{\Sigma_1}((l_j\beta_1^{j-1}, r_j\beta_1^{j-1})) = \langle\lambda(\bar{w}_j, \bar{w}'_j).(l^\circ_j, r^\circ_j); (\theta^\circ_{l_j}, \theta^\circ_{r_j}); (\phi^\circ_{l_j}, \phi^\circ_{r_j})\rangle$, for $1 \leq j \leq m$, such that $dom(\tau^\circ) = dom(\tau') \cup V_{\hat{\bar{w}},\hat{\bar{w}}'}$, $\tau' =_{E_0} \tau^\circ_{dom(\tau')}$, $\bar{l}^\circ\tau^\circ =_E \bar{r}^\circ\tau^\circ$, $\tau^\circ \ll_E \beta_{dom(\tau^\circ)}$, call $\xi = \mu\eta\beta$ and $\psi_2 = \psi_1\eta\beta \wedge \bigwedge_{j=1}^m(\phi^\circ_{l_j} \wedge \phi^\circ_{r_j})\beta_j^m$, also $\mathbf{G_1} \rightsquigarrow^m_{[d1]} \mathbf{\Theta}\xi \mid \psi_2 \mid V, \xi_V = \mathbf{G_2}$, and for every pair of substitutions ρ and γ such that $\tau^\circ \ll_E (\beta\rho)_{dom(\tau^\circ)}$ and $\tau^\circ =_E (\beta\rho)_{dom(\tau^\circ)} \cdot \gamma$ it holds that $E_0 \vDash \psi_2\rho\gamma$ and $\mathbf{\Theta}\mu\eta\tau' =_E \mathbf{\Theta}\xi\rho\gamma$ (†).

Consider the problem $P' = \mathbf{\Theta}'\xi \mid \psi_2 \mid (\hat{y} \cup \hat{x} \cup V)^\xi, none$ in $\mathcal{R}^{\xi_V}$ and $Call^{\xi_V}_{\mathcal{R}}$, whose corresponding goal is $\mathbf{G}' = \mathbf{\Theta}\xi \mid \psi_2 \mid (\hat{y} \cup \hat{x} \cup V)^\xi, none$, and take $\rho = none$. As $\tau^\circ \ll_E \beta_{dom(\tau^\circ)}$, then there exists γ' such that $\tau^\circ =_E \beta_{dom(\tau^\circ)} \cdot \gamma'$ and $ran(\tau^\circ) = ran(\beta_{dom(\tau^\circ)} \cdot \gamma')$, so as τ° is ground then γ' is ground. By (†), $E_0 \vDash \psi_2\gamma'$ and $\mathbf{\Theta}\mu\eta\tau' =_E \mathbf{\Theta}\xi\gamma'$, so also $\mathbf{\Theta}'\mu\eta\tau' =_E \mathbf{\Theta}'\xi\gamma'$, where all the terms and formulas are ground. Now, $\mathbf{\Theta}'\xi\gamma' =_E \mathbf{\Theta}'\mu\eta\tau' = (\bigwedge_{j=1}^k(x_j \to y_j/S_j) \wedge t[\bar{y}]_{\bar{p}} \to v_1/ST; \mathtt{idle} \wedge \Delta)\mu\eta\tau' = (\bigwedge_{j=1}^k(z_j\delta \to t_j/S_j) \wedge t[\bar{t}]_{\bar{p}} \to v_1/ST; \mathtt{idle} \wedge \Delta)\mu\eta\tau =_B (\bigwedge_{j=1}^k(z_j\delta \to t_j/S_j) \wedge t[\bar{t}]_{\bar{p}} \to v_1/ST; \mathtt{idle} \wedge \Delta)\mu\sigma^\circ =_{E_0} (\bigwedge_{j=1}^k(z_j\delta \to t_j/S_j) \wedge t[\bar{t}]_{\bar{p}} \to v_1/ST; \mathtt{idle} \wedge \Delta)\mu\sigma'\delta = (\bigwedge_{j=1}^k(z_j\delta \to t_j/S_j) \wedge t[\bar{t}]_{\bar{p}} \to v_1/ST; \mathtt{idle} \wedge \Delta)\sigma\delta = \bigwedge_{j=1}^k(z_j\delta \to t_j/S_j^\sigma\delta) \wedge t[\bar{t}]_{\bar{p}} \to v_1/ST^\sigma; \mathtt{idle} \wedge \Delta^\sigma = \mathbf{\Theta}''$. As we have closed proof trees for all the open goals in $\mathbf{\Theta}''$ then, by Lemma 5, there are closed proof trees for all the open goals in $\mathbf{\Theta}'\xi\gamma'$, each c.p.t. having the same depth and number of nodes as its correspondent c.p.t. in $\mathbf{\Theta}''$. As $E_0 \vDash \psi_2\gamma'$, then γ' is a solution for P'. The difference with respect to the closed proof trees in the answer σ for the reachability problem P, is that we have two less nodes, $t\sigma\delta \to t\sigma\delta[\bar{t}]_{\bar{p}}/ST_1^\sigma$ and $t\sigma\delta \to v_1\sigma/ST_1^\sigma; ST^\sigma$, so we can apply the I.H. to $\mathbf{\Theta}'\xi\gamma'$, and there exist a formula ψ and substitutions ν' and ρ', such that $\mathbf{G}' = \mathbf{\Theta}\xi \mid \psi_2 \mid (\hat{y} \cup \hat{x} \cup V)^\xi, none \rightsquigarrow^+ nil \mid \psi \mid (\hat{y} \cup \hat{x} \cup V)^\xi, \nu'$, $\gamma' =_E \nu' \cdot \rho'$, and $\psi\rho'$ is satisfiable, where $dom(\nu') \subseteq (\hat{y} \cup \hat{x} \cup V)^\xi \subseteq ran(\xi)$. But then, call $\nu = (\xi\nu')_V$ and $\rho = \rho'_{V \cup ran(\nu)}$, also $\mathbf{G_2} = \mathbf{\Theta}\xi \mid \psi_2 \mid V, \xi_V \rightsquigarrow^+ nil \mid \psi \mid V, \nu$.

As $dom(\nu) \subseteq V$, then $\nu \cdot \rho = (\nu\rho)_{\setminus ran(\nu)} = (\nu\rho'_{V \cup ran(\nu)})_{\setminus ran(\nu)} = (\rho'_V \cup \nu\rho'_{ran(\nu)})_{\setminus ran(\nu)} = \rho'_{V \setminus ran(\nu)} \cup (\nu\rho'_{ran(\nu)})_{\setminus ran(\nu)} = \rho'_V \cup (\nu\rho')_{V \setminus dom(\rho')} = (\nu\rho')_V = (\xi\nu'\rho')_V = (\mu\eta\beta\nu'\rho')_V =_E (\mu\eta\beta\gamma')_V =_E (\mu\eta\tau^\circ)_V =_{E_0} (\mu\eta\tau')_V =_B (\mu\sigma^\circ)_V =_{E_0} (\mu\sigma'\delta)_V = (\sigma\delta)_V = \sigma$, i.e., $\sigma =_E \nu \cdot \rho$.

Finally, as $\psi\rho'$ is satisfiable and ρ is more general than ρ' then $\psi\rho$ is also satisfiable.

8. $ST_1 = c[\gamma]\{\bar{S}\}$, with $c : l \to r \textit{ if } C$ a rule in R, $C = \bar{l} \to \bar{r} \mid \chi$, $\bar{S} = S_1, \ldots, S_m$, and $dom(\gamma) \cap vars(\bar{S}) = \emptyset$.

   As $[v_1\sigma]_E \in ST_0^\sigma@[u_1\sigma]_E$ then there is a c.p.t. T, with respect to $\mathcal{D}^\sigma_{\mathcal{R},Call_{\mathcal{R}}}$, of the form $\frac{\frac{T_1 \cdots T_m}{u_1\sigma \to w/ST_1^\sigma} \quad (T_0)}{u_1\sigma \to v_1\sigma/ST_0^\sigma}$, where $T_i = \frac{F_i}{l_i\gamma\sigma\delta \to r_i\gamma\sigma\delta/S_i^\sigma\delta}$, for $1 \leq i \leq m$, $T_0 = \frac{F}{w \to v_1\sigma/ST^\sigma}$, $[w]_E \in c^\sigma[(\gamma\sigma)_{dom(\gamma)}]@[u_1\sigma]_E$, where $\delta : vars(c\gamma\sigma) \to \mathcal{T}_\Sigma$, with $E_0 \vDash \chi\gamma\sigma\delta$, there is $p \in pos(u_1\sigma)$ s.t. $u_1\sigma =_E u_1\sigma[l\gamma\sigma\delta]_p$, and $w = u_1\sigma[r\gamma\sigma\delta]_p$ if T_0 exists or $w = v_1\sigma$, otherwise. By Lemma 5.13, $\bar{l}\gamma\sigma\delta \to_{R^\sigma/E} \bar{r}\gamma\sigma\delta$, so $u_1\sigma \xrightarrow[c^\sigma,p,(\gamma\sigma)_{dom(\gamma)}\delta \; R^\sigma/E]{1} w$. Call $\alpha = \gamma\sigma\delta$ ($= \gamma\delta\sigma$ since $dom(\delta) \cap dom(\sigma) = \emptyset$ and both substitutions are ground), $\alpha' = \gamma\sigma$, and $c' = c^\sigma(\gamma\sigma)_{dom(\gamma)}$ ($= c\alpha'$ because σ is ground and, by definition, $dom(\gamma) \cap dom(\sigma) = \emptyset$). As $u_1\sigma \xrightarrow[c'\delta,p \; R^\sigma/E]{1} w$ then, By Theorem 1, $u_1\sigma \xrightarrow[c'_1\delta,p' \; R^\sigma,B]{1} w$, since $\mathcal{R}$ is closed under B-extensions, with $c'_1 \in c'_B$ and proper p', as seen in the proof of Lemma 2, so also $u_1\sigma \xrightarrow[c'_1\delta,p' \; R^\sigma/E]{1} w$, hence we can assume that $c' = c'_1$, $p = p'$, and

T is the c.p.t. for $[w]_E \in c_1'@[u_1\sigma]_E$ using $u_1\sigma \xrightarrow[c_1',p',\delta]{1}{}_{R^\sigma/E}\ w$. Since $\sigma = \mu\sigma'$, if we call $l_1 = l\gamma\mu$ and $r_1 = r\gamma\mu$ then $c'(= c\alpha')$ has also the form $c\alpha' : l_1\sigma' \to r_1\sigma'$ *if* $C\alpha'$.
Let $c_2 : l_2 \to r_2$ *if* C_2 be a fresh version of c^μ except for $dom(\gamma) \cup V^\mu$ $(= dom(\gamma) \cup dom(\sigma'))$, and let τ be the renaming that verifies $c_2 = c^\mu\tau$, so $(l_2, r_2, C_2) = (l, r, C)(\mu \uplus \tau)$, where $(dom(\tau) \cup ran(\tau)) \cap (dom(\gamma) \cup V^\mu) = \emptyset$. Then $l_2(\gamma\mu)_{dom(\gamma)} = l(\mu \uplus \tau)(\gamma\mu)_{dom(\gamma)} = l((\gamma\mu)_{dom(\gamma)} \uplus \mu \uplus \tau) = l((\gamma\mu)_{dom(\gamma)} \uplus \mu)\tau = l\gamma\mu\tau = l_1\tau$, so also $r_2(\gamma\mu)_{dom(\gamma)} = r_1\tau$ and $C_2(\gamma\mu)_{dom(\gamma)} = C\gamma\mu\tau$. Call $l_c = l_2(\gamma\mu)_{dom(\gamma)}$ and $\sigma'' = \tau^{-1}\sigma'$. Then $l_c\sigma'' = l_1\tau\tau^{-1}\sigma' = l_1\sigma'$. Now:

(a) $abstract_{\Sigma_1}(l_c) = \langle \lambda\bar{y}.l^\circ; \theta_l^\circ; \phi_l^\circ \rangle$, where $\bar{y} = y_1, \ldots, y_{i_y}$, $l^\circ = l_c[\bar{y}]_{\bar{p}}$, $\bar{p} = p_1, \ldots, p_{i_y}$, $\hat{p} = top_{\Sigma_0}(l_c)$, $\theta_l^\circ = \bigcup_{i=1}^{i_y}\{y_i \mapsto l_c|_{p_i}\}$, and $\phi_l^\circ = \bigwedge_{i=1}^{i_y} y_i = l_c|_{p_i}$;

(b) since $l_1\sigma' = l_c\sigma''$ and $top_{\Sigma_0}(l_c) \subseteq top_{\Sigma_0}(l_c\sigma'')$, then we have that $abstract_{\Sigma_1}(l_1\sigma') = abstract_{\Sigma_1}(l_c\sigma'') = \langle \lambda\bar{y}\bar{z}.l^\circ_{c\sigma''}; \theta^\circ_{c\sigma''}; \phi^\circ_{c\sigma''} \rangle$, where $\bar{z} = z_1, \ldots, z_{i_z}$, $l^\circ_{c\sigma''} = l_c\sigma''[\bar{y}]_{\bar{p}}[\bar{z}]_{\bar{q}}$, $\hat{q} = top_{\Sigma_0}(l_c\sigma'') \setminus top_{\Sigma_0}(l_c)$, $\theta^\circ_{c\sigma''} = \bigcup_{i=1}^{i_y}\{y_i \mapsto l_c|_{p_i}\sigma''\} \cup \bigcup_{j=1}^{i_z}\{z_j \mapsto l_c\sigma''|_{q_j}\}$, and $\phi^\circ_{c\sigma''} = (\bigwedge_{i=1}^{i_y} y_i = l_c|_{p_i}\sigma'' \wedge \bigwedge_{j=1}^{i_z} z_j = l_c\sigma''|_{q_j})$;

(c) as $u_1\sigma \xrightarrow[c',p,\delta]{1}{}_{R^\sigma,B}\ w$, then there is a substitution $\delta' : \hat{y} \cup \hat{z} \cup V_{c'} \to \mathcal{T}_\Sigma$, such that $\delta'_{V_{c'}} = \delta$, $rep(u_1\sigma|_p) =_B l^\circ_{c\sigma''}\delta'$, $w =_E u_1\sigma[r_1\sigma'\delta']_p = u_1\sigma[r_1\sigma'\delta]_p = u_1\sigma[r\gamma\mu\sigma'\delta]_p = u_1\sigma[r\gamma\sigma\delta]_p = u_1\sigma[r\alpha]_p$, and $E_0 \vDash (\chi\alpha' \wedge \phi^\circ_{c\sigma''})\delta'$, so $E_0 \vDash \chi\alpha$ (since $\chi\alpha'\delta' = \chi\alpha'\delta = \chi\alpha$), i.e., $E_0 \vDash \chi\gamma\delta\mu\sigma'$, $\bar{y}\delta' =_{E_0} l_c|_{\bar{p}}\sigma''\delta'$ and $\bar{z}\delta' =_{E_0} l_c\sigma''|_{\bar{q}}\delta'$;

(d) as $p \in pos_{\Sigma_1}(u_1\sigma)$ and σ is R/E-normalized, hence R, E-normalized by Theorem 1, then $p \in pos_{\Sigma_1}(u_1)$, so $u_1\sigma|_p = u_1|_p\sigma = u_1|_p\mu\sigma' = u_1\mu|_p\sigma' = u|_p\sigma'$; and

(e) as τ is a fresh renaming then $\emptyset = V_u \cap ran(\tau) = V_u \cap dom(\tau^{-1})$, so $u|_p\tau^{-1}\sigma' = u|_p\sigma' = u_1\sigma|_p =_{E_0} rep(u_1\sigma|_p) =_B l^\circ_{c\sigma''}\delta' = l_c\sigma''[\bar{y}]_{\bar{p}}[\bar{z}]_{\bar{q}}\delta' =_{E_0} l_c\sigma'' = l_c\tau^{-1}\sigma'$, i.e., $u|_p\tau^{-1}\sigma' =_E l_c\tau^{-1}\sigma'$;

Let $abstract_{\Sigma_1}(u|_p) = \langle \lambda\bar{x}.u^\circ; \theta_u^\circ; \phi_u^\circ \rangle$. As $dom(\tau^{-1}\sigma') = ran(\tau) \cup V^\mu$ then, by Lemma 4, there exists a ground substitution σ° such that $u^\circ\sigma^\circ =_B l^\circ\sigma^\circ$, $E_0 \vDash (\phi_u^\circ \wedge \phi_l^\circ)\sigma^\circ$, $dom(\sigma^\circ) = dom(\tau^{-1}\sigma') \cup \hat{x} \cup \hat{y} = ran(\tau) \cup V^\mu \cup \hat{x} \cup \hat{y}$, and $\tau^{-1}\sigma' =_{E_0} \sigma^\circ_{dom(\tau^{-1}\sigma')} = \sigma^\circ_{ran(\tau)\cup V^\mu}$, so $(\tau^{-1}\sigma')_{V^\mu} =_{E_0} \sigma^\circ_{V^\mu}$ and $\tau^{-1} = (\tau^{-1}\sigma')_{\setminus V^\mu} =_{E_0} \sigma^\circ_{\setminus V^\mu}$. As $(dom(\tau) \cup ran(\tau)) \cap V^\mu = \emptyset$ and $dom(\sigma') = V^\mu$ then $\sigma' = \sigma'_{V^\mu} = (\tau^{-1}\sigma')_{V^\mu} =_{E_0} \sigma^\circ_{V^\mu}$.

As $u^\circ\sigma^\circ =_B l^\circ\sigma^\circ$, then there exist substitutions ϑ and ζ' such that $\vartheta \in CSU_B(u^\circ = l^\circ)$ and $\sigma^\circ =_B \vartheta \cdot \zeta'$, call $\xi = \mu \cdot \vartheta$ and $\zeta = \zeta'_{ran(\xi_V)\cup(V\setminus dom(\xi_V))}$. As $dom(\mu) \subseteq V$ and $\sigma' =_{E_0} \sigma^\circ_{V^\mu}$ then $\sigma = \mu \cdot \sigma' =_{E_0} \mu \cdot \sigma^\circ_{V^\mu} =_B \mu \cdot (\vartheta \cdot \zeta')_{V^\mu} = \mu \cdot (\vartheta\zeta')_{V^\mu} = (\mu\vartheta\zeta')_V = \xi_V \cdot \zeta'_{ran(\xi_V)\cup(V\setminus dom((\xi_V))} = \xi_V \cdot \zeta'_{ran(\xi_V)\cup(V\setminus dom(\xi_V))} = \xi_V \cdot \zeta$, i.e., $\sigma =_E \xi_V \cdot \zeta$, so also $\sigma =_E (\mu\vartheta\zeta')_V$.

As $\chi_2(\gamma\mu)_{dom(\gamma)} = \chi\gamma\mu\tau$, $dom(\sigma^\circ) = ran(\tau) \cup V^\mu \cup \hat{x} \cup \hat{y}$, and $\tau^{-1}\sigma' =_{E_0} \sigma^\circ_{ran(\tau)\cup V^\mu}$, then $\chi_2(\gamma\mu)_{dom(\gamma)}\sigma^\circ\delta' = \chi\gamma\mu\tau\sigma^\circ_{ran(\tau)\cup V^\mu}\delta' =_{E_0} \chi\gamma\mu\tau\tau^{-1}\sigma'\delta' = \chi\gamma\mu\sigma'\delta' = \chi\gamma\mu\sigma'\delta = \chi\gamma\delta\mu\sigma'$ so, as $E_0 \vDash \chi\gamma\delta\mu\sigma'$, also $E_0 \vDash \chi_2(\gamma\mu)_{dom(\gamma)}\sigma^\circ\delta'$.

As $E_0 \vDash \phi\sigma$, $V_{\phi\mu} \subseteq V^\mu$, and $\sigma' =_{E_0} \sigma^\circ_{V^\mu}$ then $\phi\mu\sigma^\circ =_{E_0} \phi\mu\sigma' = \phi\sigma$, so $E_0 \vDash \phi\mu\sigma^\circ$. Now, as $E_0 \vDash (\phi_u^\circ \wedge \phi_l^\circ)\sigma^\circ$, then $E_0 \vDash (\phi\mu \wedge \phi_u^\circ \wedge \phi_l^\circ)\sigma^\circ$ ground formula, so $E_0 \vDash (\phi\mu \wedge \phi_u^\circ \wedge \phi_l^\circ)\sigma^\circ\delta'$ and $E_0 \vDash (\phi\mu \wedge \phi_u^\circ \wedge \phi_l^\circ \wedge \chi_2(\gamma\mu)_{dom(\gamma)})\sigma^\circ\delta'$. Call $\varphi^\circ = \phi\mu \wedge \phi_u^\circ \wedge \phi_l^\circ \wedge \chi_2(\gamma\mu)_{dom(\gamma)}$, and let $\varphi = \varphi^\circ\vartheta$. As $\sigma^\circ =_B \vartheta \cdot \zeta'$, so $\varphi^\circ\sigma^\circ = \varphi^\circ\vartheta\zeta' = \varphi\zeta'$, then $E_0 \vDash \varphi\zeta'\delta'$, call $\delta'' = \zeta'\delta'$, hence φ is also satisfiable. Call $\mathbf{\Theta} = \bar{l}\gamma\tau \to \bar{r}\gamma\tau/\bar{S}\tau; \texttt{idle}(\wedge u_1[r\gamma\tau]_p \to v_1/ST; \texttt{idle}) \wedge \Delta$ and $\mathbf{\Theta'} = \bar{l}\gamma\tau \to \bar{r}\gamma\tau/\bar{S}\tau(\wedge u_1[r\gamma\tau]_p \to v_1/ST) \wedge \Omega$.

Now, $\mathbf{G_0} = (u_1 \to v_1/ST_1; ST; \texttt{idle} \wedge \Delta)^\mu \mid \phi\mu \mid V, \mu \rightsquigarrow_{[t]}$
$u \to^1 x_0, x_0 \to v/ST_1^\mu; ST^\mu; \texttt{idle} \wedge \Delta^\mu \mid \phi\mu \mid V, \mu \rightsquigarrow_{[c]}^*$
$u|_p \to^1 x, u[x]_p \to v/c^\mu[(\gamma\mu)_{dom(\gamma)}]\{\bar{S}^\mu\}; ST^\mu; \texttt{idle} \wedge \Delta^\mu \mid \phi\mu \mid V, \mu = \mathbf{G_1}$,
where $u|_p$ cannot be a variable, say x_u, because as $p \in pos_\Sigma(u_1)$ then, by (c), also $x_u\sigma' \to^1_{R,B} r\alpha$, so σ would not be R/E-normalized.
As $\vartheta \in CSU_B(u^\circ = l^\circ)$ and $c_2 : l_2 \to r_2\ \mathit{if}\ C_2$, where $r_2(\gamma\mu)_{dom(\gamma)} = r_1\tau = r\gamma\mu\tau$ and $C_2(\gamma\mu)_{dom(\gamma)} = C\gamma\mu\tau$, call $\vartheta' = \vartheta \cup \{x \mapsto r_1\tau\vartheta\}$, then $\mathbf{G_1} \rightsquigarrow_{[r],\vartheta'} \mathbf{\Theta}\xi \mid \varphi \mid V, \xi_V = \mathbf{G_2}$, call $V_0 = (\hat{y} \cup \hat{z} \cup V \cup V_{c\gamma\tau})^\xi$.

Consider the problem $P' = \mathbf{\Theta}'\xi \mid \varphi \mid V_0, \mathit{none}$ in $\mathcal{R}^{\xi_V}$ and $\mathit{Call}_{\mathcal{R}}^{\xi_V}$, whose corresponding goal is $\mathbf{G}' = \mathbf{\Theta}\xi \mid \varphi \mid V_0, \mathit{none}$. Now, $\mathbf{\Theta}'\xi\delta'' = \mathbf{\Theta}'\mu\vartheta\zeta'\delta' =_E \mathbf{\Theta}'(\sigma \cup (\vartheta\zeta'\delta')_{\setminus V^\mu}) =_B \mathbf{\Theta}'(\sigma \cup (\sigma^\circ\delta')_{\setminus V^\mu}) =_{E_0} \mathbf{\Theta}'(\sigma \cup \tau^{-1}\delta') = \mathbf{\Theta}'(\sigma \cup (\tau^{-1}\delta)_{V_{c'}} \cup \delta'_{\hat{y}\cup\hat{z}}) = \mathbf{\Theta}'(\sigma \cup (\tau^{-1}\delta)_{V_{c'}}) = \mathbf{\Theta}'\tau^{-1}\sigma\delta = \bar{l}\gamma\sigma\delta \to \bar{r}\gamma\sigma\delta/\bar{S}^\sigma\delta(\wedge u_1\sigma[r\gamma\sigma\delta]_p \to v_1\sigma/ST^\sigma) \wedge \Omega^\sigma = \mathbf{\Theta}''$.
We have closed proof trees $T_1, \ldots, T_m$ (and T_0 if ST_0 is a concatenation) for the open goals before Ω^σ, whose sum of nodes is two less that number of nodes in T. As we have closed proof trees for all the other open goals in $\mathbf{\Theta}''$ then, by Lemma 5, there are closed proof trees for all the open goals in $\mathbf{\Theta}'\xi\delta''$, each c.p.t. having the same depth and number of nodes as its correspondent c.p.t. in $\mathbf{\Theta}''$. As $E_0 \vDash \varphi\delta''$, then δ'' is a solution for P', so we can apply the I.H. to $\mathbf{\Theta}'\xi\delta''$, and there exist a formula φ_2 and substitutions ν'' and ρ'', such that $\mathbf{\Theta}\xi \mid \varphi \mid V_0, \mathit{none} \rightsquigarrow^+_{\nu''} \mathit{nil} \mid \varphi_2 \mid V_0, \nu''_{V_0}$, $\delta'' =_E \nu''_{V_0} \cdot \rho''$, and $\varphi_2\rho''$ is satisfiable, where $dom(\nu''_{V_0}) \subseteq V_0 \subseteq ran(\xi)$. Call $\lambda' = \vartheta'\nu''$, $\nu' = (\xi\nu'')_V$, and $\rho' = \rho''_{V \cup ran(\nu')}$. As ρ' is more general than ρ'' and $\varphi_2\rho''$ is satisfiable then $\varphi_2\rho'$ is satisfiable. Also, as $V \subseteq V_0$ and $\sigma =_E (\mu\vartheta\zeta')_V$, $\nu' \cdot \rho' = (\xi\nu'')_V \cdot \rho''_{V \cup ran(\nu')} = (\xi\nu''\rho'')_V = (\mu\vartheta\nu''\rho'')_V =_E (\mu\vartheta\delta'')_V = (\mu\vartheta\zeta'\delta')_V =_E (\sigma\delta')_V = \sigma$, i.e., $\sigma =_E \nu' \cdot \rho'$. Now:

- if $n = 1$ then $\mathbf{G_0} \rightsquigarrow^+_{\vartheta'} \mathbf{G_2} \rightsquigarrow^+_{\nu''} \mathit{nil} \mid \varphi_2 \mid V, \nu'$, i.e., $\mathbf{G_0} \rightsquigarrow^+_{\lambda'} \mathit{nil} \mid \varphi'_2 \mid V, \nu'$, $\sigma =_E \nu' \cdot \rho'$, and $\varphi'_2\rho'$ is satisfiable, so $\psi = \varphi'_2$, $\lambda = \lambda'$, $\nu = \nu'$, and $\rho = \rho'$;
- else $\mathbf{G_0} \rightsquigarrow^+_{\vartheta'} \mathbf{G_2} \rightsquigarrow^+_{\nu''} \Delta(\mu\lambda') \mid \varphi'_2 \mid V, \nu'$, i.e., $\mathbf{G_0} \rightsquigarrow^+_{\lambda'} \Delta(\mu\lambda') \mid \varphi'_2 \mid V, \nu'$, $\sigma =_E \nu' \cdot \rho'$, and $\varphi'_2\rho'$ is satisfiable. The rest of the proof is the one given at the end of the induction step for the base cases.

9. $ST_1 = \texttt{top}(c[\gamma]\{\bar{S}\})$.

   The proof is almost exactly the same as the previous one, particularized for the case $p = \epsilon$, so $u|_p = u$, $u_1|_p = u_1$, $u_1[r\gamma\tau]_p = r\gamma\tau$, et cetera. The only difference is found in the initial narrowing steps, where instead of $\mathbf{G_0} \rightsquigarrow_{[t]}\rightsquigarrow^*_{[c]} \mathbf{G_1} \rightsquigarrow_{[r],\vartheta'} \mathbf{G_2}$ now we have $\mathbf{G_0} \rightsquigarrow_{[tp],\vartheta'} \mathbf{G_2}$.

□