Dynamic Networks of Timed Petri Nets

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Abstract. We study dynamic networks of infinite-state timed processes, modelled as unbounded Petri nets. These processes can evolve autonomously, synchronize with each other (e.g., in order to gain access to some shared resources) and be created or become garbage dynamically. We introduce dense time in two different ways. First, we consider that each token in each process carries a real valued clock. We prove that this model can faithfully simulate Turing-complete formalisms and, in particular, safety properties are undecidable for them. Second, we consider locally-timed processes, where each process carries a single real valued clock. For them, we prove decidability of safety properties by a non-trivial instantiation of the framework of Well-Structured Transition Systems.

1 Introduction

Perhaps the most widely known model of real-time systems is that of Timed Automata [6]. Several tools like UPPAAL or KRONOS are available for them. Natural extensions of Timed Automata are Networks of Timed Automata (NTA) [6] or the Networks of Timed Processes in [3]. Both models consider parameterized systems of finite-state processes, each of which is endowed with a real clock.

Petri nets are one of the best known models for concurrent and distributed systems. They have been extended with discrete or continuous time in many works [19, 17, 20, 18, 9, 5]. In some, transitions have a duration, while in others they fire atomically in some time interval. They also differ in whether time is considered relative to places, transitions or arcs. In [8] an exhaustive comparison of these models is done, and in particular it is proved that the class of Petri nets obtained by adding time constrains to the arcs is more expressive than the classes obtained from adding them to places or transitions. Among this class, in Timed-Arc Petri Nets (*TPN*) [5] each token is endowed with a real-valued clock. In particular, clocks can be dynamically created or destroyed.

Under the so called counting abstraction, one can think that each token in a place s of a Petri net represents a process in state s. Hence, Petri nets can be seen as networks (or products) of finite-state automata. With this intuition

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in mind, in *TPN* each (finite-state) process has a real-valued clock. Therefore, they encompass two infinite dimensions: infinitely-many (finite-state) processes and clocks over an infinite (uncountable) domain. In *TPN* arcs are labeled with intervals. Thus, when a token is taken from a place by a transition the value of the clock of the token must be in the interval labeling the corresponding arc, and when a token is put in a place, its clock is set to any real value in the corresponding interval. Moreover, it follows the so called weak semantics in timed systems, in which time delays may happen even when they disable transitions.

We extend the work in [5] by allowing each process to be infinite-state in turn. Hence, our model manages infinitely-many timed processes, each of which is infinite-state (a potentially unbounded Petri net). In this way we can for example easily model dynamic networks of processes accessing shared resources, that can be potentially unbounded.

Dynamic process creation is closely related to parametric verification, when the number of processes is a parameter of the system. Indeed, a standard approach for parametric verification is the addition of an "initialization phase" that spawns an unbounded number of processes (see e.g. [11] for a recent discussion). Hence, our results on verifying dynamic systems can also be seen as results on parametric verification of systems with a fixed number of processes.

As a starting point we consider an untimed model we have developed in previous work [21, 22], called ν-*PN*. In ν-*PN* tokens are names, that can be created fresh and matched with other names. Therefore, there can be an unbounded number of different names, each of which can appear an unbounded number of times. Each name can be understood as a process identifier. Hence, ν-*PN* encompass infinitely-many (untimed) processes, each of which can be infinite-state.

We consider two ways in which to introduce time. In the first way we assign a clock to each token in each process. Then, each process is a *TPN*, that can be created fresh and can synchronize with others. We call this model *Timed* ν-*PN* (ν-*TPN*). As in *TPN*, the clock value of each token consumed by a transition must belong to a given interval, as well as for the produced tokens. Moreover, as in ν-*PN*, names can be created and matched. We prove that this model can simulate Turing-complete formalisms and, in particular, even the control-state reachability problem (that of deciding if a given place can be marked) is undecidable.

In the second variant we consider that each process has a single real-valued clock. Since each process is a (concurrent) Petri net, we say these are locally-timed processes, and call them *locally-timed* ν-*PN* (ν-*lTPN*). Hence, we still encompass infinitely-many processes, each of which is infinite-state and is endowed with a real-valued clock.

For ν-*lTPN* we successfully apply the theory of regions of [3]. More precisely, we prove that working with regions we can give ν-*lTPN* a well-structure, so that they belong to the class of Well-Structured Transition Systems [10, 1], for which the coverability problem is decidable. This proves that control-state reachability (which can be reduced to coverability) is decidable for them. Moreover, safety properties can be reduced to control-state reachability by standard techniques.

Outline: Section 2 gives notations and results we use throughout the paper. Section 3 defines ν-TPN and proves undecidability of control-state reachability for them. In Section 4 we define ν-$lTPN$, and we prove decidability of control-state reachability for them. Finally, in Section 5 we present our conclusions.

2 Preliminaries

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and for each $n \in \mathbb{N}$ let us denote $n^+ = \{1, \ldots, n\}$ and $n^* = \{0, \ldots, n\}$. We denote open, closed and mixed intervals of real numbers as (a, c), $[a, b]$ and $[a, c)$ or $(a, b]$, respectively, where $a, b \in \mathbb{N}$ and $c \in \mathbb{N} \cup \{\infty\}$. The set of intervals is denoted by $\mathcal{I}$. Let $\mathbb{R}_{\geq 0} = [0, \infty)$ and for each $x \in \mathbb{R}_{\geq 0}$ we denote by $\lfloor x \rfloor$ and $\mathit{frct}(x)$ the integer and the fractional part of x, respectively.
Well orders: $(X, \leq)$ is a *partial order* (po)[1] if $\leq$ is a reflexive, transitive and antisymmetric binary relation on X. Let $A \subseteq X$. An element $x \in A$ is *minimal* in A if $x' \in A$ with $x' \leq x$ implies $x = x'$. We denote by $\min(A)$ the set of minimal elements in A. The *upward closure* of $A \subseteq X$ is defined as $\uparrow A = \{x \in X \mid \exists x' \in A,\ x' \leq x\}$. We say A is *upward closed* iff $\uparrow A = A$. A po $(X, \leq)$ is a *well partial order* (wpo) if for every infinite sequence $x_0, x_1, \ldots \in X$ there are i and j with $i < j$ such that $x_i \leq x_j$. Equivalently, a po is a wpo iff $\min(U)$ is finite for every upward closed set U. If X is finite, then $(X, =)$ is a wpo. If $(X, \leq_X)$ and $(Y, \leq_Y)$ are wpos, their product $X \times Y$ is well ordered by $(x, y) \leq (x', y')$ iff $x \leq_X x'$ and $y \leq_Y y'$.
Multisets: A (finite) *multiset* m over X is a mapping $m : X \to \mathbb{N}$ with finite support, that is, such that $\mathit{supp}(m) = \{x \in X \mid m(x) > 0\}$ is finite. We denote by $X^{\oplus}$ the set of finite multisets over X. For $m_1, m_2 \in X^{\oplus}$ we define $m_1 + m_2 \in X^{\oplus}$ by $(m_1 + m_2)(x) = m_1(x) + m_2(x)$ and $m_1 \subseteq m_2$ if $m_1(x) \leq m_2(x)$ for every $x \in X$. When $m_1 \subseteq m_2$ we can define $m_2 - m_1 \in X^{\oplus}$ by $(m_2 - m_1)(x) = m_2(x) - m_1(x)$. We denote by $\emptyset$ the empty multiset, that is, $\emptyset(x) = 0$ for every $x \in X$, and $|m| = \sum_{x \in \mathit{supp}(m)} m(x)$. We use set notation for multisets when convenient, with repetitions to account for multiplicities greater than one. Given a po $\leq$ over X, we define the po $\leq^{\oplus}$ over $X^{\oplus}$ as $\{x_1, \ldots, x_n\} \leq^{\oplus} \{y_1, \ldots, y_m\}$ if there is an injection $h : n^+ \to m^+$ such that $x_i \leq y_{h(i)}$ for each $i \in n^+$. If $(X, \leq)$ is a wpo then so is $(X^{\oplus}, \leq^{\oplus})$ [13].
Words: Any $u = x_1 \cdots x_n$ with $n \geq 0$ and $x_i \in X$ for all $i \in n^+$ is a (finite) *word* over X. We denote by $X^{\circledast}$ the set of words over X. If $n = 0$ then u is the empty word, denoted by ϵ. If X is a wpo then so is $X^{\circledast}$ [13] ordered by $\leq^{\circledast}$, defined as $x_1 \ldots x_n \leq^{\circledast} y_1 \ldots y_m$ if there is a strictly increasing mapping $h : n^+ \to m^+$ such that $x_i \leq y_{h(i)}$ for each $i \in n^+$.
Transition systems: A *transition system* is a tuple $\mathcal{S} = \langle X, \to, x_0 \rangle$ where X is the set of states, $x_0 \in X$ is the initial state and $\to \subseteq X \times X$ is the transition relation. We write $x \to x'$ instead of $(x, x') \in \to$ and we denote by $\to^*$ the reflexive and transitive closure of $\to$. We say $A \subseteq X$ is *reachable* if $x_0 \to^* x$ for some $x \in A$. For $x \in X$ we define $\mathit{Pre}(x) = \{x' \mid x' \to x\}$ and

[1] We only work with po (and not quasi-orders).

$Pre^*(x) = \{x' \mid x' \to^* x\}$, and extend them pointwise to sets of states. If X is a po, we can define the *coverability problem*, that of deciding, given U upward closed, whether U is reachable, or equivalently, whether $x_0 \in Pre^*(U)$. All the models in the paper induce transition systems in the obvious way if we provide them with an initial state.

Timed-Arc Petri Nets A *Timed-Arc Petri Net* (*TPN*) is a tuple $N = \langle P, T, F, H\rangle$, where P and T are finite disjoint sets of places and transitions, respectively, and $F, H : P \times T \to \mathcal{I}^{\oplus}$. A marking of a *TPN* is a finite multiset M over $P \times \mathbb{R}_{\geq 0}$. Abusing notation, we define $M(p)$ as the multiset of clocks of tokens in place p at M. There are two types of transitions: timed transitions and discrete transitions. Given a marking $M = \{(p_1, d_1) \ldots, (p_n, d_n)\}$ and $d \geq 0$, we write $M \xrightarrow{d} M'$ if $M' = \{(p_1, d_1 + d) \ldots, (p_n, d_n + d)\}$. Given $t \in T$ and a marking M we write $M \xrightarrow{t} M'$, if for each $p \in P$ with $F(p, t) = \{I_1, \ldots, I_n\}$ and $H(p, t) = \{J_1, \ldots, J_m\}$, there are $In = \{r_1, \ldots, r_n\}$ and $Out = \{r'_1, \ldots, r'_m\}$ such that $In \subseteq M(p)$, $r_i \in I_i$ for any $i \in n^+$, $r'_j \in J_j$ for any $j \in m^+$ and $M'(p) = (M(p) - Out) + In$. Finally, we write $M \to M'$ iff there is $d \geq 0$ with $M \xrightarrow{d} M'$ or $t \in T$ with $M \xrightarrow{t} M'$.

For an example see the nets in Fig. 1, in which F and H are represented by labelled arcs. Disregard the variables labelling the arcs and the different names in places, considering that all the names are plain tokens $\bullet$ instead. The superscripts of the tokens represent their clocks. In the first net, transition t cannot be fired, as the clocks of tokens do not fit in the intervals of the arcs. However, after an elapse of 1 unit of time, transition t can be fired from the marking represented in the second net, reaching the marking in the third one.

ν-Petri Nets: We fix infinite sets *Id* of names, *Var* of variables and a subset of special variables $\Upsilon \subset Var$ for fresh name creation. A *ν-Petri Net* (*ν-PN*) [22] is a tuple $N = \langle P, T, F, H\rangle$, where P and T are finite disjoint sets, and $F, H : T \to (P \times Var)^{\oplus}$ are the input and output functions, respectively. If $(p, x) \in F(t)$ ($(p, x) \in H(t)$), we say that there is an arc from p to t (from t to p) labelled by x (among possibly other variables).[2] We call $Var(t) = \{x \in Var \mid \exists p \in P,\ (p, x) \in F(t) + H(t)\}$. A *marking* is a multiset over $P \times Id$. A *mode* is an injection $\sigma : Var(t) \to Id$. Modes are extended homomorphically to $(P \times Var(t))^{\oplus}$. A transition t is *enabled* with mode σ for a marking M if $\sigma(F(t)) \subseteq M$ and for every $\nu \in \Upsilon$, $(p, \sigma(\nu)) \notin M$ for any p. In that case we have $M \xrightarrow{t} M'$, where $M' = (M - \sigma(F(t)) + \sigma(H(t))$ and $M \to M'$ if $M \xrightarrow{t} M'$ for some $t \in T$. We interpret each name as (the identifier of) a process that can be created, synchronize with other processes or become garbage.

Again, for an example see the second and the third nets in Fig. 1. Tokens are represented as names in places. Disregard the intervals in the arcs and the superscripts of the tokens. Transition t can be fired from the marking represented in the second net, reaching the marking in the third one, with mode σ, with $\sigma(x) = a$, $\sigma(y) = b$ and $\sigma(\nu) = c$. In particular, note that the firing of t creates a new name c in place p_4. See [22, 21] for more details.

[2] We use this notation following [15].

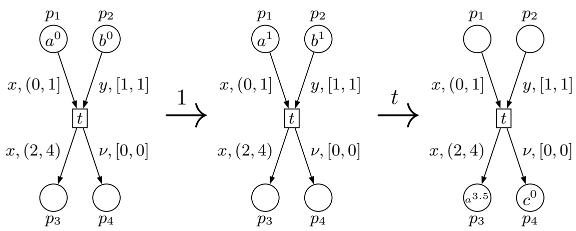

Fig. 1. Firing of transitions in a ν-*TPN*

3 Timed ν-Petri Nets

In this section we define the first extension of ν-*PN* with time, namely Timed ν-Petri nets (ν-*TPN* for short) and we prove the undecidability of control-state reachability for them.

Basically, a ν-*TPN* is a ν-*PN* in which each token has a clock, or equivalently, a *TPN* in which each token has a name. Arcs are labelled by intervals, meaning that the value of the clock of the tokens consumed and produced by the transition must be in these intervals. In Fig. 1 the nets depicted show the same ν-*TPN* with three different markings, in which tokens are depicted as names with its clock as superscript. In the first marking the transition t is not enabled, since the value of the clock of the only token in p_2 is not in $[1, 1]$. After a delay of one unit of time, t becomes enabled, and can be fired reaching, for example, the marking depicted in the right.

Let us define ν-*TPN* formally. Let *Var* be a set of variables with $\Upsilon \subset Var$.

Definition 1 (Timed ν-Petri Nets). *A* Timed ν-Petri net *(ν-TPN) is a tuple $N = \langle P, T, F, H \rangle$, where:*

- *P and T are finite disjoint sets,*
- *$F : T \to (P \times Var \times \mathcal{I})^{\oplus}$ is the input function,*
- *$H : T \to (P \times Var \times \mathcal{I})^{\oplus}$ is the output function.*

For a transition $t \in T$, we take $Var(t)$ as the set of variables adjacent to t, that is, $Var(t) = \{x \in Var \mid \exists p \in P, I \in \mathcal{I},\ (p, x, I) \in F(t)+H(t)\}$. In figures, for each $(p, x, I) \in F(t)$ we draw an arc from p to t, labeled by x, I (and analogously for postconditions). For the next definition we consider a fixed infinite set Id of names.

Definition 2 (Markings). *A* token *of a ν-TPN is an element of $P \times Id \times \mathbb{R}_{\geq 0}$. A* marking *is a finite multiset of tokens.*

We write $p(a, r)$ instead of (p, a, r) to denote tokens. Intuitively, $p(a, r)$ is a token in p, carrying the name a, with clock value r. We use M, M', M_1,... to range over markings. We say M *marks* $p \in P$ if there are $a \in Id$ and $r \in \mathbb{R}_{\geq 0}$ such that $p(a, r) \in M$. We denote $Id(M) = \{a \mid \exists p, r,\ p(a, r) \in M\}$. We assume $\bullet \in Id$, so that black tokens can appear in markings as in ordinary Petri nets.

If an arc is not labeled by any variable we assume that the token involved is $\bullet$. Moreover, in figures we do not write the interval $[0, \infty)$. Hence, ordinary notations in Petri nets can be used.

Now, let us define the semantics of ν-*TPN*. As expected, markings may evolve in two different ways: time elapsing and firing of transitions. Time elapsing is accomplished by simply adding the same amount of time to each token in the net. In order to fire a transition $t \in T$, we assign an identifier to each of the variables in $Var(t)$, and we need to ensure that for each $(p, x, I) \in F(t)$, there is a token $p(a, r)$ in the current marking such that $r \in I$.

Definition 3 (Semantics of ν-*TPN*). Time elapsing*: Given a marking* $M = \{p_1(a_1, r_1), \ldots, p_n(a_n, r_n)\}$ *and a delay* $d \in \mathbb{R}_{\geq 0}$*, we write* M^{+d} *to denote the marking* $\{p_1(a_1, r_1 + d), \ldots, p_n(a_n, r_n + d)\}$ *in which the value of the clocks of all tokens has increased by* d*. Then we write* $M \xrightarrow{d} M^{+d}$*.*

Firing of transitions*: Let* $t \in T$ *be a transition with* $F(t) = \{p_1(x_1, I_1), \ldots, p_n(x_n, I_n)\}$ *and* $H(t) = \{q_1(y_1, J_1), \ldots, q_m(y_m, J_m)\}$*. We say* t *is* enabled *or* can be fired *in marking* M*, evolving to* M'*, and we denote it by* $M \xrightarrow{t} M'$*, if there is an injection* $\sigma : Var(t) \to Id$*,* $r_1, \ldots, r_n \in \mathbb{R}_{\geq 0}$ *and* $r'_1, \ldots, r'_m \in \mathbb{R}_{\geq 0}$ *such that:*

- $r_i \in I_i$ *for all* $i \in n^+$ *and* $r'_j \in J_j$ *for all* $j \in m^+$*,*
- $\sigma(\nu) \notin Id(M)$ *for all* $\nu \in \Upsilon$*,*
- $\{p_1(\sigma(x_1), r_1), \ldots, p_n(\sigma(x_n), r_n)\} \subseteq M$*,*
- $M' = (M - \{p_1(\sigma(x_1), r_1), \ldots, p_n(\sigma(x_n), r_n)\}) + \{q_1(\sigma(y_1), r'_1), \ldots, q_m(\sigma(y_m), r'_m)\}$*.*

We write $M \to M'$ *if* $M \xrightarrow{t} M'$ *for some* $t \in T$ *or* $M \xrightarrow{d} M'$ *for some* $d \in \mathbb{R}_{\geq 0}$*.*

As an example, let M_1, M_2 and M_3 be the markings represented in the first, second and third nets in Fig. 1, respectively. Note that $M_1 \xrightarrow{1} M_1^{+1} = M_2$ and $M_2 \xrightarrow{t} M_3$ with mode σ, where $\sigma(x) = a$, $\sigma(y) = b$ and $\sigma(\nu) = c$. We remark that we are defining a weak semantics, in which time elapsings can happen even if they disable transitions. For instance, from M_1 in Fig. 1 two units of time can elapse, which disables the firing of t forever.

The *control-state reachability* problem is that of deciding, given a place p, whether p is marked in some reachable marking (the same definition applies to the rest of the models in the paper).[3] Let us prove undecidability of control-state reachability for ν-*TPN*. Instead of giving a reduction from a well-known Turing-complete model (as Minsky or Turing machines), we first present a Turing complete model based on Petri nets with identifiers, called ν-*RN* systems in [21]. Then we reduce control-state reachability in ν-*RN*, which is undecidable, to our problem. Considering ν-*RN* considerably simplifies our reduction, since the

[3] We use this terminology, even if places are not necessarily control-states.

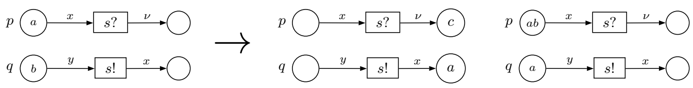

Fig. 2. A ν-*RN* system and the synchronous firing of the compatible transitions t_1 and t_2, assuming it creates M with $M(p) = \{a, b\}$ and $M(q) = \{a\}$

representation gap between both models is certainly smaller than that obtained if we considered a better known Turing-complete formalism.[4]

We briefly present ν-*RN* systems. For more details see Appendix A. Intuitively, a ν-*RN* system is just a collection of ν-*PN* that can synchronize with each other, and that can create replicas of themselves (hence the name, ν-Replicated Nets). For synchronization purposes, we consider a set $\mathcal{L}$ of transition labels.

A ν-*RN system* is a tuple $N = \langle P, T, F, H, \lambda \rangle$, where $\langle P, T, F, H \rangle$ is a ν-*PN* and $\lambda : T \to \mathcal{L}$ labels transitions for two different purposes. On the one hand, it specifies how a transition can be fired: whether it is an autonomous transition, that can be fired in isolation, or a synchronizing transition, that must be fired synchronously with another transition. On the other hand, it indicates which new instances (if any) are created by its firing. An *instance* of N is a multiset over $P \times Id$ (i.e., a marking of the underlying ν-*PN*). A *marking* of N is a multiset of instances of N. Therefore, in ν-*RN* each instance contains tokens, possibly with different names. A synchronous firing can happen whenever two compatible transitions (having labels $s?$ and $s!$) are enabled, according to the enabling condition of ν-*PN*. In that case they can both be fired simultaneously, following the ordinary token game of ν-*PN*. In particular, names can be moved along the nets, be communicated between instances and be created fresh. Moreover, firings may create new instances (see Fig. 2).

The control-state reachability problem for ν-*RN* is that of deciding whether some reachable marking marks a given place. The model of ν-*RN* is Turing-complete [21], and termination for Turing machines can be easily reduced to control-state reachability for ν-*RN*. Hence control-state reachability is undecidable for ν-*RN*.

Proposition 1. *Control-state reachability is undecidable for ν-TPN.*

Proof. We reduce control-state reachability for ν-*RN* systems to our problem. Given a ν-*RN* $N = \langle P, T, F, H, \lambda \rangle$, we build a ν-*TPN* $N' = \langle P', T', F', H' \rangle$ which simulates it. In particular, we build N' such that $P \subset P'$, and a place $p \in P$ can be marked in N iff it can be marked in N'. Without loss of generality, we suppose that the initial marking of every instance consists of a single (black) token in a place $p_0 \in P$. Moreover, we assume that only autonomous transitions may create new instances.

[4] Even a reduction from Petri nets with inhibitor arcs, which are also Turing-complete, needs to fill a much bigger representation gap. Informally, inhibitor nets are close to counter machines, while ν-*RN* systems and ν-*TPN* are somewhat close to Turing machines.

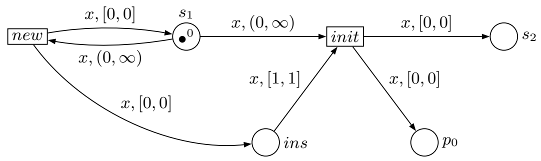

Fig. 3. Creation of instances

Intuitively, we represent each instance of N by a multiset of tokens with the same clock value in N'. The construction guarantees that all the transitions in N' use only tokens with clocks set to 1. Hence, tokens with clocks older than 1 are dead tokens, that cannot be used for the firing of transitions. In order to allow instances not to become dead, we will add transitions that reset tokens with clock 1 to 0. These transitions may not reset every token with clock 1, in which case some tokens are lost (after the elapsing of time). Therefore, in some simulations some tokens are lost, but there are also perfect simulations in which no tokens are lost. In this sense our simulation is lossy, though it preserves control-state reachability, since loosing tokens can only remove behavior(no spurious behavior is introduced). We also guarantee in our construction that we do not merge instances, that is, that no two tokens with different clock values may end up having the same value.

Executions in N' simulate executions of N in two steps: In the first step N' creates an unbounded number of tokens with different clock values, which represent all the instances that may take part in the simulation. The second step is the simulation itself. We consider in N' two places s_1 and s_2 (marked in mutual exclusion) to specify in which of the two steps the simulation currently is.

Step 1 (creation of instances): In the first step, depicted in Fig. 3, we repeatedly fire a transition *new*, which creates new tokens with clock 0 in place *ins*. The clock of each token in *ins* will represent a different instance of N, so that we need to ensure that they are all different. We do that by forcing some time elapsing between two consecutive firings of *new*, by demanding that the token in s_1 is strictly older than 0 when *new* is fired (and setting it back to 0). Initially, there is only one token in place s_1, with clock 0.

The firing of a transition *init* concludes step 1, by moving the token in s_1 to s_2 when the token in s_1 has a non-null clock. It also sets the initial marking of N, by taking a token of clock vale 1 from *ins* and putting it in p_0, with clock 0. Notice that this guarantees that the clock value of the token in the initial instance is different from all the clock values of the tokens in *ins*.

Step 2 (simulation of transitions): As mentioned before, only tokens with clocks between 0 and 1 (both included) are valid tokens, that represent a token in some instance. Step 1 guarantees that at the beginning of step 2 there are no two tokens having clocks set to 0 and 1, respectively. Moreover, at any point in step 2, two tokens in P with clocks 0 and 1 belong to the same instance. Now we show how we reset the clock of tokens, and how we simulate the firing of autonomous

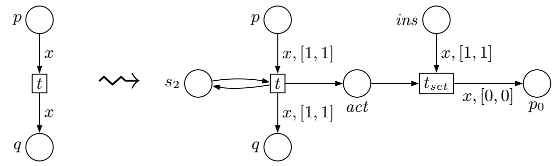

Fig. 4. Simulation of the firing of t, assuming t creates a fresh instance

transitions (possibly creating a fresh instance), and the synchronization of two compatible transitions.

Reseting tokens: In order to be able to perform perfect (non-lossy) simulations, we need to be able to reset the clock of tokens with value 1. For that purpose, for each place $p \in P'$ we add a transition t_p which takes from p a token of clock 1 and puts it back with clock set to 0.[5] Formally, $F'(t_p) = \{(p, x, [1, 1])\}$ and $H'(t_p) = \{(p, x, [0, 0])\}$. Notice that this is correct because before reseting there are no tokens with clock set to 0.

Simulation of the firing of a transition: The simulation of the (autonomous) transition $t \in T$ is simply achieved by demanding that the clock of all tokens involved in the firing is set to 1. Thus, we consider $t \in T'$, and we attach the interval $[1, 1]$ to every arc adjacent to t. More precisely, if $(p, x) \in F(t)$ then $(p, x, [1, 1]) \in F'(t)$ (and analogously for postconditions). We also add s_2 as pre/postcondition of t. Moreover, if t creates a fresh instance, it puts a token in a new place act. Intuitively, we store in act a token for each instance that the simulation has created, but that has not been initialized yet. In order to initialize new instances, we add a new transition t_{set}, which takes a token from act and a token with clock value 1 from ins, and puts a token in p_0 with clock set to 0, analogously as $init$ (see Fig.4). Again, notice that when there is a token with clock value 1 in ins there is no token with clock 0 in the whole net, so that we are correctly creating the new instance.

Simulation of synchronizing transition: Let us see how we simulate the firing of $u = (t_1, t_2) \in T \times T$, where t_1 and t_2 are two compatible transitions according to λ.(see Fig. 5). We simulate u by means of the consecutive firing of transitions $start_u^1$, $start_u^2$, $\overline{u}$, end_u^1 and end_u^2 in T'. We guarantee (thanks to s_2 and new control places, not shown in Fig. 5) that these transitions can only be fired in the order shown, and that $start_u^1$ can only be fired when there is a token in s_2 (no simultaneous simulations of firings can take place).

Let us consider in P' new places, $role^1$ and $role^2$ (whose content can also be reseted, as explained above), and for each $p \in P$ let us consider $\overline{p} \in P'$. The firing of $start_u^1$ removes the tokens from the preconditions p of t_1 with clock value 1 and puts them in the corresponding $\overline{p}$ (with any value for the clock). More precisely, if $(p, x) \in F(t_1)$ then $(p, x, [1, 1]) \in F'(start_u^1)$ and $(\overline{p}, x, [0, \infty)) \in H'(start_u^1)$. Moreover, a token (with any name, e.g. a black token) is added to $role^1$ with clock value 1. The case of $start_u^2$ is analogous.

[5] It is enough to reset places in which the clock is meaningful, unlike e.g. s_2.

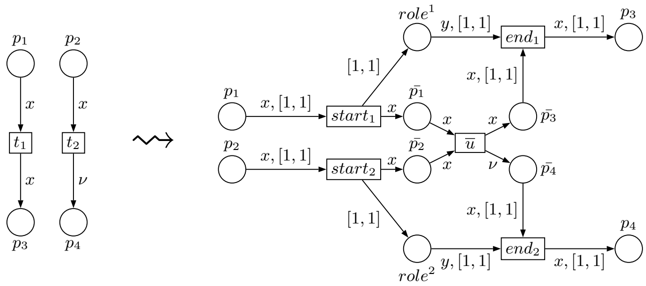

Fig. 5. Synchronizing transitions

The firing of $\overline{u}$ simulates the firing of u (that is, the simultaneous firing of t_1 and t_2) in the overlined places. More precisely, if $(p, x) \in F(t_i)$ for $i \in \{1, 2\}$ then $(\overline{p}, x, [0, \infty)) \in F'(\overline{u})$ (and analogously for postconditions). In particular, it checks that names in different places are matched according to the variables in the arcs, and new names are created if needed. Notice that if the names selected by $start_u^1$ and $start_u^2$ do not match then $\overline{u}$ is disabled. Hence, our simulation may introduce deadlocks, though it still preserves control-state reachability. Notice also that this firing can take place independently of the clocks of the tokens involved.

Finally, transitions end_u^1 and end_u^2 set the clocks of the tokens involved in the firing of u to their correct values. For that purpose, end_u^i takes the token from $role^i$ with clock value 1, and for every p postcondition of t_i it takes the token in $\overline{p}$ and puts it in p with clock value 1. More precisely, for $i = 1, 2$, $(role^i, y, [1, 1]) \in F'(t_i)$ (where y is a *fresh* variable), and if $(p, x) \in H(t_i)$ then $(\overline{p}, x, [0, \infty)) \in F'(end_u^i)$ and $(p, x, [1, 1]) \in H'(end_u^i)$.

The previous simulation preserves control-state reachability. Indeed, if p is marked by some execution of N, then that execution can be perfectly simulated, ending up in a marking that marks p. Conversely, if p is marked by some execution of N', by construction that execution corresponds to the simulation of some execution of N which also marks p (possibly with more tokens, if some were lost).

4 Locally-timed ν-*PN*

In the previous section we have seen that even control-state reachability is undecidable for ν-*TPN*. Now we define the class of locally-timed ν-*PN* (ν-*lTPN*), for which each instance has a single clock. ν-*lTPN* can be obtained as a syntactic restriction of ν-*TPN*, ensuring that each instance uses only one clock. One way to do it is to consider a special place in which we store a token of each name in the net, whose clocks represents the age of the corresponding instance. However, in order to have simpler notations, we prefer to define ν-*lTPN* from scratch[6].

[6] Read-only constraints could also be considered within the same setting. However, for simplicity, we do not consider them in this paper.

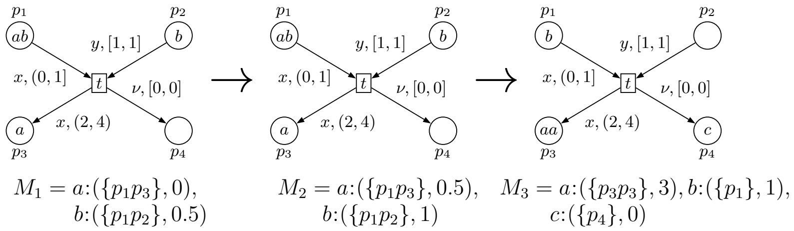

Fig. 6. Firing of a transition in a ν-*lTPN*.

Definition 4 (Locally-timed ν-*PN*). *A* locally-timed ν-*PN* *(ν-lTPN) is a tuple $N = \langle P, T, F, H, \mathcal{G} \rangle$, where:*

- *P and T are finite disjoint sets,*
- *for $t \in T$, $F_t, H_t : Var \to P^{\oplus}$ are the input and output functions of t,*
- *for $t \in T$, $\mathcal{G}_t : Var \to \mathcal{I} \times \mathcal{I}$ is the time constraints function of t.*

For each $t \in T$ we define $Var(t) = \{x \in Var \mid F_t(x) + H_t(x) \neq \emptyset\}$, which is assumed to be finite, and we split it into $nfVar(t) = Var(t) \setminus \Upsilon$ and $fVar(t) = Var(t) \cap \Upsilon$. In fact, $\mathcal{G}_t$ only needs to be defined in $Var(t)$.

Definition 5 (Markings). *A* marking M *of a ν-lTPN is an expression of the form $a_1{:}(m_1, r_1), ..., a_n{:}(m_n, r_n)$, where $Id(M) = \{a_1, ..., a_n\} \subset Id$ are pairwise different names, and for each $i \in n^+$, $\emptyset \neq m_i \in P^{\oplus}$ and $r_i \in \mathbb{R}_{\geq 0}$.*

We treat markings of ν-*lTPN* as multisets over elements of the form $a{:}(m, r)$, which we call *instances*. Hence, $a : (m, r)$ is an instance with name a, tokens according to m, and clock with value r. We assume that each m_i in each instance is not empty. We use M, M',... to range over markings, and say a marking M *marks* $p \in P$ if there is $a{:}(m, r) \in M$ such that $p \in m$.

Definition 6 (Time delay). *Given $M = a_1{:}(m_1, r_1), ..., a_n{:}(m_n, r_n)$ and $d \in \mathbb{R}_{\geq 0}$, we write M^{+d} to denote the marking $a_1{:}(m_1, r_1 + d), ..., a_n{:}(m_n, r_n + d)$, in which the clock of every instance has increased by d. We write $M \xrightarrow{d} M^{+d}$.*

Again, we are defining a weak timed semantics. Now we define the firing of transitions, for which we need the following notations. We denote by $\mathcal{G}_t^1(x)$ and $\mathcal{G}_t^2(x)$ the first and second component of $\mathcal{G}_t(x)$, respectively. Intuitively, for a transition to fire the instance corresponding to x must have a clock value in $\mathcal{G}_t^1(x)$. This clock is set to any value in $\mathcal{G}_t^2(x)$. We say M' is an $\emptyset$-*expansion* of a marking M (or M is the $\emptyset$-*contraction* of M') if M' is obtained by adding instances $a{:}(\emptyset, r)$ to M.

Definition 7 (Firing of transitions). *Let $t \in T$ with $nfVar(t) = \{x_1, ..., x_n\}$ and $fVar(t) = \{\nu_1, ..., \nu_k\}$. We say t is* enabled *at marking M if:*

- *$M = a_1 : (m_1, r_1), ..., a_n : (m_n, r_n) + \overline{M}$, for some $\overline{M}$,*
- *for each $i \in n^+$, $F_t(x_i) \subseteq m_i$ and $r_i \in \mathcal{G}_t^1(x_i)$.*

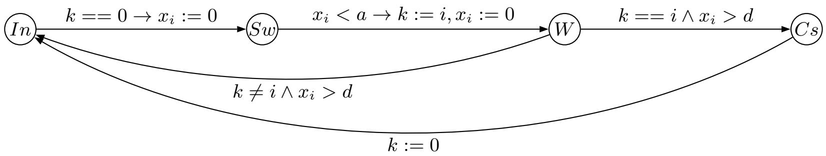

Fig. 7. Timed automaton modelling the i-th process of Fischer's mutual exclusion protocol

Then, t *can be* fired, *and taking*

- $\{b_1, ..., b_k\}$ *pairwise different names not in* $Id(M)$,
- $m'_i = (m_i - F_t(x_i)) + H_t(x_i)$ *for all* $i \in n^+$,
- $m''_j = H_t(\nu_j)$ *for all* $j \in k^+$,
- r'_i *any value in* $\mathcal{G}_t^2(x_i)$, *for all* $i \in n^+$,
- r''_j *any value in* $\mathcal{G}_t^2(\nu_j)$, *for all* $j \in k^+$,

we can reach M', *denoted by* $M \xrightarrow{t} M'$, *where* M' *is the* $\emptyset$*-contraction of*

$$a_1{:}(m'_1, r'_1), ..., a_n{:}(m'_n, r'_n), b_1{:}(m''_1, r''_1), ..., b_k{:}(m''_k, r''_k) + \overline{M}$$

Let us give two examples to illustrate the previous definitions.

Example 1. Fig. 6 depicts a *ν-lTPN* with three different markings. In the first marking the transition t is not fireable, because no instance with a clock value in $[1, 1]$ has a token in place p_2. However, after waiting 0.5 units of time, the marking M_2 is reached, and t becomes enabled. Then, we can fire t reaching, for example, the marking M_3 in the figure.

Example 2. Fischer's protocol: We model a parameterized version of Fischer's protocol for mutual exclusion, which considers n processes p_i (where n is a parameter), each of those endowed with a real clock x_i. Moreover, a shared integer variable $k \in \{1, \dots, n\}$ is considered, in order to set the turn for entering the critical section. Each process p_i can be modelled by the timed automaton of Fig. 7, and behaves as follows:

```
1  repeat
2    non critical section
3    repeat
4      await k==0;
5      k:=i;
6      delay(d);
7    until k==i;
8    critical section;
9    k:=0;
10   non critical section
11 until false;
```

Process p_i repeatedly tries to enter the critical section (state In). For that purpose, it waits until $k = 0$, which means that no other process is in the critical section (state Sw). Then, it sets $k := i$, to ask for permission to enter (state W). After a delay of d units of time, if k is still i, the process enters the critical section (state Cs), setting $k = 0$ when it leaves. Otherwise, it repeats lines $4-6$. In order to make the algorithm satisfy the mutual exclusion property, it is important to fix a proper delay d, greater than the time a it takes each process to execute line 5. Then in Fig. 7 we take $a < d$.

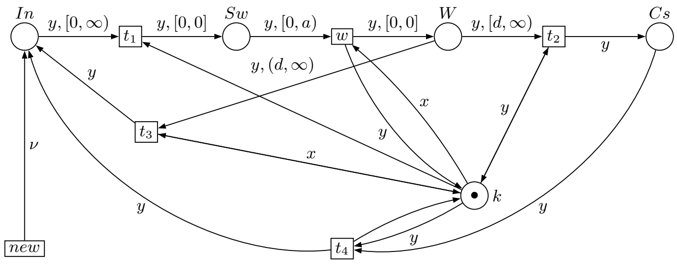

Fig. 8. Fischer's mutual exclusion protocol as a ν-*lTPN*

Let us define our model: We consider the net depicted in Fig 8. Intuitively, each token in places In, Sw, W and Cs represents a different instance. The variable k is represented by a place k that contains a black token if $k = 0$ or a token with the identifier that changed its value last. When a transition t is fired, if there are two different variables $x, y \in Var(t)$, then the names of the tokens associated to x and y in the firing are different (hence checking if $k == i$). Notice the transition *new*, that can create any number of processes in their initial state. To prove mutual exclusion, we have to prove that no marking with two tokens in place Cs can be reached, which can be easily reduced to control-state reachability.

You can note that the timed automaton in Fig. 7 modelling Fischer's protocol and our parametric model are very similar. In [2], the authors model Fischer's protocol using *TPN*. As they do not use colors, they need to use the counting abstraction (hence considering the state space of each process and the shared variable), and the obtained model is far more complicated than ours.

The state space of ν-*lTPN* is infinite in various dimensions. It encompasses infinitely-many instances, each of which is potentially unbounded, and contains a clock over an uncountable domain. Moreover, as any marking has infinitely-many successors due to time delays, the transition system induced by a ν-*lTPN* is not finitary. Next, we use the theory of regions to obtain a finitary transition system over a countable domain. Moreover, this transition system will be a Well-Structured Transition System [1, 10], so that we can solve the control-state reachability problem by reducing it to a coverability problem. The proofs omitted from this section can be found in Appendix B.

We fix a ν-*lTPN* $N = \langle P, T, F, H, \mathcal{G}\rangle$ and simply denote by max the maximum integer bound appearing in the intervals of the net. Also, we write n^*_∞ to denote $n^* \cup \{\infty\}$. Following [3, 5], we represent markings of N using *regions*.

Definition 8 (Regions). *A* region *is an expression of the form* $A_0 * A_1 * \ldots A_n * A_\infty$ *with* $n \geq 0$*, where* $A_i \in (P^\oplus \times I_i)^\oplus$ *for every* $i \in n^*_\infty$ *and* $I_0 = \max^*$, $I_i = (\max - 1)^*$ *for* $i \in n^+$ *and* $I_\infty = \{\max + 1\}$*. We write* $|R| = \sum_{i \in n^*_\infty} |A_i|$.

We assume $A_i \neq \emptyset$ for any $i \in n^+$, and $m \neq \emptyset$ for all $(m, r) \in A_i$, for any $i \in n^*_\infty$. We use R, R',... to range over regions and $\mathcal{R}$, $\mathcal{R}'$,... to range over sets of regions. Let us intuitively explain their meaning. Each marking M of a ν-*lTPN* has a region R_M associated to it. To obtain it, we partition the instances in M into three multisets:

- The multiset M_1 of instances with an integer clock value of at most max,
- The multiset M_2 of instances younger than max, with a non-integer clock value,
- The multiset M_3 of instances older than max.

Then we put instances in M_1 in A_0, with the information about their clocks (though forgetting their names). Moreover, we keep in $A_1 \dots A_n$ the instances in M_2, ordered according to the fractional part of their clocks, and storing only their integer part. Finally, we put instances in M_3 in A_∞, abstracting its clocks to $\max + 1$. Let us see it formally.

Definition 9 (Region of a marking). *Let M be a marking. We define the region $R_M = A_0 * A^{x_1} * ... * A^{x_n} * A_\infty$ where:*

- $|R_M| = |M|$, $x_1, ..., x_n \in (0, 1)$ *and* $i < j$ *iff* $x_i < x_j$,
- $A_0 = \{(m, r) \mid a{:}(m, r) \in M,\ r \in \max^*\}$,
- $A^x = \{(m, \lfloor r \rfloor) \mid a{:}(m, r) \in M, r < \max, \mathit{frct}(r) = x\}$,
- $A_\infty = \{(m, \max + 1) \mid a{:}(m, r) \in M,\ r > \max\}$.

Example 3. Let $M = a_1{:}(\{p\}, 1), a_2{:}(\{p\}, 1.1), a_3{:}(\{q\}, 2.1), a_4{:}(\{p, q\}, 1.2), a_5{:}(\{pq\}, 3.1), a_6{:}(\{p, q\}, 3)$, and $\max = 3$. Then, $R_M = A_0 * A_1 * A_2 * A_\infty$, with $A_0 = \{(\{p\}, 1), (\{p, q\}, 3)\}$ (which represents the two instances with integer clock value), $A_1 = \{(\{p\}, 1), (\{q\}, 2)\}$, $A_2 = \{(\{p, q\}, 1)\}$ (corresponding to the two different fractional parts, ordered) and $A_\infty = \{(\{pq\}, 4)\}$ (the only instance with clock value greater than max). Note that $x_1, ..., x_n$ above are not part of the definition of R_M.

Let us define the transition system over regions induced by N.

Time elapsing: There are two ways in which time may elapse in regions. If $A_0 \neq \emptyset$, the region may evolve to $\emptyset * A_0^< * A_1 * \dots * A_n * (A_\infty + A_0^=)$, where $A^< = \{(m, r) \in A \mid r < \max\}$ and $A^= = \{(m, \max + 1) \mid (m, \max) \in A\}$, which corresponds to a small elapsing of time that makes all the instances in A_0 to have a non-integer clock value, and so that the instances in A_n do not reach an integer value. Notice that instances in A_0 with clock max are added to A_∞. Otherwise, when $A_0 = \emptyset$, the region may evolve to $A_n^{+1} * A_1 * \dots * A_{n-1} * A_\infty$, where $A^{+1} = \{(m, r + 1) \mid (m, r) \in A\}$, which represents an elapsing of time that causes the instances in A_n (those with a higher fractional part) to reach the next integer part. Formally:

Definition 10 (Time elapsing for regions). *Let $R = A_0 * A_1 * \dots * A_n * A_\infty$ be a region. We write $R \overset{\delta}{\twoheadrightarrow} R'$, where*

$$R' = \begin{cases} \emptyset * A_0^< * A_1 * \dots * A_n * (A_\infty + A_0^=) & \textit{if } A_0 \neq \emptyset \\ A_n^{+1} * A_1 * \dots * A_{n-1} * A_\infty & \textit{otherwise} \end{cases}$$

Example 4. Consider the region $R_M = A_0 * A_1 * A_2 * A_\infty$ of Ex.3. As $A_0 \neq \emptyset$ and $max = 3$, according to the first case of the previous definition, it holds that $R_M \overset{\delta}{\twoheadrightarrow} R'$, where $R' = \emptyset * A_0^< * A_1 * A_2 * (A_\infty + A_0^=)$, $A_0^< = \{(\{p\}, 1)\}$ and $A_0^= = \{(\{p, q\}, 4)\}$.

Firing of transitions: In order to define the firing of transitions for regions we first need to define $\emptyset$-expansions/contractions for them.

Definition 11 ($\emptyset$-expansion/contraction). *We say R' is an* $\emptyset$-expansion *of a region $R = A_0 * A_1 * \ldots * A_n * A_\infty$ (or R is the* $\emptyset$-contraction *of R') if R' is of the form $A'_0 * u_0 * A'_1 * u_1 * \ldots * A'_n * u_n * A'_\infty$ and for each i:*

- $A'_i = A_i + B_i$ *with* $m = \emptyset$ *for all* $(m, r) \in B_i$,
- $u_i = B_1^i * \ldots * B_{k_i}^i$ *with* $k_i \geq 0$ *and* $m = \emptyset$ *for all* $(m, r) \in B_j^i$.

Example 5. Consider again the region R_M of Ex.3. $R_\emptyset = A_0 * A'_1 * B * A_2 * A_\infty$, with $A'_1 = \{(\{p\}, 1), (\{q\}, 2), (\emptyset, 1)\}$ and $B = \{(\emptyset, 1), (\emptyset, 2)\}$ is an $\emptyset$-expansion of R_M. Note that we have added to A_1 the pair $(\emptyset, 1)$, and a new multiset B, with only empty instances.

Now, we define the firing of transitions for regions. Intuitively, a transition t is enabled at a region if we can assign to each variable $x \in Var(t)$ with $x \notin \Upsilon$ a pair (m, r) in some multiset A_i of the region, in such a way that $F_t(x) \subseteq m$ and the clock that represents the pair is in $\mathcal{G}_t^1(x)$. Then, the transition can be fired, reaching a new region in which we update the markings of the pairs assigned to each variable according to F_t and H_t, and we update the clocks of the pair according to $\mathcal{G}_t^2$. Moreover, we possibly need to remove some of the pairs we have chosen from some A_i they are in, and put them in a different A_j, according to one of the possible clocks they may represent. Finally, for each $\nu \in \Upsilon$, we put a new pair $(H_t(\nu), r)$ in a proper (and maybe new) multiset of the region. In order to make the previous assignments, we define *modes* for regions. For any interval I, we call *left closure of* I the result of replacing the left delimiter of I by a closed one (for instance, the left closure of (a, b) is $[a, b)$).

Definition 12. *Given a transition $t \in T$ and an $\emptyset$-expansion $A_0 * A_1 * \ldots * A_n * A_\infty$ of a region R, let $l = |Var(t)|$. A mode for t and R is any tuple $\tau = (\tau_1, \tau_2, \tau_3)$ where $\tau_1 : Var(t) \to (n_\infty^* \times l^+)$ is an injection, and $\tau_2 : Var(t) \to (\max + 1)^*$ and $\tau_3 : Var(t) \to n_\infty^* \cup (n^* \times l^+)$ are mappings such that:*

- *For all $x \in Var(t)$, $\tau_2(x)$ is in the left closure of $\mathcal{G}_t^2(x)$,*
- *if $\tau_2(x) > \max$ then $\tau_3(x) = \infty$,*
- *if $\mathcal{G}_t^2(x) = (a, b]$ or $\mathcal{G}_t^2(x) = (a, b)$ and $\tau_2(x) = a$ then $\tau_3(x) \neq 0$.*

Intuitively, τ_1, τ_2 and τ_3 assign to each variable of t an instance of the region to perform the firing, the clock value to which we update this instance and the new position in the region that the instance takes, respectively. The first condition above ensures that the integers we choose to update the clocks of the instances are correct according to $\mathcal{G}_t^2(x)$. The second condition makes sure that instances older than max are stored in A_∞. The third condition ensures that the created instances with a clock value of integer part a, but not exactly a, are not stored in A_0. Let us now define the firing of transitions for regions.

Definition 13. *We say a transition t is enabled at a region R if there is an $\emptyset$-expansion $A_0 * A_1 * \ldots * A_n * A_\infty$ of R and a mode $\tau = (\tau_1, \tau_2, \tau_3)$ such that for each $i \in n^*_\infty$ there is $\bar{A}_i = \{(m_{ij}, r_{ij}) \mid \tau_1(x) = (i,j)\} \subseteq A_i$ and for each $x \in Var(t)$ with $\tau_1(x) = (i,j)$:*

- *If $x \in \Upsilon$, then $m_{ij} = \emptyset$,*
- *$F_t(x) \subseteq m_{ij}$,*
- *$r_{ij} \in \mathcal{G}^1_t(x)$ if $i \in \{0, \infty\}$, and $r_{ij} + 0.5 \in \mathcal{G}^1_t(x)$, otherwise.*

Then, we define $m'_{ij} = (m_{ij} - F_t(x)) + H_t(x)$ and take for all $k \in n^*$ and $b \in l^+$:

- *$B_k = A_k - \bar{A}_k$,*
- *$D_k = \{(m'_{ij}, r) \mid \exists x \in Var(t)$ with $\tau_1(x) = (i,j), \tau_2(x) = r, \tau_3(x) = k\}$,*
- *$C_k = B_k + D_k$ and*
- *$C_{kb} = \{(m'_{ij}, r) \mid \exists x \in Var(t)$ with $\tau_1(x) = (i,j), \tau_2(x) = r, \tau_3(x) = (k,b)\}$.*

Then, we write $R \xrightarrow{t} R'$, where R' is the $\emptyset$-contraction of $C_0 * C_{01} * \ldots * C_{0l} * C_1 * C_{11} * \ldots * C_{1l} * \ldots * C_n * C_{n1} * \ldots * C_{nl} * C_\infty$.

Intuitively, for each $i \in n^*_\infty$, $\bar{A}_i$ represents the multiset of instances selected by τ_1 from the multiset A_i which take part in the firing. Therefore, B_i is the multiset obtained after removing from A_i the instances corresponding to the preconditions. Finally, C_i represents B_i after adding the postconditions, and the C_{ij}s represent multisets of instances which we assign to clocks with fractional parts not appearing in R. Note that between two C_is we add l C_{ij}s. This is so to handle the case in which all the instances update their clocks to values with a fractional part between the ones represented by C_i and C_{i+1}.

Example 6. Let t be a transition with:

- $F_t(x) = \{p\}$,
- $H_t(x) = \{q\}$, $H_t(\nu) = \{pq\}$,
- $\mathcal{G}^1_t(x) = (0, 3]$, $\mathcal{G}^2_t(x) = (1, 2)$ and $\mathcal{G}^2_t(\nu) = (1, 3)$.

Then, we could consider the mode $\tau = (\tau_1, \tau_2, \tau_3)$ for t and the region R_M in Ex.3, where:

- $\tau_1(x) = (2, 1)$,
- $\tau_2(x) = 1$, $\tau_2(\nu) = 2$,
- $\tau_3(x) = (2, 1)$ and $\tau_3(\nu) = 1$.

According to the previous definition, we can fire t from R_M with mode τ, reaching a new marking $R' = A_0 * A'_1 * A_{21} * A_\infty$, where $A'_1 = \{(\{p\}, 1), (\{q\}, 2), (\{pq\}, 2)\}$ and $A_{21} = \{(\{q^2\}, 1)\}$. Note that A'_1 comes from adding $(\{pq\}, 2)$ to A_1, which represents that the new instance (associated to ν) is created with a clock with fractional part as the ones represented in A_1. Moreover, the multiset A_2 dissapears, as we have removed from it the only instance it contained. Finally, a new multiset A_{21} is created, with the instance associated to variable x.

Let $\xrightarrow{\Delta}\!\!\!\rightarrow$ be the reflexive and transitive closure of $\overset{\delta}{\twoheadrightarrow}$ and $\twoheadrightarrow = \overset{\Delta}{\twoheadrightarrow} \cup \bigcup_{t\in T} \overset{t}{\twoheadrightarrow}$.

Proposition 2. *The following relations between $\rightarrow$ and $\twoheadrightarrow$ hold:*

- *If $M \longrightarrow^* M'$ then $R_M \twoheadrightarrow^* R_{M'}$,*
- *If $R_M \twoheadrightarrow^* R'$ then there is M' with $R' = R_{M'}$ and $M \longrightarrow^* M'$.*

The proof of the previous proposition can be found in Appendix B. We omit it from this paper because it is rather technical. However, there is a point which would be interesting to focus in, which is how we manage the elapsings of time. In Def. 10 there are two ways time may elapse, depending on the region we consider. Both ways correspond to a small elapse, of less than a unit of time. However, we need to be able to represent longer elapsings. Prop. 2 can be proved because $\overset{\Delta}{\twoheadrightarrow}$ is defined as the reflexive and transitive closure of $\overset{\delta}{\twoheadrightarrow}$, and therefore, we can concatenate as many small elapsings as we need, in order to represent longer elapsings of time.

Let us next see that we can reduce the control-state reachability problem to a coverability problem in *ν-lTPN* using regions. In the first place, we must define an order over regions, which induces the corresponding coverability problem.

Definition 14 (Order over regions). *We define $(m, r) \leq (m', r')$ iff $m \subseteq m'$ and $r = r'$. Then, we define $A_0 * A_1 * \ldots * A_n * A_\infty \sqsubseteq B_0 * B_1 * \ldots * B_m * B_\infty$ iff $A_0 \leq^{\oplus} B_0$, $A_\infty \leq^{\oplus} B_\infty$ and $A_1 \ldots A_n \leq^{\oplus\circledast} B_1 \ldots B_m$.*

Notice that we are using the word order induced by the multiset order, and therefore $\sqsubseteq$ is a decidable wpo. The order $\sqsubseteq$ induces a coverability problem in the transition system with regions as states, and we can reduce control-state reachability to it.

Proposition 3. *Given $p \in P$ we can compute a finite set of regions $\mathcal{R}_p$ such that p is marked by some reachable marking iff $\uparrow\mathcal{R}_p$ can be reached.*

A *Well Structured Transition System* (WSTS) is a tuple $\mathcal{S} = \langle X, \rightarrow, x_0 \leq \rangle$, where $\langle X, \rightarrow, x_0 \rangle$ is a transition system, and $\leq$ is a decidable wpo on X, such that (i) for all $x_1, x_2, x_1' \in X$ such that $x_1 \leq x_1'$ and $x_1 \rightarrow x_2$ there is $x_2' \in X$ such that $x_1' \rightarrow x_2'$ and $x_2 \leq x_2'$ (compatibility); (ii) $\min(\uparrow\mathit{Pre}(\uparrow x))$ is computable for every $x \in X$ (effective *Pre*-basis).[7] Coverability is decidable for WSTS [1, 10].

Since $\sqsubseteq$ is a decidable wpo, in order to prove that the transition system over regions induced by a *ν-lTPNN* is a WSTS, it only remains to prove that the transition relation is compatible with the order, and that the effective *Pre*-basis property holds. We prove both properties by first proving that they hold for $\overset{\Delta}{\twoheadrightarrow}$ and $\overset{t}{\twoheadrightarrow}$, and then considering the union of them. In particular, in order to prove the effective *Pre*-basis, we split *Pre* into $\mathit{Pre}_\Delta(R) = \{R' \mid R' \overset{\Delta}{\twoheadrightarrow} R\}$ and $\mathit{Pre}_t(R) = \{R' \mid R' \overset{t}{\twoheadrightarrow} R\}$, and we we define $\overline{\mathit{Pre}}_\Delta$ and $\overline{\mathit{Pre}}_t$ for each $t \in T$, so

[7] This class is actually referred to as effective WSTS with strong compatibility and effective *Pred*-basis in the literature.

that $Pre_\Delta(\uparrow R) = \uparrow\overline{Pre}_\Delta(R)$ and $Pre_t(\uparrow R) = \uparrow\overline{Pre}_t(R)$. These proofs are rather technical, and therefore, we prefer to omit them from this paper. From the fact that the transition system we have defined is a WSTS and Prop. 3 above, we obtain the following result:

Corollary 1. *Control-state reachability is decidable for ν-lTPN.*

5 Conclusions and future work

We have introduced real time in a model of dynamic networks of processes that encompasses two sources of infinity: processes can be infinite-state, and there can be infinitely-many such processes. Despite there are previous works in which real time is studied in such Turing-powerful concurrent systems, as in [7], up to our knowledge, this is the first work in which real time is considered for this kind of concurrent systems, in which safety properties are still decidable. In the first model considered, ν-*TPN*, each process is endowed with an arbitrary amount of real clocks, while in the second one, ν-*lTPN*, only one clock per process is allowed. While control-state reachability (whether a given place can be marked) is undecidable in the first model, we have shown that we can use the theory of regions to prove decidability of this property in the second. With regions as state space, we prove that ν-*lTPN* belong to the class of WSTS, for which coverability is decidable. In [16], we compare ν-*lTPN* with other classes of WSTS, proving that they are the most expressive of the studied classes. In particular, we prove that *TPN* are strictly less expressive than ν-*lTPN*, using coverability languages for their comparison.

As future work, we plan to study the expressive power of models in between ν-*TPN* and ν-*lTPN*, in which a fixed number (possibly greater than one) of clocks is allowed, and the relation of ν-*lTPN* to the existing works that model GALS (globally asynchronous locally synchronous) systems using Petri nets [14]. In a different line, we have assumed that processes (or their identifiers) are not ordered in any way. It would be interesting to see whether our work scales in the case of ordered processes, which amounts to extend Data Nets [15] with time.

Regarding complexity, since ν-*lTPN* are more expressive than Data Nets or *TPN* [16], we can already obtain a lower bound for coverability and termination at level $F_{\omega^{\omega^\omega}}$ [12] in the fast-growing hierarchy. It would be interesting to know if this lower bound is tight, though we may expect it is not, due to the higher order types of the state space in ν-*lTPN*.

Although we have not discussed properties other than control-state reachability, the properties of termination and boundedness are still decidable for ν-*lTPN*. Indeed, termination is decidable for WSTS under rather general hypothesis, as well as boundedness.[8] Other directions for further study include other properties, as the existence of Zeno behaviors [4] (actually, the first step in the proof of Prop. 1 exhibits such behavior), or liveness properties, although negative results in the untimed case are discouraging [22].

[8] For boundedness compatibility must be strict, as we claim is the case for ν-*lTPN*.

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Appendix A. Definition of ν-*RN* systems

We fix an arbitrary set $\mathcal{S}$ of service names and let $Sync = \{s?, s! \mid s \in \mathcal{S}\}$.

Definition 15 (ν-*RN* systems). *A ν-RN* system *is a tuple $N = \langle P, T, F, H, \lambda\rangle$, where:*

- $\langle P, T, F, H\rangle$ *is a ν-PN,*
- $\lambda : T \to \mathcal{L}$ *assigns a label to each transition,*

where $\mathcal{L} = (\mathcal{S} \cup Sync) \times (P \times Id)^{\oplus}$. An instance *of N is an element of $(P \times Id)^{\oplus}$. A* marking *of N is a multiset of instances of N.*

As for ν-*TPN*, we write $Var(t)$ to denote the set of variables in arcs adjacent to t. For two instances M and M' we write $M \xrightarrow{t(\sigma)} M'$ if M can reach M' after the firing of t with mode σ, following the semantics of ν-*PN*. We write $Id(\mathcal{M})$ to denote the set of names that appear in marking $\mathcal{M}$. We identify any marking $\mathcal{M}$ with $\mathcal{M} + \{\emptyset\}$.[9]

Definition 16 (Firing of autonomous transitions). *Let $t \in T$ such that $\lambda(t) = (s, \overline{M})$, and M and M' be two instances such that $M \xrightarrow{t(\sigma)} M'$ with $\sigma(\nu) \notin Id(\overline{M})$ for $\nu \in \Upsilon$. Then $\{M\} + \mathcal{M} \xrightarrow{t} \{M', \overline{M}\} + \mathcal{M}$ for any marking $\mathcal{M}$ such that $\sigma(\nu) \notin Id(\mathcal{M})$ for $\nu \in \Upsilon$.*

Definition 17 (Firing of synchronizing transitions). *Let $t_1, t_2 \in T$ such that $\lambda(t_1) = (s?, \overline{M}_1)$ and $\lambda(t_2) = (s!, \overline{M}_2)$ for some $s \in \mathcal{S}$, and let M_1, M_1', M_2 and M_2' be instances such that $M_i \xrightarrow{t_i(\sigma_i)} M_i'$ with $\sigma_1(x) = \sigma_2(x)$ for all $x \in Var(t_1) \cap Var(t_2)$ and $\sigma(\nu) \notin Id(\overline{M}_i)$ for $i = 1, 2$. Then $\{M_1, M_2\} + \mathcal{M} \xrightarrow{(t_1,t_2)} \{M_1', M_2', \overline{M}_1, \overline{M}_2\} + \mathcal{M}$ for any marking $\mathcal{M}$ such that $\sigma(\nu) \notin Id(\mathcal{M})$ for $\nu \in \Upsilon$.*

Notice that when $\overline{M}$, $\overline{M}_1$ or $\overline{M}_2$ are empty then no instance is created.

Appendix B. Proofs of Section 4.

For simplicity, we sometimes extend firings of transitions in regions, to firings in $\emptyset$-expansions of regions in the natural way.

B1. Proof of Proposition 2 and Proposition 3

Let us denote $\mathcal{C}(M) = \{r \mid a{:}(m, r) \in M\} \in {\mathbb{R}_{\geq 0}}^{\oplus}$.

Lemma 1. *Let M be a marking such that $\mathcal{C}(M) \cap \mathbb{N} \neq \emptyset$ and $\epsilon = \max\{frct(r) \mid r \in \mathcal{C}(M)\}$. If $0 < d < 1 - \epsilon$ then $R_M \overset{\delta}{\twoheadrightarrow} R_{M^{+d}}$. Moreover, $\mathcal{C}(M^{+d}) \cap \mathbb{N} = \emptyset$.*

[9] This is equivalent to the mechanism of $\emptyset$-expansions/contractions, though we prefer to use the later in the rest of the paper in order to deal only with wpo (and not wqo).

Proof. Suppose that $R_M = A_0 * A^{x_1} * ... * A^{x_n} * A_\infty$, where $0 < x_1 < ... < x_n < 1$ are the fractional parts of the ages of the instances younger than max in M. Then, as $x_n = \epsilon = \max\{\mathit{frct}(r) \mid r \in \mathcal{C}(M)\}$, $0 < d < 1 - \epsilon$ and $\mathcal{C}(M) \cap \mathbb{N} \neq \emptyset$, the fractional parts of the ages of the instances younger than max in M^{+d} are $d, x_1 + d, ..., x_n + d$ (with $x_n + d < 1$). For each $i \in n^+$, the instances and markings with fractional parts of its ages $x_i + d$ in M^{+d} are the same as the ones in M with fractional parts of its ages x_i. Moreover, the instances and markings with fractional parts of its ages d in M^{+d} are the instances with natural ages younger than max in M. Therefore, $R_{M^{+d}} = \emptyset * A^d * A^{x_1+d} * ... * A^{x_n+d} * A'_\infty$ as defined in Def. 9, where in A'_∞ are represented the instances in A_∞ and the instances in A_0 with age max and in A^d are represented the instances in A_0 younger than max. By the first case of Def. 10, we have that $R_M \overset{\delta}{\twoheadrightarrow} R'$, where $R' = \emptyset * A_0^< * A_{x_1} * ... * A_{x_n} * (A_\infty + A_0^=) = \emptyset * A^d * A^{x_1+d} * ... * A^{x_n+d} * A'_\infty = R_{M^{+d}}$.

Lemma 2. *Let M be a marking such that $\mathcal{C}(M) \cap \mathbb{N} = \emptyset$ and $\epsilon = \max\{\mathit{frct}(r) \mid r \in \mathcal{C}(M)\}$. If $d < 1 - \epsilon$ then $R_M = R_{M^{+d}}$.*

Proof. Suppose that $R_M = \emptyset * A^{x_1} * ... * A^{x_n} * A_\infty$, where $0 < x_1 < ... < x_n < 1$ are the fractional parts of the ages of the instances younger than max in M. Then, as $x_n = \epsilon = \max\{\mathit{frct}(r) \mid r \in \mathcal{C}(M)\}$, $d < 1 - \epsilon$ and $\mathcal{C}(M) \cap \mathbb{N} = \emptyset$, the fractional parts of the ages of the instances younger than max in M^{+d} are $x_1 + d, ..., x_n + d < 1$, and moreover, for each $i \in n^+$, the instances and markings with fractional parts of its ages $x_i + d$ in M^{+d} are the same as the ones in M with fractional parts of its ages x_i. Therefore, by the definition of region, $R_{M^{+d}} = \emptyset * A^{x_1+d} * ... * A^{x_n+d} * A_\infty = \emptyset * A^{x_1} * ... * A^{x_n} * A_\infty = R_M$.

Lemma 3. *Let M be a marking such that $\mathcal{C}(M) \cap \mathbb{N} = \emptyset$ and $\epsilon = \max\{\mathit{frct}(r) \mid r \in \mathcal{C}(M)\}$. If $d = 1 - \epsilon$ then $R_M \overset{\delta}{\twoheadrightarrow} R_{M^{+d}}$. Moreover, $\mathcal{C}(M^{+d}) \cap \mathbb{N} \neq \emptyset$.*

Proof. Suppose that $R_M = \emptyset * A^{x_1} * ... * A^{x_n} * A_\infty$, where $x_1, ..., x_n \in (0, 1)$ with $i < j$ iff $x_i < x_j$, are the fractional parts of the ages of the instances younger than max in M. Then, as $x_n = \epsilon = \max\{\mathit{frct}(r) \mid r \in \mathcal{C}(M)\}$, $d = 1 - \epsilon$ and $\mathcal{C}(M) \cap \mathbb{N} = \emptyset$, the fractional parts of the ages of the instances younger than max in M^{+d} are $0, x_1 + d, ..., x_{n-1}$. For each $i \in (n-1)^+$, the instances and markings with fractional parts of its ages $x_i + d$ in M^{+d} are the same as the ones in M with fractional parts of its ages x_i. Moreover, the instances and markings with natural ages in M^{+d} are the instances with ages with fractional part x_n in M. Therefore, $R_{M^{+d}} = A_0 * A^{x_1+d} * ... * A^{x_{n-1}+d} * A_\infty$ as defined in Def. 9. By the second case of Def. 10, we have that $R_M \overset{\delta}{\twoheadrightarrow} R'$, where $R' = A_n^{+1} * A_{x_1} * ... * A_{x_{n-1}} * A_\infty = A_0 * A^{x_1+d} * ... * A^{x_{n-1}+d} * A_\infty = R_{M^{+d}}$.

Lemma 4. *Let M be a marking such that $\mathcal{C}(M) \cap \mathbb{N} \neq \emptyset$. Then $R_M \overset{\Delta}{\twoheadrightarrow} R_{M^{+1}}$.*

Proof. Let $R_M = A_0 * A_1 * ... * A_n * A_\infty$ and let x_i be the fractional part of the ages of instances in A_i for $i \in n^*$ (so $x_0 = 0$) and take $x_{n+1} = 1$. Then $x_0 < x_1 < ... < x_n < x_{n+1}$. We define $\epsilon_i = (x_{i+1} - x_i)/2$ for $i \in n^*$. Let $M_{n+1} = M$,

$M'_{i+1} = M_{i+1}^{+\epsilon_i}$ for $i \in n^*$ and $M_{i-1} = (M'_i)^{+\epsilon_i}$ for $i \in (n+1)^+$. Then we have $M = M_{n+1} \overset{\epsilon_n}{\twoheadrightarrow} M'_{n+1} \overset{\epsilon_n}{\twoheadrightarrow} M_n \overset{\epsilon_{n-1}}{\twoheadrightarrow} \ldots \overset{\epsilon_1}{\twoheadrightarrow} M_1 \overset{\epsilon_0}{\twoheadrightarrow} M'_1 \overset{\epsilon_0}{\twoheadrightarrow} M_0$. Notice that $\sum_{i \in n^*} 2\epsilon_i = 1$, so that $M_0 = M^{+1}$. It also holds that $\mathcal{C}(M_i) \cap \mathbb{N} \neq \emptyset$ for all $i \in (n+1)^*$ and $\mathcal{C}(M'_i) \cap \mathbb{N} = \emptyset$ for all $i \in (n+1)^+$. Moreover, the maximum fractional part of the reals in M_{i+1} is $1 - 2\epsilon_i$ for $i \in n^+$, and that of M'_{i+1} is $1 - \epsilon_i$ for $i \in n^*$. Then M_{i+1} and ϵ_i are in the hypothesis of Lemma 1, and M'_{i+1} and ϵ_i in the ones of Lemma 3. Therefore, $R_{M_i} \overset{\delta}{\twoheadrightarrow} R_{M'_i}$ for $i \in (n+1)^+$ and $R_{M'_{i+1}} \overset{\delta}{\twoheadrightarrow} R_{M_i}$ for $i \in n^*$, so that $R_M = R_{M_{n+1}} \overset{\Delta}{\twoheadrightarrow} R_{M_0} = R_{M^{+1}}$.

Lemma 5. *Given two markings M and M', if $M \xrightarrow{t} M'$ then $R_M \overset{t}{\twoheadrightarrow} R_{M'}$.*

Proof. Let us suppose that $M \xrightarrow{t} M'$. Then, if $nfVar(t) = \{x_1, ..., x_{n_1}\}$ and $fVar(t) = \{x_{n_1+1}, ..., x_{n_2}\}$ then we have $M = a_1 : (m_1, r_1), ..., a_{n_1} : (m_{n_1}, r_{n_1}) + \overline{M}$ and for each $i \in n_1^+$, $F_t(x_i) \subseteq m_i$ (1) and $r_i \in \mathcal{G}_t^1(x_i)$ (2). Moreover, let $R_M = A_0 * A_1 * \ldots * A_n * A_\infty$, and $R_M^\emptyset = A_0 * A_1 * \ldots * A_n * A_{n+1} * A_\infty$ be the $\emptyset$-expansion of R_M, where $A_i = \{(m_{i1}^\emptyset, r_{i1}^\emptyset), \ldots, (m_{ik_i}^\emptyset, r_{ik_i}^\emptyset)\}$ for $i \in n_\infty^*$ and A_{n+1} which only contains empty instances. Let $l = max\{k_i \mid i \in n_\infty^*\}$.

Let us define a mode $\tau = (\tau_1, \tau_2, \tau_3)$ for firing t from R_M, obtaining the region R'_M. First of all, we define τ_1, and prove that t is enabled at R_M with a mode with τ_1 as first component. Let $\Phi_1 : n_1^+ \to n_\infty^* \times l$, be the function which associates each $i \in n_1^+$ to the location of the pair which represents the instance a_i in R_M. Note that the instances in $\overline{M}$ are not selected by τ_1, and therefore, they remain in the same A_is after firing t in the region. Then, we define τ_1 such that $\tau_1(x_i) = \Phi_1(i)$ if $x_i \in nfVar(t)$ and $\tau_1(x_i) = (n+1, i)$ otherwise. Then, we have that, for each $x \in Var(t)$:

- If $x \in \Upsilon$ then $\tau_1(x) = (n+1, i)$, and $m_{n+1,i} = \emptyset$.
- If $\tau_1(x) = (i, j)$, then $F_t(x) \subseteq m_{ij}^\emptyset$, because of (1).
- $r_{ij} \in \mathcal{G}_t^1(x)$ if $i \in \{0, \infty\}$, and $r_{ij} + 0.5 \in \mathcal{G}_t^1(x)$, otherwise, because of (2).

Therefore, t is enabled at R_M. Now, we prove that we can fire it in such a way that we reach $R_{M'}$. For that purpose, we first need to define the proper functions τ_2 and τ_3 of the mode for the firing.

Let us call $\Phi_2 : Var(t) \to \mathbb{R}_{\geq 0}$ the function which associates each $x_i \in Var(t)$ to the age to which the instance a_i of M is updated in the firing. Then, we define τ_2 such that, for each $x \in Var(t)$, $\tau_2(x) = \lfloor \Phi_2(x) \rfloor$ if $\Phi_2(x) \leq max$ or $\tau_2(x) = max + 1$. With this definition, for each $x \in Var(t)$, $\tau_2(x)$ is in the left closure of $\mathcal{G}_t^2(x)$, as Def. 12 demands. Finally, we define τ_3 such that:

- If $\tau_2(x) = n + 1$ then $\tau_3(x) = \infty$ (and therefore, the second condition demanded by Def. 12) else,
- if $\tau_2(x) = \Phi_2(x)$, that is, if $\Phi_2(x) \in \mathbb{N}$ then $\tau_3(x) = 0$ else,
- if $frct(\Phi_2(x))$, is a fractional part which has names represented in A_i then $\tau_3(x) = i$ else,

– if A_i represents the names with the greatest fractional part f lower than $\tau_2(x)$, and $\tau_2(x)$ if the j^{th} fractional part greater than f of all the $frct(x_i)$, $\tau_3(x) = (i, j)$.

The third condition of Def. 12 holds for the mode τ we have defined, because if $\mathcal{G}_t^2(x) = (a, b]$ or $\mathcal{G}_t^2(x) = (a, b)$ and $\tau_2(x) = a$, then $\tau_2(x) \neq \Phi_2(x)$ and therefore $\tau_3(x) \neq 0$.

Now, we prove that the region R that we reach by firing t from R_M with mode τ is $R_{M'}$. We do it by proving that A'_i is in R'_M iff it is in R.

We analyse different cases:

– First, we consider A'_0 of R'_M. $A'_0 = \{(m, r) \mid \exists a \text{ with } a\colon (m, r) \in \bar{M}, r \in \mathbb{N}, r \leq max\} + \{(m, r) \mid \exists k \in n_1^+ \text{ with } \Phi_1(k) = (i, j), m = (m_{ij}^{\emptyset} - F_t(x_k)) + H_t(x_k), \Phi_2(k) = r, r \in \mathbb{N}, r \leq max\} = \{(m_{0j}^{\emptyset}, r_{0j}^{\emptyset}) \in A_0 \mid \nexists x \in Var(t) \text{ with } \tau_1(x) = (0, j)\} + \{(m, r) \mid \exists x \text{ with } \tau_1(x) = (i, j), m = (m_{ij}^{\emptyset} - F_t(x)) + H_t(x), \tau_2(x) = r, \tau_3(x) = 0\}$ which is the first set in R.
– Consider A'_∞ of R'_M. $A'_\infty = \{(m, max + 1) \mid \exists a \text{ with } a\colon (m, r) \in \bar{M}, r > max\} + \{(m, max + 1) \mid \exists k \in n_1^+ \text{ with } \Phi_1(k) = (i, j), m = (m_{ij}^{\emptyset} - F_t(x_k)) + H_t(x_k), \Phi_2(k) = r, r > max\} = \{(m_{\infty j}^{\emptyset}, max{+}1) \in A_\infty \mid \nexists x \in Var(t) \text{ with } \tau_1(x) = (\infty, j)\} + \{(m, max + 1) \mid \exists x \text{ with } \tau_1(x) = (i, j), m = (m_{ij}^{\emptyset} - F_t(x)) + H_t(x), \tau_2(x) = max + 1\}$, which is the last set in R.
– Now, let us consider the case in which A is a set of R'_M which represents instances in M with the fractional part of the age ρ. Moreover, let us suppose A_k is the set of R_M which represents instances with this fractional part of the age. Then $A = \{(m, r) \mid \exists a \text{ with } a\colon (m, r') \in \bar{M}, r = \lfloor r' \rfloor, frct(r') = \rho, r' \leq max\} + \{(m, r) \mid \exists k' \in n_1^+ \text{ with } \Phi_1(k') = (i, j), m = (m_{ij}^{\emptyset} - F_t(x_{k'})) + H_t(x_{k'}), \Phi_2(k') = r', r = \lfloor r' \rfloor, frct(r') = \rho, r' \leq max\} =$ $= \{(m_{kj}^{\emptyset}, r_{kj}^{\emptyset}) \in A_k \mid \nexists x \in Var(t) \text{ with } \tau_1(x) = (k, j)\} + \{(m, r) \mid \exists x \text{ with } \tau_1(x) = (i, j), m = (m_{ij}^{\emptyset} - F_t(x)) + H_t(x), \tau_2(x) = r, \tau_3(x) = k\}$, which is in R.
– Finally, we consider the case in which A is a set of R'_M which represents instances with fractional part of the age ρ different to all the ones in M. Then, $A = \{(m, r) \mid \exists k \in n_1^+ \text{ with } \Phi_1(k) = (i, j), m = (m_{ij}^{\emptyset} - F_t(x_k)) + H_t(x_k), \Phi_2(k) = r', r = \lfloor r' \rfloor, frct(r') = \rho, r' \leq max\} = \{(m, r) \mid \exists x \text{ with } \tau_1(x) = (i, j), m = (m_{ij}^{\emptyset} - F_t(x)) + H_t(x), \tau_2(x) = r, \tau_3(x) = (k, k)$ where A_{k1} represents the names with the greatest fractional part f lower than r, and r if the k^{th} fractional part greater than f of all the $frct(x_i)\}$, which is in R.

Finally, note that the order of the sets of R correspond to the order of the corresponding A'_i of R'_M. That is because we have defined τ, in such a way that we order the different sets depending on the fractional part of $r_g''s$ younger than max, setting the instances older than max in A'_∞, as in R'_M.

Lemma 6. *Given two markings M and M' and $d > 0$, if $M \xrightarrow{d} M'$ then $R_M \overset{\Delta}{\twoheadrightarrow} R_{M'}$*

Proof. Let $d > 1$ (the other case is easier) and $M \xrightarrow{d} M'$. Suppose that $\mathcal{C}(M) \cap \mathbb{N} \neq \emptyset$ (otherwise, by lemma 3 we know that there exist ϵ such that $R_M \overset{\delta}{\twoheadrightarrow} R_{M^{+\epsilon}}$

and $\mathcal{C}(M^{+\epsilon}) \cap \mathbb{N} \neq \emptyset$, and we start from $M^{+\epsilon}$). Then, we have that $M \xrightarrow{1} M^{+1} \xrightarrow{1} M^{+2} \xrightarrow{1} \ldots \xrightarrow{1} M^{+\lfloor d \rfloor} \xrightarrow{frct(d)} M'$. Because of Lemma 4 we know that $R_M \overset{\Delta}{\twoheadrightarrow} R_{M^{+1}} \overset{\Delta}{\twoheadrightarrow} R_{M^{+2}} \overset{\Delta}{\twoheadrightarrow} \ldots \overset{\Delta}{\twoheadrightarrow} R_{M^{+\lfloor r \rfloor}}$. Therefore, we only need to prove that $R_{M^{+\lfloor d \rfloor}} \overset{\Delta}{\twoheadrightarrow} R_{M'}$. As in Lemma 4, let $R_{M^{+\lfloor d \rfloor}} = A_0 * A_1 * ... * A_n * A_\infty$ and let x_i be the fractional part of the ages of instances in A_i for $i \in n^*$, $x_{n+1} = 1$, $\epsilon_i = (x_{i+1} - x_i)/2$ for $i \in n^*$, $M_{n+1} = M^{+\lfloor d \rfloor}$, $M'_{i+1} = M_{i+1}^{+\epsilon_i}$ and $M_{i-1} = (M'_i)^{+\epsilon_i}$. Now, we select k such that $1 - x_k \leq \mathit{frct}(d)$ and $1 - x_{k-1} > \mathit{frct}(d)$, and we define $y = x_k - (1 - \mathit{frct}(d))$ and $M_y = M_k^{+y}$. Note that $\sum_{i=k}^{n} 2 * \epsilon + y = (1 - x_k) + x_k - (1 - \mathit{frct}(d)) = \mathit{frct}(d)$, so that $M_y = M^{+\lfloor d \rfloor + \mathit{frct}(d)} = M'$. Repeating the same reasoning as in Lemma 4, we can conclude that $R_{M^{+\lfloor d \rfloor}} = R_{M_{n+1}} \overset{\Delta}{\twoheadrightarrow} R_{M_y} = R_{M'}$.

Lemma 7. *Given a markings M, if $R_M \overset{t}{\twoheadrightarrow} R'$ then there is M' with $R' = R_{M'}$ and $M \xrightarrow{t} M'$.*

Proof. Let us suppose that $R_M \overset{t}{\twoheadrightarrow} R'$ with mode $\tau = (\tau_1, \tau_2, \tau_3)$. We define a marking M' with $R' = R_{M'}$, and then we prove that $M \xrightarrow{t} M'$. Let us suppose $R_M = A_0 * A_1 * \ldots * A_n * A_\infty$ and $R' = A'_0 * A'_1 * \ldots * A'_{n'} * A_\infty$. We define M' as the marking such that:

- For each $(m, r) \in A'_0$, $a{:}(m, r) \in M'$.
- For each $(m, max + 1) \in A'_\infty$, $a : (m, r) \in M'$, where r is the age of the corresponding instance $a{:}(m, r) \in M$ if $(m, max + 1)$ is in A_∞, and $max + 1$ otherwise.
- Analogously, we consider $(m, r) \in A'_i$, where A'_i is a set obtained in the firing from the set A of R_M which represents instances with age of fractional part $\rho(A'_i)$. Then, $a{:}(m, r') \in M'$, where $r' = r + \rho$.
- Finally, for each A'_i which is a new set obtained in the firing from the set A, we define some $\rho(A'_i) \in (0, 1)$ such that if $i < j$ then $\rho(A'_i) < \rho(A'_j)$. Then, for each $(m, r) \in A'_i$, $a{:}(m, r') \in M'$, where $r' = r + \rho$.

Clearly, $R' = R_{M'}$. Now, we prove that t is enabled at M and $M \xrightarrow{t} M'$. We know that t is enabled at R_M with mode τ, and therefore, we have that for each $x \in \mathit{nfVar}(t)$ with $\tau_1(x) = (i, j)$:

- $F_t(x) \subseteq m_{ij}$ and
- $r_{ij} \in \mathcal{G}_t^1(x)$ if $i \in \{0, \infty\}$, and $r_{ij} + 0.5 \in \mathcal{G}_t^1(x)$.

Therefore, if $\mathit{nfVar}(t) = \{x_1, ..., x_n\}$, we can rename $M = a_1 : (m_1, r_1), ..., a_n : (m_n, r_n) + \overline{M}$, and for each $i \in n^+$, $F_t(x_i) \subseteq m_i$ and $r_i \in \mathcal{G}_t^1(x_i)$, where m_i is m_{jk} of R_M if $\tau_1(x_i) = (j, k)$. Therefore, t is enabled at M, so there is M'' such that $M \xrightarrow{t} M''$. We prove that for each instance in M'', the same instance is in M', and therefore, as the number of instances of M' and M'' are the same (because of the definition of firing of transition for region), $M'' = M'$. Let $a{:}(m, r) \in M''$. We consider different cases:

– If $a{:}(m,r) \in \overline{M}$ then it is in M too. Let us suppose that $(m_{ij}, r_{ij}) \in A_i$ is the pair which represents this instance in R_M. Then, because of how we have renamed M, there is not $x \in \mathit{Var}$ with $\tau_1(x) = (i,j)$, and therefore $(m_{ij}, r_{ij}) \in A'_k$, where A'_k is the set of R' which represents instances that remain in A_k after firing. Therefore, $a{:}(m,r) \in M'$, because of the third point in the definition of M'.
– Suppose $a{:}(m,r) \notin \overline{M}$. Then, $a{:}(m,r)$ is associated to some $x_i \in \mathit{Var}(t)$ in the firing, that is, there is x_i such that $a = a_i$ and $m = (m_i - F_t(x_i)) + H_t(x_i)$ if $x_i \in \mathit{nfVar}(t)$, $m = H_t(x_i)$ if $x_i \in \mathit{fVar}(t)$. We analize the first case, and suppose $\tau_3(x_i) \neq \infty$ (the other cases are analogous). If $\tau_1(x_i) = (j,k)$ then $(m_{jk}, r_{jk}) \in R_M$, where $m_{jk} = m_i$ and $r_{jk} = \lfloor r_i \rfloor$. Then, after firing t from R_M with mode τ, we have that $(m, \tau_2(x_i)) \in A'_{\tau_3(x)}$ (with the notations of the firings for R'). Therefore, because of how we have defined M', the instance $a{:}(m,r)$ is in M' (note that $r = \tau_2(x_i) + \rho$, where ρ represents the fractional part of the ages of the instances represented in $A'_{\tau_3(x)}$.

Therefore, $M'' = M'$, so $M \xrightarrow{t} M'$.

Lemma 8. *Given a marking M, if $R_M \overset{\delta}{\twoheadrightarrow} R'$ there is M' with $R' = R_{M'}$ and $M \xrightarrow{d} M'$ for some $d \in (0,1)$.*

Proof. Let us suppose that $R_M \overset{\delta}{\twoheadrightarrow} R'$. Let us first consider the case in which $R_M = \emptyset * A^{x_1} * \ldots * A^{x_n} * A_\infty$. Then, $R' = A_n^{+1} * A^{x_1} * \ldots * A^{x_n} * A_\infty$. Let d be the fractional part of the ages of the instances in A_n. And let M' be a marking such that $M \xrightarrow{d} M'$. We are going to prove that $R' = R_{M'}$. By the definition of region of a marking, $R_{M'} = A'_0 * A^{x'_1} * \ldots * A^{x'_n} * A'_\infty$ where:

– $A'_0 = \{(m,r) \mid a{:}(m,r) \in M',\ r \in \max^*\} = \{(m,r) \mid a{:}(m, r-(1-d)) \in M,\ r \in \max^*\} = A_n^{+1}$.
– $A^{x'_i} = \{(m, \lfloor r \rfloor) \mid a{:}(m,r) \in M', r < \max, \mathit{frct}(r) = x'_i\} = \{(m, \lfloor r \rfloor) \mid a{:}(m, r-d) \in M, r-d < \max, \mathit{frct}(r-d) = x_i\} = A^{x_i}$.
– $A_\infty = \{(m, \max+1) \mid a{:}(m,r) \in M',\ r > \max\} = \{(m, \max+1) \mid a{:}(m, r-d) \in M,\ r-d > \max\} = A_\infty$.

Now, we consider the case in which $R_M = A_0 * A^{x_1} * \ldots * A^{x_n} * A_\infty$ and $A_0 \neq \emptyset$. Then, $R' = \emptyset * A_0^{<} * A^{x_1} * \ldots * A^{x_n} * (A_\infty + A_0^{=})$. Let $0 < d < 1 - x_n$, and let M' be a marking such that $M \xrightarrow{d} M'$. Again, we are going to prove that $R' = R_{M'}$. By the definition of region of a marking, $R_{M'} = A'_0 * A^{x'_1} * \ldots * A^{x'_n} * A'_\infty$ where:

– $A'_0 = \{(m,r) \mid a{:}(m,r) \in M',\ r \in \max^*\} = \{(m,r) \mid a{:}(m, r-d) \in M',\ r \in \max^*\} = \emptyset$.
– $A^{x'_1} = \{(m, \lfloor r \rfloor) \mid a{:}(m,r) \in M', r < \max, \mathit{frct}(r) = x'_1\} = \{(m, \lfloor r \rfloor) \mid a{:}(m, r-d) \in M, r < \max, \mathit{frct}(r) = x'_1\} = A_0^{<}$.
– $A^{x'_i} = \{(m, \lfloor r \rfloor) \mid a{:}(m,r) \in M', r < \max, \mathit{frct}(r) = x'_i\} = \{(m, \lfloor r \rfloor) \mid a{:}(m, r-d) \in M, r-d < \max, \mathit{frct}(r-d) = x_i\} = A^{x_i}$.

– $A_\infty = \{(m, \max+1) \mid a{:}(m,r) \in M',\ r > \max\} = \{(m, \max+1) \mid a{:}(m,r) \in M,\ r - d > \max\} \cup \{(m, \max+1) \mid a{:}(m,r) \in M,\ r = \max\} = A_\infty + A_0^{=}$.

Lemma 9. *Given a marking M, if $R_M \overset{\Delta}{\twoheadrightarrow} R'$ there is M' with $R' = R_{M'}$ and $M \xrightarrow{d} M'$ for some $d \in \mathbb{R}_{\geq 0}$.*

Proof. Suppose that $R_M \overset{\Delta}{\twoheadrightarrow} R'$. As $\overset{\Delta}{\twoheadrightarrow}$ is the reflexive and transitive closure of $\overset{\delta}{\twoheadrightarrow}$, we have that $R_M = R_0 \overset{\delta}{\twoheadrightarrow} R_1 \overset{\delta}{\twoheadrightarrow} \ldots \overset{\delta}{\twoheadrightarrow} R' = R_k$. We can prove by an inductive reasoning that for each $i \in n^+$, there exist M_i, d_i such that $R_i = R_{M_i}$ and $M_{i-1} \xrightarrow{d_i} M_i$, by applying the previous claim to each R_i. Therefore, we have that $M = M_0 \xrightarrow{d_1} M_1 \xrightarrow{d_2} \ldots \xrightarrow{d_k} M_k$ and $R' = R_k = R_{M_k}$. Therefore, if we take $d = \sum_{i \in k^+} d_i$, then we have that $M \xrightarrow{d} M_k$ and $R' = R_{M_k}$

As $\twoheadrightarrow = \overset{\Delta}{\twoheadrightarrow} \cup \bigcup_{t \in T} \overset{t}{\twoheadrightarrow}$, Prop. 2 easily follows from the previous lemma.

Proposition 3 *Given $p \in P$ we can compute a set of regions $\mathcal{R}_p$ such that there is a reachable marking that marks p iff $\uparrow\mathcal{R}_p$ can be reached.*

Proof. Let $R_0^r = \{(\{p\}, r)\} * \emptyset$ for each $r \in \max_\infty^*$, $R_\infty = \emptyset * \{(\{p\}, \max+1)\}$ and $R^r = \emptyset * \{(\{p\}, r)\} * \emptyset$ for $r \in (\max-1)^*$. Let us see that $\mathcal{R}_p = \{R_0^r \mid r \in \max_\infty^*\} \cup \{R^r \mid r \in (\max-1)^*\} \cup \{R_\infty\}$ satisfies the thesis. First, let us assume that $M_0 \rightarrow^* M$ with $a : (m, r) \in M$ with $p \in m$. By Prop. 2 we have $R_{M_0} \twoheadrightarrow^* R_M = A_0 * A_1 * ... * A_n * A_\infty$. Let us distinguish cases for $r \in \mathbb{R}_{\geq 0}$. If $r \in \max^*$ then by Def. 9, $(m, r) \in A_0$ and $R_0^r \sqsubseteq R_M$. If $r > \max$ also by Def. 9 we have $(m, \max+1) \in A_\infty$, so that $R_\infty \sqsubseteq R_M$. Finally, if $r \leq \max$ and $r \notin \mathbb{N}$, we have $(m, \lfloor r \rfloor) \in A_i$ for some $i \in n^+$, so that $R^{\lfloor r \rfloor} \sqsubseteq R_M$. In any case, $R_M \in \uparrow\mathcal{R}_p$.

Conversely, let us assume that $R_{M_0} \twoheadrightarrow^* R$ with $R \in \uparrow\mathcal{R}_p$. By Prop. 2 there is M reachable such that $R = R_M$. Since $R_M \in \uparrow\mathcal{R}_p$ there is $R' \in \mathcal{R}_p$ such that $R' \sqsubseteq R$. Analogously as in the converse implication, and using again Def. 9, we distinguish cases over R' obtaining in any case that $a : (m, r) \in M$ for some m with $p \in m$, and we conclude.

B2. ν-*lTPN* are Well Structured Transition systems

Proposition$\sqsubseteq$ *is a decidable wpo.*

Proof. In the first place, it is trivially decidable. To prove that it is a wpo, let us remark that a region $R = A_0 * A_1 * ... * A_n * A_\infty$ can be seen as an element of $X = X_{max^*}^{\oplus} \times (X_{(max-1)^*}^{\oplus})^{\circledast} \times X_{\{max+1\}}^{\oplus}$, where for every $I \subseteq (max+1)^*$, $X_I = P^{\oplus} \times I$. Indeed, $A_0 \in X_{max^*}^{\oplus}$, $A_\infty \in X_{\{max+1\}}^{\oplus}$ and $u = A_1 * ... * A_n$ can be seen as a word over $X_{(max-1)^*}^{\oplus}$. Therefore, $\sqsubseteq$ is just the standard order in X, as defined in the preliminaries. Then, $\sqsubseteq$ is a wpo because it is built from wpos (finite sets with equality[10]) using operators that preserve well-orders (multisets, words and the product).

Lemma 10. *If $R_1 \xrightarrow{\delta} R_2$ and $R_1 \sqsubseteq R_1'$ then there is R_2' such that $R_1' \sqsubseteq R_2'$ and $R_1' \xrightarrow{\Delta} R_2'$.*

Proof. Let $R_1 = A_0 * A_1 * ... * A_n * A_\infty$ and $R_1' = B_0 * u_0 * B_1 * ... * B_n * u_n * B_\infty$ with $A_i \leq^\oplus B_i$. First we assume that $A_0 \neq \emptyset$, so that $B_0 \neq \emptyset$. By Def. 10 we have $R_2 = \emptyset * A_0^< * A_1 * ... * A_n * (A_\infty + A_0^=)$ and since $B_0 \neq \emptyset$ we also have $R_1' \xrightarrow{\delta} R_2' = \emptyset * B_0^< * u_0 * B_1 * ... * B_n * u_n * (B_\infty + B_0^=)$. Since $A_0 \leq B_0$ we also have $A_0^< \leq B_0^<$, $A_0^= \leq B_0^=$ and thus $(A_\infty + A_0^=) \leq (B_\infty + B_0^=)$. Then $R_1' \sqsubseteq R_2'$.

Now let us assume that $A_0 = \emptyset$, so that $R_2 = A_n^{+1} * A_1 * ... * A_{n-1} * A_\infty$. We also assume that $B_0 \neq \emptyset$ (the other case is only slightly more simple). If $u_n = C_1 * ... * C_k$ then $R_1' \xrightarrow{\Delta}{}^* R_2' = B_n^{+1} * (C_1^{+1})^< * ... * (C_k^{+1})^< * B_0^< * u_0 * B_1 * u_1 * ... * u_{n-1} * (B_\infty + B_0^= + (C_1^{+1})^= + ... + (C_k^{+1})^=)$ in $2k+2$ steps, and clearly $R_2 \sqsubseteq R_2'$.

Lemma 11. *If $R_1 \xrightarrow{\Delta} R_2$ and $R_1 \sqsubseteq R_1'$ then there is R_2' such that $R_1' \sqsubseteq R_2'$ and $R_1' \xrightarrow{\Delta} R_2'$.*

Proof. Since $\xrightarrow{\Delta}$ is the reflexive and transitive closure of $\xrightarrow{\delta}$, it follows from the previous lemma.

Lemma 12. *Let R and R' be two regions such that $R \sqsubseteq R'$. If $A_0 * A_1 * ... * A_n * A_\infty$ is an $\emptyset$-expansion of R, then there is an $\emptyset$-expansion of R' of the form $B_0 * u_0 * B_1 * u_1 * ... * u_{n-1} * B_n * u_n * B_\infty$ such that for all $i \in n_\infty^*$:*

- *$A_i \leq^\oplus B_i$,*
- *$(\emptyset, r) \in A_i$ iff $(\emptyset, r) \in B_i$.*

Proof. Indeed, the $\emptyset$-expansion of R' in the lemma can be obtained by adding to R' the same empty instances added to R in order to obtain $A_0 * A_1 * ... * A_n * A_\infty$.

Lemma 13. *If $R_1 \xrightarrow{t} R_1'$ and $R_1 \sqsubseteq R_2$ then there is R_2' such that $R_2 \sqsubseteq R_2'$ and $R_2 \xrightarrow{t} R_2'$.*

Proof. Assume $R_1 \xrightarrow{t} R_1'$ with mode τ . Let $A_0 * A_1 * \ldots * A_n * A_\infty$ be the $\emptyset$-expansion of R_1 in the firing of t. By the previous lemma, there is an $\emptyset$-expansion of R_2 of the form $A_0^2 * u_0 * A_1^2 * u_1 * ... * u_{n-1} * A_n^2 * u_n * A_\infty^2$ such that $A_i \leq^\oplus A_i^2$ and $(\emptyset, r) \in A_i$ iff $(\emptyset, r) \in A_i^2$ for all i. Without loss of generality, we can suppose that for each $i \in n_\infty^*$, if $A_i = \{(m_1, r_1), \ldots, (m_{n'}, r_{n'})\}$ and $A_i^2 = \{(m_1^2, r_1^2), \ldots, (m_{n''}^2, r_{n''}^2)\}$ then, for each $j \in n'^+$, $m_{ij} \subseteq m_{ij}^2$ and $r_{ij} = r_{ij}^2$. Let us see that we can fire t from R_2 with mode τ, obtaining R_2' greater than R_1' (in fact, mode τ for R_2 is an abuse of notation, since we are forgetting about the u_is) . First, we prove that t is enabled at R_2 with mode τ. Let $x \in Var(t)$ with $\tau_1(x) = (i, j)$. Then:

[10] Multiset containment is the multiset order induced by the equality.

– If $x \in \Upsilon$, then $m_{ij} = \emptyset$, and therefore, $m^2_{ij} = \emptyset$ (because of how we have defined the $\emptyset$-expansion of R_2).
– $F_t(x) \subseteq m_{ij} \subseteq m^2_{ij}$.
– As $r_{ij} = r^2_{ij}$, the conditions for r^2_{ij} hold trivially.

Therefore, t is enabled at R_2. Let us see that $R_2 \xrightarrow{t}\!\!\!\!\rightarrow R'_2 \leq R'_1$. Suppose $R'_1 = A'_0 * A'_{01} * \ldots * A'_1 * \ldots * A'_{n'} * A'_{n'1} * \ldots * A'_\infty$ and $R'_1 = A^{2'}_0 * A^{2'}_{01} * \ldots * u_1 * \ldots * A^{2'}_1 * \ldots * A^{2'}_n * A^{2'}_{n1} * \ldots * u_n * \ldots * A^{2'}_\infty$, as in the definition of firings of transitions for regions. Then, we prove that for each index i there is $A^{2'}_{i'}$ with $A'_i \leq^{\oplus} A^{2'}_{i'}$. Let i be one of the indices in R'_1. We prove that for each $(m'_{ij}, r'_{ij}) \in A'_i$, $(m^{2'}_{ij}, r^{2'}_{ij})$ is such that $m'_{ij} \subseteq m^{2'}_{ij}$ and $r^{2'}_{ij} = r^{2'}_{ij}$. Note that, again, this is an abuse of notation, because we are forgetting about the us. This could be fixed by defining another τ'_3, by simply doing a renumeration. However, for the ease of understanding, we prefer to keep using τ_3 and only consider the $A^{2'}_i$s (in fact, the us do not take part in the firing). We consider two cases:

– Suppose that there is not x with $\tau_3(x) = (i, j)$. Then, $(m'_{ij}, r'_{ij}) \in A_i$, and therefore there is $(m^{2'}_{ij}, r^{2'}_{ij}) \in A^2_i$, with $m'_{ij} \subseteq m^{2'}_{ij}$ and $r^2_{ij} = r^{2'}_{ij}$. Moreover, as there is not x with $\tau_3(x) = (i, j)$, $(m^{2'}_{ij}, r^{2'}_{ij}) \in A^{2'}_i$.
– Suppose that there is x with $\tau_3(x) = (i, j)$. Then, if $\tau_1(x) = (k, l)$ then $m'_{ij} = (m_{kj} - F_t(x)) + H_t(x) \subseteq (m'_{kj} - F_t(x)) + H_t(x) = m^{2'}_{ij}$. Moreover, $r'_{ij} = \tau_2(x) = r^{2'}_{ij}$.

Finally, we obtain:

Proposition 4. $\twoheadrightarrow$ is compatible with $\sqsubseteq$.

Proof. It follows as a corollary of Lemma 11 and Lemma 13.

Therefore, we have proved that ν-$lTPN$ are Well Structured Transition Systems. Now, we prove that the effective *Pre*-basis property holds for them.

For that purpose, we need to compute $\min(\uparrow Pre(\uparrow R))$ for any region R. We split Pre into $Pre_\Delta(R) = \{R' \mid R' \xrightarrow{\Delta}\!\!\!\!\rightarrow R\}$ and $Pre_t(R) = \{R' \mid R' \xrightarrow{t}\!\!\!\!\rightarrow R\}$, and we we define $\overline{Pre}_\Delta$ and $\overline{Pre}_t$ for each $t \in T$, so that $Pre_\Delta(\uparrow R) = \uparrow \overline{Pre}_\Delta(R)$ and $Pre_t(\uparrow R) = \uparrow \overline{Pre}_t(R)$. First, we define $\overline{Pre}_\Delta$, the function that computes the predecessors corresponding to time delays, using in turn $\overline{Pre}_\delta$ as an auxiliary function, which corresponds to small time delays. Then $\overline{Pre}_\Delta$ is the reflexive and transitive closure of $\overline{Pre}_\delta$.

Definition 18 ($\overline{Pre}_\delta$). *Let $R = A_0 * A_1 * \ldots * A_n * A_\infty$. We define $\overline{Pre}_\delta(R)$ (and extend it pointwise) as*

$$\begin{cases} \{(A_1 + B_0^{-1}) * A_2 * ... * A_n * B_\infty, \\ \quad B_0^{-1} * A_1 * ... * A_n * B_\infty \mid A_\infty = B_0 + B_\infty\} \text{ if } A_0 = \emptyset \\ \\ \{A_1 * A_2 * ... * A_n * A_0^{-1} * A_\infty \mid A_0^{-1} \text{ defined}\}, \text{ otherwise} \end{cases}$$

Let $\overline{Pre_\delta}^0(\mathcal{R}) = \mathcal{R}$, $\overline{Pre_\delta}^{i+1}(\mathcal{R}) = \overline{Pre_\delta}^i(\mathcal{R}) \cup \overline{Pre_\delta}(\overline{Pre_\delta}^i(\mathcal{R}))$, and $\overline{Pre}_\Delta(R) = \bigcup_{i\geq 0} \overline{Pre_\delta}^i(\{R\})$.

Lemma 14. *Given a region R, $\overline{Pre}_\Delta(R)$ is finite and $\uparrow\overline{Pre}_\Delta(R) = Pre_\Delta(\uparrow R)$*

We split the previous lemma in the following two lemmas, that we prove separately.

Lemma 15. *Given a region R, $\overline{Pre}_\Delta(R)$ is finite.*

Proof. For any $R = A_0 * A_1 * ... * A_n * A_\infty$ we define $\mathit{size}(R) = (r, i, |A_\infty|) \in n^* \times n^*_\infty \times \mathbb{N}$, where $(r, i) = \min\{(r, i) \mid (m, r) \in A_i,\ i \in n^*\}$, where the pairs (r, i) are ordered lexicographically, and we also compare tuples $\mathit{size}(R)$ lexicographically. If $\mathit{size}(R) > (0, 0, 0)$ one of the following holds:

- $\mathit{size}(R) = (r, 0, s)$: then $\overline{Pre}_\delta(R) = \{R'\}$ with $R' = \emptyset * A_1 * A_2 * ... * A_n * A_0^{-1} * A_\infty$ and $\mathit{size}(R') = (r-1, n, s)$.
- $\mathit{size}(R) = (r, i, s)$ with $0 < i \leq n$: then the ages in $A_0, ..., A_{i-1}$ are at least $r+1$. The case $A_0 \neq \emptyset$ is analogous to the previous one: R' as in the previous case is the only region in $\overline{Pre}_\delta(R)$, but now $\mathit{size}(R') = (r, i-1, s)$. If $A_0 = \emptyset$ then any R' in $\overline{Pre}_\delta(R)$ is either of the form $(A_1 + B_0^{-1}) * A_2 * ... * A_n * B_\infty$ or $B_0^{-1} * A_1 * ... * A_n * B_\infty$, with $A_\infty = B_0 + B_\infty$, so that $\mathit{size}(R')$ is either $(r, i-1, s')$ in the first case, or (r, i, s') in the second case. Notice also that in the second case, if $R \neq R'$ then $s' < s$.
- If $\mathit{size}(R) = (\max + 1, \infty, s)$ then $R = \emptyset * A_\infty$ and every R' in $\overline{Pre}_\delta(R)$ is of the form $R' = B_0^{-1} * B_\infty$ with $A_\infty = B_0 + B_\infty$. Notice that if $B_0 = \emptyset$ then $R = R'$. Otherwise, $\mathit{size}(R') = (\max, 0, s')$.
- If $\mathit{size}(R) = (0, 0, s)$ then A_0^{-1} is undefined, and $\overline{Pre}_\delta(R) = \emptyset$.

Assume by contradiction that $\overline{Pre}_\Delta(R)$ is infinite. Then there is a sequence $(R_i)_{i\geq 0}$ of pairwise different regions such that $R_{i+1} \in \overline{Pre}_\delta(R_i)$. By the previous items notice that $\mathit{size}(R_{i+1}) < \mathit{size}(R_i)$, which is a contradiction because the lexicographic order is well-founded in $n^* \times n^*_\infty \times \mathbb{N}$.

Lemma 16. *Given a region R, $\uparrow\overline{Pre}_\Delta(R) = Pre_\Delta(\uparrow R)$*

Proof. Let us first see that $Pre_\Delta(\uparrow R) \subseteq \uparrow\overline{Pre}_\Delta(R)$, for which it is enough to see that $Pre_\delta(\uparrow R) \subseteq \uparrow\overline{Pre}_\Delta(R)$. Let $R = A_0 * A_1 * ... * A_n * A_\infty$, R' and R'' such that $R'' \xrightarrow{\delta} R'$ with $R \sqsubseteq R'$. Since $R \sqsubseteq R'$ we can write $R' = B_0 * u_0 * B_1 * ... * B_n * u_n * B_\infty$ with $A_i \leq^\oplus B_i$. We distinguish three cases: (i) If $A_0 \neq \emptyset$ then $B_0 \neq \emptyset$, in which case $R'' = \emptyset * u_0 * B_1 * ... * B_n * u_n * B_0^{-1} * B_\infty$, which is greater than $\emptyset * A_1 * ... * A_n * A_0^{-1} * A_\infty \in \overline{Pre}_\delta(R) \subseteq \overline{Pre}_\Delta(R)$. (ii) If $A_0 = \emptyset$ and $B_0 \neq \emptyset$ then $R'' = \emptyset * u_0 * B_1 * ... * B_n * u_n * B_0^{-1} * B_\infty \in \uparrow R \subseteq \uparrow\overline{Pre}_\Delta(R)$. (iii) Finally, if $A_0 = B_0 = \emptyset$ we distinguish two subcases. If $u_0 = \epsilon$ then $R'' = (B_1 + C_1) * u_1 * B_2 * ... * B_n * u_n * C_2$ with $C_1^{+1} + C_2 = B_\infty$, which is greater than $(A_1 + D_1) * A_2 * ... * A_n * D_2 \in \overline{Pre}_\delta(R)$ for some $D_1^{+1} + D_2 = A_\infty$. If $u_0 \neq \epsilon$ then $u_0 = B * u_0'$, in which case $R'' = (B + C_1) * u_0' * B_1 * ... * B_n * u_n * C_2$

with $C_1^{+1} + C_2 = B_\infty$, which is greater than $D_1 * A_1 * ... * A_n * D_2 \in \overline{Pre}_\delta(R)$ for some $D_1^{+1} + D_2 = A_\infty$. For the other containment, it is enough to see that $\uparrow\overline{Pre}_\delta(R) \subseteq Pre_\Delta(\uparrow R)$. Notice that for any $R' \in \overline{Pre}_\delta(R)$ we have $R' \xrightarrow{\Delta} R$. Hence, given $R'' \in \uparrow\overline{Pre}_\delta(R)$ we have $R' \sqsubseteq R''$ for some $R' \in \overline{Pre}_\delta(R)$ such that $R' \xrightarrow{\Delta} R$. Then, by compatibility of $\xrightarrow{\delta}$ (see Lemma 10), $R'' \xrightarrow{\Delta} \uparrow R$, so that $R'' \in Pre_\Delta(\uparrow R)$.

Now, we define $\overline{Pre}_t$ to compute the predecessors corresponding to firings of transitions. We first define for $t \in T$ and each region R a family $\mathcal{F}(t, R)$ of functions which assign to each variable in $Var(t)$ a part of the region taking part of the firing. Then we define $\overline{Pre}_t(R)$ considering every $f \in \mathcal{F}(t, R)$.

Definition 19 ($\mathcal{F}(t, R)$). *Let $t \in T$ and a region $R = A_0 * A_1 * \ldots * A_n * A_\infty$, with $A_i = \{(m_{i1}, r_{i1}), \ldots, (m_{ik_i}, r_{ik_i})\}$. Suppose that $l = |Var(t)|$ and $q = \max\{k_i \mid i \in n^*_\infty\}$. A function $f : Var(t) \to (n+1)^*_\infty \times (q+1)^* \times (max+1)^* \times (n^* \cup \{\infty\}) \times l^*$ is in $\mathcal{F}(t, R)$ iff for all $x \in Var(t)$, $f(x) = (b_1, b_2, b_3, b_4, b_5)$ with:*

- *If $b_1 \in n^+$ and $b_2 \leq k_{b_1}$ then $r_{b_1 b_2} + 0.5 \in \mathcal{G}_t^2(x)$.*
- *If $b_1 = 0$ and $b_2 \leq k_0$ then $r_{b_1 b_2} \in \mathcal{G}_t^2(x)$.*
- *If $b_1 = \infty$ then $\max + 0.5 \in \mathcal{G}_t^2(x)$.*
- *If $x \in \Upsilon$ then $m_{b_1 b_2} \subseteq H_t(x)$.*
- *If $b_4 \in n^+ \cup \{\infty\}$ then $b_3 + 0.5 \in \mathcal{G}_t^1(x)$.*
- *If $b_4 = 0$ then $b_3 \in \mathcal{G}_t^1(x)$.*
- *If $b_4 = \infty$ iff $b_3 = max + 1$.*
- *If $y \neq x$ and $f(y) = (b'_1, b'_2, b'_3, b'_4, b'_5)$, then $(b_1, b_2) \neq (b'_1, b'_2)$.*

Intuitively, the first two numbers that the previous functions assign to a variable x, correspond to the selection of the part of the region we assign to x to remove the effects of H_t. Analogously, the two last components manage the effects of F_t. The third number assigns to each variable the natural number that correspond to the age of the instance in the predecessor.

Clearly, the family $\mathcal{F}(t, R)$ is finite. We define $\overline{Pre}_t(R)$ as the effects of computing the predecessors of R according to all the functions in $\mathcal{F}(t, R)$.

Definition 20 ($\overline{Pre}_t$). *Let $l = |Var(t)|$. Given $t \in T$, $f \in \mathcal{F}(t, R)$ and $R = A_0 * A_1 * \ldots * A_n * A_\infty$, with $A_i = \{(m_{i1}, r_{i1}), ..., (m_{ik}, r_{ik})\}$, we define $\overline{Pre}_{ft}(R)$ as follows:*

- *First, we define $R'' = A'_{00} * A'_{01} * \ldots * A'_{0l} * A'_{10} * A'_{11} * ... * A'_{nl} * A'_{\infty 0}$, where:*
  - *$A'_{j0} = A_j - \{(m_{jk}, r_{jk}) \mid \exists x$ with $f(x) = (j, k, b_3, b_4, b_5)$ for some $b_3, b_4, b_5\}$*
  - *$A'_{\infty 0} = A_\infty - \{(m_{\infty k}, max+1) \mid \exists x$ with $f(x) = (\infty, k, b_3, b_4, b_5)$ for some $b_3, b_4, b_5\}$*
  - *$A'_{ij} = \emptyset$ elsewhere.*
- *For each $x \in Var(t)$, if $f(x) = (b_1, b_2, b_3, b_4, b_5)$, then we define $m'_x = (m_{b_1 b_2} \ominus H_t(x)) + F_t(x)$ and $r'_x = b_3$, where $(m_1 \ominus m_2)(x) = \max(0, m_1(x) - m_2(x))$.*
- *Finally, $\overline{Pre}_{ft}(R)$ is the $\emptyset$-contraction of $B_{00} * B_{01} * \ldots * B_{0l} * B_{10} * B_{11} * \ldots * B_{nl} * B_{\infty 0}$, where for each $i \in n^*_\infty$ and $j \in l^*$, $B_{ij} = A'_{ij} + \{(m'_x, r'_x) \mid f(x) = (b_1, b_2, b_3, i, j)$ for some $b_1, b_2, b_3\}$.*

Then, we define $\overline{Pre}_t(R) = \{\overline{Pre}_{ft}(R) \mid f \in \mathcal{F}(t, R)\}$.

Intuitively, in R'' we have removed the instances corresponding to the effects of H_t, and added l empty multisets of instances between each A_i and A_{i+1} in order to be able to add tokens with new fractional parts as predecessors.

Lemma 17. *Given a region* R*, we can compute a finite set* $\overline{Pre}_t(R)$ *such that* $Pre_t(\uparrow R) = \uparrow\overline{Pre}_t(R)$

Proof. Clearly, $\overline{Pre}_t(R)$ as defined above is computable. First, we prove $\uparrow\overline{Pre}_t(R) \subseteq Pre_t(\uparrow R)$. Let $R' \in \uparrow\overline{Pre}_t(R)$, we are going to prove that $R' \in Pre_t(\uparrow R)$. Therefore, we need to prove that there is R''' with $R \sqsubseteq R'''$ such that $R' \xrightarrow{t} R'''$. Let us call $R = A_0 * A_1 * ... * A_{n_A} * A_\infty$ with $A_i = \{(m^A_{ij}, r^A_{ij}) \mid j \in |A_i|^+\}$ and $R' = E_0 * E_1 * ... * E_{n_E} * E_\infty$ with $E_i = \{(m^E_{ij}, r^E_{ij}) \mid j \in |E_i|^+\}$.

As $R' \in \uparrow\overline{Pre}_t(R)$, there is $R'' \in \overline{Pre}_t(R)$ such that $R'' \sqsubseteq R'$. Then, there is $f \in \mathcal{F}(t, R)$ such that $R'' = \overline{Pre}_{ft}(R)$. Suppose that $C_{00} * C_{01} * \ldots * C_{0l} * C_{10} * C_{11} * \ldots * C_{nl} * C_{\infty 0}$, is the $\emptyset$-expansion of R'' obtained in the definition of $\overline{Pre}_{ft}(R)$. For simplicity, we consider the firing from this $\emptyset$-expansion. Abusing notation, if $x \in Var(t)$ with $f(t, R) = (b_1, b_2, b_3, b_4, b_5)$, we call $(m^C_{b_4 b_5 x}, r^C_{b_4 b_5 x})$ to the pair $(m^A_{b_1 b_2} \ominus H_t(x)) + F_t(x), b_3)$ we add in $C_{b_4 b_5}$. We define the mode $\tau = (\tau_1, \tau_2, \tau_3)$ to fire t from R'' such that, for each $x \in Var(t)$, if $f(t, R) = (b_1, b_2, b_3, b_4, b_5)$:

- $\tau_1(x) = (b_4 b_5, x)$
- $\tau_2(x) = r_{b_4 b_5}$
- $\tau_3(x) = b_1 0$

Now, we prove that t is enabled at R''. Indeed, for each $x \in Var(t)$, if $f(t, R) = (b_1, b_2, b_3, b_4, b_5)$, $\tau_1(x) = (b_4 b_5, x)$, so:

- if $x \in \Upsilon$ then $m^A_{b_1 b_2} \leq H_t(x)$, so $m^C_{b_4 b_5 x} = (m^A_{b_1 b_2} \ominus H_t(x)) = \emptyset$
- $m^C_{b_4 b_5 x} = (m^A_{b_1 b_2} \ominus H_t(x)) + F_t(x)$, and therefore $m^C_{b_4 b_5 x} \geq F_t(x)$
- $r_{b_4 b_5 x} = b_3$. If $b_4 = 0$ or $b_4 = \infty$ then $b_3 \in \mathcal{G}^1_t(x)$. Otherwise, $b_3 + 0.5 \in \mathcal{G}^1_t(x)$, because of the definition of $\mathcal{F}$.

Therefore, t is enabled at R'', so it is enabled in R' too. Because of how we have defined τ, $R'' \overset{t}{\twoheadrightarrow} \bar{R} \sqsupseteq R$ with this mode. As $R'' \subseteq R'$, and the transition system is monotonic, there is R''' such that $R \sqsubseteq R'''$, with $R' \overset{t}{\twoheadrightarrow} R'''$.

Now, we prove that $Pre_t(\uparrow R) \subseteq \uparrow\overline{Pre}_t(R)$. Let $R = A_0 * A_1 * ... * A_n * A_\infty$, R' and R'' such that $R'' \overset{t}{\twoheadrightarrow} R' \sqsupseteq R$ with mode τ. It is enough to see that there exist $f \in \mathcal{F}(t, R')$ such that $R'' \in \Uparrow(\overline{Pre}_{ft}(R))$. In order to define f, we give some notations and renamings for these regions.

Suppose $R'' = F_0 * F_1 * \ldots * F_{\bar{n}} * F_\infty$. We denote by $\mathcal{S}$ the set $\{F_i \mid F_i - \{(m_{ij}, r_{ij}) \mid \tau_1(x) = (i, j)\} = \emptyset\}$ and $l = max\,|A_i|$ for $i \in \bar{n}^*$. Let us consider that we fire t from the $\emptyset$-expansion of R'' $F_{00} * F_{01}^* \ldots * F_{0l+1} * \ldots * F_{\bar{n}0} * \ldots * F_{\bar{n}l} * F_\infty$ such that $F_{i0} = F_i$ if $F_i \notin \mathcal{S}$, $F_{ij} = F_k$ if $F_k \in \mathcal{S}$ and F_k is the j^{th} set in $\mathcal{S}$ after F_i; and $F_{ij} = \emptyset$ otherwise. For simplicity, let us consider τ for this renaming (defining the renumbering of τ is trivial, but we do not do that for simplicity). Intuitively, we have renamed R'' in order to define b_4 and b_5 of $f(x)$, so that we have renamed the F_ks which do not "disappear" in the firing of t, have indices $j0$ for some j.

Consider the $\emptyset$-expansion of R' $E_0 * E_{01} * \ldots * E_n * E_{n1} * \ldots * E_\infty$ obtained when firing t. Since $R \sqsubseteq R'$, we can rewrite it as $E_{k_0} * u_0 * E_{k_1} * ... * E_{k_n} * u_n * E_\infty$, and $k_i \in n_\infty^* \cup n_\infty^* \times |Var(t)|^+$, where $A_i \subseteq E_{k_i}$ for $i \in n_\infty^*$. Without loss of generality, for each $i \in n_\infty^*$, $j \in |A_i|$, $m_{ij} \leq m^E_{k_i j}$ and $r_{ij} = rk - ij^E$.

With these notations, let us define f. Given $x \in Var(t)$, suppose that $\tau_1(x) = (i,j)$. Then, we define $f(x) = (b_1, b_2, b_3, b_4, b_5)$ with:

- Suppose that $\tau_3(x) = i'$, $(m^E_{i'j'}, r^E_{i'j'})$ in $E_{i'}$ is such that $m^E_{i'j'} = (m_{ij} - F_t(x)) + H_t(x)$ and $r^E_{ij} = \tau_2(x)$ is the pair we add in the firing of t, associated to x. Then $b_i = i'$ and $b_2 = j'$.
- $b_3 = r_{ij}$.
- If $\tau_1(x) = (i,j)$ with the considered renaming of the $\emptyset$-expansion of R'', then $b_4 = i$ and $b_5 = j$.

Finally, we prove that if $R_f = B_{00} * B_{01} * \ldots * B_{20} * \ldots * B_{\infty 0}$ is the $\emptyset$-expansion obtained when calculating $\overline{Pre}_{ft}(R)$, then $R_f \subseteq R''$. We see that each B_{ij} there is $F_{i'j'}$ with $B_{ij} \subseteq E_{i'j'}$. We consider the case in which $i = 0$, because the other one is easier.

First, we consider B_{i0}, for some i. For each $x \in Var(t)$, if $f(x) = (b_1, b_2, b_3, b_4, b_5)$, we call $m'_x = (m_{b_1 b_2} \ominus H_t(x)) + F_t(x)$ and $r'_x = b_3$. Then, $B_{i0} = A_i - \{(m_{ik}, r_{ik}) \mid \exists x$ with $f(x) = (i, k, b_3, b_4, b_5)$ for some $b_3, b_4, b_5\} + \{(m'_x, r'_x) \mid \exists x \in Var(t)$ with $f(x) = (b_1, b_2, b_3, i, 0)$ for some $b_1, b_2, b_3\} \subseteq E_i - \{(m_{ik}, t_{ik}) \mid \exists x \in Var(t)$ with $f(x) = (i, k, b_3, b_4, b_5)$ for some $b_3, b_4, b_5\} + \{(m'_x, r'_x) \mid \exists x$ with $f(x) = (b_1, b_2, b_3, i, 0)$ for some $b_1, b_2, b_3\}$, because $A_i \subseteq E_i$. Moreover, for each $x \in Var(t)$ if $b_i = i'$ and $b_2 = j'$, there are i', j' such that $\tau_3(x) = i'$, $(m^E_{i'j'}, r^E_{i'j'})$ in $E_{i'}$ is such that $m^E_{i'j'} = (m_{ij} - F_t(x)) + H_t(x)$ and $r^E_{ij} = \tau_2(x)$ is the pair we add in the firing of t, associated to x. Therefore, $\{(m_{ik}, r_{ik}) \mid \exists x \in Var(t)$ with $f(x) = (j, k, b_3, b_4, b_5)$ for some $b_3, b_4, b_5\} = \{(m_{ik}, r_{ik}) \mid \exists j'$ such that, if $\tau_3(x) = i'$, $(m^E_{i'j'}, r^E_{i'j'})$ in $E_{i'}$ is such that $m^E_{i'j'} = (m_{ik} - F_t(x)) + H_t(x)$ and $r^E_{ik} = \tau_2(x)$ is the pair we add in the firing of $t\}$. Analogously, $\{(m'_x, r'_x) \mid \exists x \in Var(t)$ with $f(x) = (b_1, b_2, b_3, i, 0)$ for some $b_1, b_2, b_3\} \subseteq \{(m'_x, r'_x) \mid \exists x \in Var(t)$ with $\tau_1(x) = (i, 0)\}$. Therefore, we obtain that there is i', j' with $B_{i0} \subseteq F_{i'j'}$, as we required.

Proposition 5. $\min(\uparrow Pre(\uparrow R))$ *is computable for any* R.

Proof. Indeed, by the two previous lemmas, we can compute it as $\min(\overline{Pre}_\Delta(R) \cup \bigcup_{t \in T} \overline{Pre}_t(R))$.

Then, we obtain the following result:

Corollary 2. *Control-state reachability is decidable for ν-lTPN.*

Proof. We have proved that the transition system over regions is a WSTS, so coverability is decidable. Since the control-state reachability problem can be reduced to coverability (Prop. 3) the thesis follows.