An axiomatic study of infinite basis

David de Frutos Escrig, Carlos Gregorio Rodríguez, and Miguel Palomino

Departamento de Sistemas Informáticos
Universidad Complutense de Madrid

Abstract. We present a new technique for studying the w-axiomatizability of the semantics in the ltbt-spectrum. Although Fokkink et al. have recently solved most of the problems still open, our main goal is to shed light on them, and to simply and unify their proofs. Besides, we will focus on preorders (and make a fundamental use of the axiom for ready simulation semantics) instead of equivalences, since they give rise to much simpler proofs.

1 Introduction

In a recent collection of papers [7, 8, 4, 2, 3], Fokkink, Aceto et al. have studied the ω-axiomatizability of the semantics in the ltbt-spectrum presented by Van Glabbeek [12]. In a survey on the subject [1], they presented several techniques to prove or disprove the existence of finite basis for an equational theory and applied them to the basic process algebra BCCSP under the different semantics in the ltbt-spectrum; the survey concluded with a collection of open problems, but most of them have been recently solved by members of their research group [2, 3].

We will concentrate in this paper on the collection [7, 4, 2, 3], where negative results for many of the semantics in the spectrum are presented; [8] was the only paper of the saga where a positive new result, corresponding to the failure semantics, was proved. Previously, Groote presented in [9] the first positive results for finite alphabets for the weakest semantics in the spectrum, traces and completed traces, while Moller [11] had proved that result for the semantics in the other side of the spectrum: bisimulation semantics.

We will mainly present a new technique to prove negative results about the existence of finite basis which is (almost) one hundred per cent proof-theoretic, thus avoiding rather complex case analysis like those appearing in the proofs in [7, 4, 2, 3]. Besides, our method is the same for all the semantics, making use of families of inequalities quite similar in all cases, whereas those in [7, 4, 2, 3] are rather different from each other and seem to be quite close to the peculiarities of each semantics. Hence, it is not our main goal to present new results but to shed light on them by unifying and simplifying their proofs.

The foundations of our technique is the study of the similarities among the different semantics in terms of their axiomatization. Although unrelated to our present subject, our work on bisimulations up-to and simulations up-to [5, 6], by means of which we can characterize all the semantics in the ltbt spectrum, has showed to us that simulation semantics play an important role. In particular, the ready simulation preorder, which corresponds to the axiom (RS) $ax \sqsubseteq ax + ay$, was fundamental in [6] and is also related to several of the arguments we will need in our proofs here.

Another key point is our use of the semantic preorders instead of directly working with the equivalences. Although we won't be able to dwell on their usefulness due to lack of space, let us at least mention that the preorders that appear in the literature usually have pleasant properties that will be exploited in our proofs. We have recently proved [6] that those properties do not arise by chance, but as a consequence of the characterization of these preorders as the canonical orders obtained from the corresponding equivalences by a simulation up-to procedure in which ready simulation again plays an important role.

We will present our results pedagogically, justifying all our steps, and looking for the simplest counterexamples, thus not only showing why most of the semantics are not finitely based but also why weaker ones, such as failure semantics, are. In this way we will also explain the differences between the semantics in terms of the corresponding axiomatizations so that the reader will (hopefully!) easily grasp the idea behind the general technique, so as to apply it in other scenarios.

1.1 Notation

We briefly present standard notation on the subject. Given a set of actions A, the basic process algebra BCCSP is defined by the following BNF-expression: $p ::= \mathbf{0} \mid ap \mid p + q$, where $a \in A$. $\mathbf{0}$ represents the term that performs no action; for every action in A, there is a prefix operator; and $+$ is a choice operator.

We denote the set of ground (closed) terms by $\mathrm{T}_{\mathrm{BCCSP}}(A)$. Given a set of variables V, the set of open terms $\mathbf{T}_{\mathrm{BCCSP}}(A)$ is defined as usual. In general, $A = \{b_1, \ldots, b_n\}$; $p, q, r \ldots$ range over $\mathrm{T}_{\mathrm{BCCSP}}(A)$; $t, u, v, x, y \ldots$ range over $\mathbf{T}_{\mathrm{BCCSP}}(A)$; $X, Y \ldots$ denote set of variables.

We use the function $\mathit{depth}(\_)$ to compute the depth of a process; the depth of a variable is 1. The function $I(\_)$ computes the set of initial actions of a term; a variable has no initial action. For the sake of clarity we also denote by $I(\_)$ the function that applies to sets of terms and computes the union of the set of initial actions. The function $\mathit{Vars}(\_)$ computes the set of variables of a term and, as before, we extend it to sets of terms. Finally, A^{-b_i} is $A \setminus \{b_i\}$ and we will sometimes use X to denote the process $x_1 + \ldots + x_n$, when $X = \{x_1, \ldots, x_n\}$ is a set of variables or actions.

2 Ready simulation: the simplest case study

Let us start by considering the ready simulation semantics, whose corresponding preorder $\preceq_{\mathrm{RS}}$ is that axiomatically defined by the axioms for bisimulation equivalence together with the axiom (RS).

Any simulation preorder satisfies the following separation property.

Proposition 1. *For every action a and $p, q, r \in \mathrm{T}_{\mathrm{BCCSP}}(A)$, if $ap \preceq_{\mathrm{RS}} q + r$ then $ap \preceq_{\mathrm{RS}} q$ or $ap \preceq_{\mathrm{RS}} r$.*

In order to prove that some preorder $\preceq$ is not finitely based we have to find, given a set E of inequations, a correct $t \preceq u$ which cannot be inferred from E. Let us study the simplest possible form of t. Obviously, it corresponds to the term ax, so that we will look for a nontrivial inequation $ax \preceq_{\mathrm{RS}} t$ *as difficult as possible* to derive. The idea to get such an inequation is to consider separately the cases $x = \mathbf{0}$ and $x \neq \mathbf{0}$. In the first case we immediately obtain $t_0 = a\mathbf{0}$, while for $x \neq \mathbf{0}$ we obtain $t' = \sum_{i=1}^{n} a(x + b_i\mathbf{0})$. This is formally justified by the following theorem.

Theorem 1. *For all actions a and terms t, $ax \preceq_{\mathrm{RS}} t$ iff $t = ax + t'$ or $t = a\mathbf{0} + t'$, where $t' = \sum_{i=1}^{n} a(x + b_i t_i) + t''$, with $I(t'') = \{a\}$.*

Proof. Let us assume that t does not contain a summand ax. Then, to have $a\mathbf{0} \preceq_{\mathrm{RS}} \sigma(t)$ when $\sigma(x) = \mathbf{0}$, the only possibility is to have a summand $a\mathbf{0}$ in t. Let us now consider the set of subsitutions $\sigma_{i,p}$ with $\sigma_{i,p}(x) = b_i p$; there must be a summand $t_{i,p}$ of t such that $\sigma_{i,p}(t_{i,p}) = ab_i r_{i,p}$ with $p \preceq_{\mathrm{RS}} r_{i,p}$. Since we can take p arbitrarily large and t has a finite number of summands, there must be a single one covering $\sigma_{i,p}(t_{i,p})$ for infinitely many p; it follows that it must be of the form $a(x + b_i t_i)$. Moreover, we have $a\sigma(x) \preceq_{\mathrm{RS}} a\mathbf{0} + \sum_{i=1}^{n} a(x + b_i t_i)$: it is immediate for $\sigma(x) = \mathbf{0}$; otherwise, we have some b_i such that $\sigma(x) = b_i q_i + q$ and then $b_i q_i + q \preceq_{\mathrm{RS}} b_i q_i + q + b_i p_i$. □

The reason why this inequation is a good candidate to generate counterexamples to disprove the existence of a finite basis for $\preceq_{\mathrm{RS}}$ is that it does not satisfy the separation property. In particular, it is crucial that $a\mathbf{0} \not\preceq_{\mathrm{RS}} \sum_{i=1}^{n} a(x + b_i\mathbf{0})$. But in order to contradict the existence of any finite axiomatization E, we need not a single inequation but a family parameterized on a value k including terms of arbitrary large sizes. Once again, we have a clear choice for the simplest lhs: we take as such the term $a^k x$. Then, the same argument as before leads us to the corresponding rhs $t = a^k\mathbf{0} + \sum_{i=1}^{n} a^k(x + b_i\mathbf{0})$. The theorem that characterizes the terms greater than $a^k x$ is a bit more complicated and needs the following definition.

Definition 1. *For $t = a^k v \in \mathbf{T}_{\mathrm{BCCSP}}(A)$, the set $GTriv_{\mathrm{RS}}^{k,a}(v)$ of trivial RS-greater elements than t is recursively defined as:*

- $GTriv_{\mathrm{RS}}^{1,a}(v) = \{av + w' \mid I(w') = \{a\}\}$
- $GTriv_{\mathrm{RS}}^{k+1,a}(v) = \{aw^k + w' \mid w^k \in GTriv_{\mathrm{RS}}^{k,a}(v), I(w') = \{a\}\}$

It is immediate to see that $GTriv_{\mathrm{RS}}^{k,a}(v)$ is the set of terms that can be obtained from $t = a^k v$ by a finite number of applications of (RS) at any of the k subterms $a^i v$ with $i = 1, \ldots, k$. Since we will also apply such a characterization under a substitution σ such that $\sigma(x) = \mathbf{0}$, we will need a generalization that allows additional summands x. Thus we define $GTriv_{\mathrm{RS},x}^{k,a}$ as $GTriv_{\mathrm{RS}}^{k,a}$, but allowing extra summands x in the last clause of the definition. For technical reasons, we define $GTriv^{k,a}_{\mathrm{RS},x^-}$ as the subset of $GTriv^{k,a}_{\mathrm{RS},x}$ without summands of the form x at the root.

Theorem 2. *For any $k \geq 1$, if $a^k x \preceq_{\mathrm{RS}} t$ then $t \in GTriv^{k,a}_{\mathrm{RS}}(x)$ or $t = t_0 + \sum_{i=1}^{n} t_i + t'$, with $t_0 \in GTriv^{k,a}_{\mathrm{RS},x^-}(\mathbf{0}), t_i \in GTriv^{k,a}_{\mathrm{RS}}(x + b_i p_i)$, $I(t') = \{a\}$.*

Proof. Since it is very similar to that of Theorem 1, we will limit ourselves to explaining that we need to introduce the sets $GTriv^{a,k}_{\mathrm{RS}}$ because once we have a subterm at' of a term t we can "enlarge" it by adding any summand at''. Although we will consider the rhs under any substitution, whenever $\sigma(x) \neq \mathbf{0}$ we can forget about the first summand thus making the x appearing in it irrelevant. This is why we have to consider $GTriv^{k,a}_{\mathrm{RS},x^-}$ instead of $GTriv^{k,a}_{\mathrm{RS},x}$, since an occurrence of x at the root of the term could lead to $I(\sigma(x)) \notin \{a\}$, invalidating the inequation. □

We also have the following simple proposition.

Proposition 2. *$x \preceq_{\mathrm{RS}} t$ iff $t = x + \ldots + x$.*

Proof. For a substitution σ with $\sigma(x) = \mathbf{0}$ and $\sigma(y) = a$ otherwise, we have $\mathbf{0} \preceq_{\mathrm{RS}} \sigma(t)$ and we conclude that $\sigma(t) = \mathbf{0}$ and thus $t = x + \ldots + x$. □

In this way we have obtained the family of inequations

$$a^k x \preceq_{\mathrm{RS}} a^k \mathbf{0} + \sum_{i=1}^{n} a^k (x + b_i \mathbf{0}) \,,$$

which is in fact the same used in [4]. But while on that paper we could find no explanation on how it was reached, here it has been obtained in a systematic way which will be used as the basis for the corresponding counterexamples in the following sections.

Theorem 3. *Let E be any finite set of correct inequations with respect to $\preceq_{\mathrm{RS}}$ and $k \geq depth(t)$ for all terms t in E; then:*

$$E \not\vdash a^k x \preceq a^k \mathbf{0} + \sum_{i=1}^{n} a^k (x + b_i \mathbf{0})$$

Proof. If we have a proof of $a^k x \preceq a^k \mathbf{0} + \sum_{i=1}^{n} a^k (x + b_i \mathbf{0})$ using a finite set of inequations E, then we have a sequence

$$a^k x = t^0 \preceq \ldots \preceq t^p = a^k \mathbf{0} + \sum_{i=1}^{n} a^k (x + b_i \mathbf{0})$$

where, at each step, a single axiom is applied to some subterm t'^q of the corresponding t^q. Let us consider the smallest index q such that $t^q \notin GTriv^{k,a}_{\mathrm{RS}}(x)$.

Then, $t^{q-1} \in GTriv^{k,a}_{\mathrm{RS}}(x)$ and the branch $a^k x$ is reduced by some inequation $e_l \preceq e_r$ such that $e_l \in GTriv^{l,a}_{\mathrm{RS}}(y)$ for some $l < k$. But then, taking k_1 as the depth at which it has been applied, $t^q \in GTriv^{k_1,a}_{\mathrm{RS}}(e_r[a^{k_2}x])$ with $k_1+l+k_2 = k$.

We cannot have $k_2 > 0$ because, by applying Theorem 2 to the subterm a^l of e_l, t^q has a completed trace $a^{k_1+l}\mathbf{0}$ which $a^k\mathbf{0} + \sum_{i=1}^n a^k(x + b_i\mathbf{0})$ has not. Then we have $k_1 > 0$ (recall $l < k$) and applying the definition of ready simulation it is easy to conclude that when $\sigma(x) \neq \mathbf{0}$ we cannot simulate an a-step of t^q because of its delayed non-determinism that we need to solve immediately in $a^k\mathbf{0} + \sum_{i=1}^n a^k(x + b_i\mathbf{0})$. □

3 Simulation semantics

Let us now consider the case of plain simulation semantics. Why didn't we start with it? Because the simple counterexample we used in the previous section does not work, since in this case there is a non trivial term t greater than ax.

Proposition 3. *For any action a and $t \in \mathbf{T}_{\mathrm{BCCSP}}(A)$, $ax \preceq_{\mathrm{S}} t$ if and only if $t \in STriv(ax)$, where $STriv(u) = \{v \mid \{x \leq x + y\} \vdash u \leq v\}$.*

Example 1. $STriv(\mathbf{0}) = \mathbf{T}_{\mathrm{BCCSP}}(A)$ and $STriv(x) = \{x + v\}$.

Equivalently, we have $v \in STriv(u)$ if u is a subtree of v with the same root, or u can be obtained from v by erasing some of its branches.

If we examine why ax cannot be improved in a nontrivial way, we realize that the difficulty lies in that whenever $\sigma(ax) \preceq_{\mathrm{S}} \sigma(t)$ for any σ such that $\sigma(x) \neq \mathbf{0}$, we have $t \in STriv(ax)$ and hence $a\mathbf{0} \preceq_{\mathrm{S}} t[x/\mathbf{0}]$ as well. This was not the case for ready simulation because then, whenever $b_i \in I(\sigma(x))$, we had $\sigma(ax) \preceq_{\mathrm{RS}} \sigma(a(x + b_i))$, but $a\mathbf{0} \not\preceq_{\mathrm{RS}} ab_i$.

Let us then try a "symmetric" approach and consider instead terms of the form $a(x + A)$. By a case analysis, it immediately follows that $a(x + A) \preceq_{\mathrm{S}} aA + \sum a(x + A^{-b_i})$, and this is indeed the inequation that will serve us to build the corresponding family of independent inequations, namely,

$$a^k(x + A) \preceq_{\mathrm{S}} a^k A + \sum a^k(x + A^{-b_i})\,.$$

Theorem 4. *For all actions a and $t \in \mathbf{T}_{\mathrm{BCCSP}}(A)$, if $a^k(x + A) \preceq_{\mathrm{S}} t$ then $t \in STriv(a^k(x + A))$ or $t = t_0 + \sum_{i=1}^n t_i$, $t_i \in STriv(a^k(x + A^{-b_i}))$.*

Proof. It is similar to that of Theorem 2. Let us only explain why the sets A^{-b_i} appear. Whenever $\sigma(x) \neq \mathbf{0}$ we have some b_i such that $\sigma(x) = b_i t + t'$. Then for ready simulation we had $\sigma(x) \preceq_{\mathrm{RS}} \sigma(x) + b_i p_i$. Instead $\sigma(x + A)$ is simulated by $\sigma(x) + A^{-b_i}$ because $\{b_i\} \cup A^{-b_i} = A$. □

The proof of the inexistence of a finite basis for the simulation semantics now follows the same steps as that for ready simulation.

Theorem 5. *Let E be a finite set of correct inequations with respect to $\preceq_{\mathrm{S}}$ and $k \geq depth(k)$ for all terms t in E; then*

$$E \not\vdash a^k(x + A) \preceq a^k A + \sum_{i=1}^{n} a^k(x + A^{-b_i})\,.$$

Proof. The proof follows similar steps to those of Theorem 4, but once we apply an axiom to reduce a subterm including a branch $a^l(x + A)$ we obtain t^q with the tree $a^l A + \sum a^l(x + A^{-b_i})$ under a branch a^{k_1}. In order to have $t^q \preceq_{\mathrm{S}} a^k A + \sum a^l(x + A^{-b_i})$ there can be no action $b \neq a$ in its first k levels. Eventually, we will have to reduce that tree and move its non-deterministic choice to the top, separating all of its branching as in $a^l A + \sum a^l(x + A^{-b_i})$. For that we will need an axiom $t(\bar{x}) \preceq t'(\bar{x})$ where $\bar{x}$ is a collection of variables and t is a term that only contains action a besides its variables. But it is easy to see that we must have $t' = t + t''$ which means that is not possible the required reduction. □

By comparing their respective lengths one can conclude that our new proof is simpler than that in [2], although its main virtue is that it has been obtained by following the same procedure used in the previous section. Instead, it seems that once the authors of [2] realized that the simple counterexample for ready simulation semantics did not work in this case, they decided to try with inequations where the lhs term of the previous case, $a(x + A)$, was replaced by using $\psi_n = \sum_{b_1 \ldots b_n \in A^n} b_1 \ldots b_n \mathbf{0}$ in place of A. Our method, which simply changes the prefix a to a^k seems to be more natural, as witnessed by the fewer technical difficulties of our proof.

4 Failure traces

The failure traces preorder requires two axioms besides those for bisimulation equivalence to be completely axiomatized: (RS) and

$$\text{(FT)} \qquad a(x + y) \preceq ax + ay\,.$$

Now, the family of inequations we used to prove that the ready simulation preorder was not finitely based does not do the trick. Given an inequation $ap \preceq aq + ar$, it follows that $aap \preceq a(aq + ar) \preceq a^2q + a^2r$ and then, by repeatedly applying axiom (FT), we can infer $a^k p \preceq a^k q + a^k r$.

We must then complicate a bit the rhs of our family of inequations so as to disable the possibility of using (FT) to reach them. The idea to achieve this in the simplest way is to decompose the prefix a^k in the original rhs so that we cannot generate new members of the family just by adding an a on top. The obvious subterm to which apply this decomposition is the first summand $a^k\mathbf{0}$; since it was introduced by considering a substitution such that $\sigma(x) = \mathbf{0}$, it is safe to extend it with an additional summand x. Thus, we have

$$a^k x \preceq_{\mathrm{RS}} a(x + a^{k-1}\mathbf{0}) + \sum_{i=1}^{n} a^k(x + b_i\mathbf{0})$$

and therefore

$$a^k x \preceq_{\mathrm{FT}} a(x + a^{k-1}\mathbf{0}) + \sum_{i=1}^{n} a^k(x + b_i\mathbf{0})\,.$$

We can then prove the following result, that simplifies considerably the rest of the proof on the nonexistence of a finite basis for the failure traces preorder.

Theorem 6. *For all actions a and $t \in \mathbf{T}_{\mathrm{BCCSP}}(A)$, $a^k x \preceq_{\mathrm{FT}} t$ if and only if $a^k x \preceq_{\mathrm{RS}} t$.*

Proof. Since the branch $a^k x$ generates a failure trace $A^{-a}aA^{-a}a \cdots A^{-a}a$ we must have the same trace in t, also terminating with x. This means that all along that trace the set of offered actions is $\{a\}$, which implies that $t \in GTriv^{k,a}_{\mathrm{RS}}(x)$, and therefore $a^k x \preceq_{\mathrm{RS}} t$. □

The cause underlying this result is the fact that for any linear term $a_1 \dots a_n\mathbf{0}$ it holds that $a_1 \dots a_n\mathbf{0} \preceq_{\mathrm{FT}} t \iff a_1 \dots a_n\mathbf{0} \preceq_{\mathrm{RS}} t$, which is justified at the axiomatic level by the fact that (FT) cannot be applied in a nontrivial way. (It could be applied trivially by making use of the idempotence axiom on any subterm, but then the resulting rhs coincides with that of directly applying the idempotence law, thus having no use.)

Theorem 7. *Let E be a finite set of correct inequations with respect to $\preceq_{\mathrm{FT}}$ and $k \geq depth(t)$ for every term in E; then*

$$E \nvdash a^k x \preceq a(x + a^{k-1}\mathbf{0}) + \sum_{i=1}^{n} a^k(x + b_i\mathbf{0})\,.$$

Proof. In order to reduce the term $a^k x$ we must apply some inequation e of the form $a^l y \leq t$, with $0 < l \leq k$. Then, by Theorem 6, after applying e we either obtain a term in $GTriv^{k,a}_{\mathrm{RS}}(x)$ or of the form $a^{k_1}(t_0 + \sum_{i=1}^{n} t_i)$, where $t_0 \in GTriv^{l,a}_{\mathrm{RS},x^-}(a_2^k\mathbf{0})$ and $k_1 + k_2 + l = k$. In the first case we are still left with a subterm $a^k x$, so eventually we will reach the second case. If $k_2 \neq 0$ we have now a completed trace $a^{k_1+l}\mathbf{0}$ which generates a failure trace not in the goal process. But if $k_1 > 0$ taking $\sigma(x) = b, b \neq a$, we cannot have any x summand in t_0 and then we ave a failure trace $A^{-a}aA^{-a}a \cdots A^{-a}a$ that the goal process has not. □

As a matter of fact, we have used in this case a collection of inequations similar to that in [3], although they introduced an additional variable that we have not needed and substituted $x+a^{k-1}x$ for $x+a^{k-1}\mathbf{0}$. Finally, they distinguish the first action a that appears in the trace of the process in the lhs of the inequation from the others, denoted with b. Actually, the name of the action in the i-th place is irrelevant since the only thing that matters is that the process is linear. Thus, in both cases $a_1 \dots a_k$ could be used instead of ab^{k-1} or $aa^{k-1} = a^k$.

5 Readiness semantics

Something that came as a surprise to us was that the most difficult case to settle following our approach was however the first solved by Fokkink et al. Indeed, after comparing the families of inequations they use in their proofs of the nonexistence of ω-axiomatizations, that for readiness semantics turns out to be the most complicated by far. We will justify this claim by proving, not only that the collections of inequations for the previous cases do no longer work, but also that no similar collection would. In fact, we will show that if the number of variables appearing in the inequations is bound, then the corresponding class can be finitely axiomatized. Therefore, in order to get a counterexample we need a familiy of inequations where the number of variables will be the increasing parameter. We will see that the family of inequations used in [7] is indeed the simplest choice to consider. Even though we will end up with the same counterexample used in [7], once again the use of our methodology to show that readiness semantics is not finitely ω-axiomatizable will produce a much shorter proof and, besides, provides us with additional interesting results that will be exploited in our final section on failure semantics.

The readiness preorder is axiomatized by the axioms for bisimulation equivalence together with (RS) and

$$\text{(R)} \qquad a(bx + by + u) \preceq a(bx + u) + a(by + v)\,.$$

This last axiom can alternatively be substituted by the axiom scheme

$$\text{(RI)} \qquad I(x) \supseteq I(y) \Rightarrow a(x + y) \preceq ax + a(y + z)\,,$$

which, in the case of a finite alphabet, gives rise to a finite family of axioms that contains the family of axioms (R).

In particular, we have $I(x) = I(y) \Rightarrow a(x + y) \preceq ax + ay$ and therefore we immediately conclude, as in Section 4, that the family we used for ready simulation semantics cannot be used for readiness semantics.

Let us now consider the general form of a basic inequation

$$a(X + \sum_i b_i B_i) \preceq \sum_j a(Y_j + \sum_{k_j} b_{k_j} C_{k_j})\,.$$

The main difference between readiness semantics and those that are stronger can be traced to the following axiom for readiness equivalence,

$$\text{(REQ)} \qquad a(bx + z) + a(by + v) = a(bx + by + z) + a(bx + by + v)\,,$$

which roughly says that the continuations after a trace have no memory, that is, they do not depend on the state after a choice nor on the states traversed along the computation. Obviously this is not true for ready traces or failure traces, where visited states have influence on the following steps of the traces.

The next proposition states that the variables appearing at different depths in the terms of an inequation $t \preceq_{\mathrm{R}} u$ can be assumed to be all different.

Proposition 4. *For all $t, u \in \mathbf{T}_{\mathrm{BCCSP}}(A)$, $t \preceq_{\mathrm{R}} u$ iff $t' \preceq_{\mathrm{R}} u'$, where t' and u' are obtained from t and u by renaming all variables that appear at depth 0.*

Proof. Let $t = X + \sum a_i t_i \preceq_{\mathrm{R}} u = Y + \sum b_j u_j$, such that $t' = X' + \sum a_i t_i \not\preceq_{\mathrm{R}} u' = Y' + \sum b_j u_j$. Then:

- $X \subseteq Y$, as can be seen by taking $\sigma(x) = a^k\mathbf{0}$ for k sufficiently large;
- $\{a_i\} = \{b_j\}$, as proved by taking $\sigma(x) = \mathbf{0}$ for all $x \in X$.

Now, there is some σ' and $(r, S) \in \mathit{Readies}(\sigma'(t'))$ such that $(r, S) \notin \mathit{Readies}(\sigma'(u'))$. Since $I(\sigma'(t')) = I(\sigma(t))$ when $\sigma(x) = \sigma'(x')$, it is clear that $r \neq \langle\rangle$ and, from $X' \subseteq Y'$ (since $X \subseteq Y$), it follows that $(r, S) = (a_i r', S)$, with $(r', S) \in \mathit{Readies}(\sigma(t_i))$ for some i, but $(r', S) \notin \mathit{Readies}(\sigma(u_j))$ for any $b_j = a_i$. But then $(r, S) \in \mathit{Readies}(\sigma'(t))$ too, so that we must have $(r, S) \in \mathit{Readies}(\sigma'(y))$ for some $y \in Y$; we distinguish two cases:

- $S \neq \emptyset$. We can remove (r, S) from $\mathit{Readies}(\sigma'(y))$ by changing $\sigma'(y)$ and turning it into σ'' so that $\sigma''(y)$ is the $|r|$-th approximation of $\sigma'(y)$, obtained from cutting all its traces at that depth. Thus, $(r, S) \in \mathit{Readies}(\sigma'(y'))$ becomes $(r, \emptyset) \in \mathit{Readies}(\sigma''(y))$. Instead, for any r' such that $|r'| = |r|$ and any set S' we have

  $$(r', S') \in \mathit{Readies}(\sigma'(a_i t_i)) \iff (r', S') \in \mathit{Readies}(\sigma''(a_i t_i)) \text{ and}$$
  $$(r', S') \in \mathit{Readies}(\sigma'(b_i u_i)) \iff (r', S') \in \mathit{Readies}(\sigma''(b_i u_i))$$

  so that we would have $(r, S) \in \mathit{Readies}(\sigma''(t)) \setminus \mathit{Readies}(\sigma''(u))$.
- $S = \emptyset$. The reasoning is analogous to the previous case, but now we obtain σ'' by expanding any complete trace r in $\sigma'(y)$ with an extra action, so that the ready set $(r, \emptyset)$ disappears from $\sigma''(y)$.

□

As a consequence of the proposition above, in order to study the set of valid inequalities $t \preceq_{\mathrm{R}} u$ we can limit ourselves to those whose terms have distinct variables at each level. But then the following factorization result applies.

Definition 2. *We say that a term $u = \sum_j a(Y_j + \sum_{i=1}^n \sum_{k \in K_{ji}} b_i u_{jik})$, where the sets K_{ji} can be empty, is in* normal form *whenever for all $i = 1, \ldots, n$, if $K_{j_1 i} \neq \emptyset$ and $K_{j_2 i} \neq \emptyset$ then $K_{j_1 i} = K_{j_2 i}$ and $u_{j_1 ik} = u_{j_2 ik}$ for all $k \in K_{j_1 i}$.*

Proposition 5. *Let $t = a(X + \sum_{i=1}^n \sum_{k \in K_i} b_i t_{ik})$, with K_i possibly empty, and $u = \sum_j a(Y_j + \sum_{i=1}^n \sum_{k \in K_{ji}} b_i u_{jik})$ a normal form term, be such that $(X \cup \bigcup_j Y_j) \cap (\mathit{Vars}(\bigcup_i t_i) \cup \bigcup_j \mathit{Vars}(\bigcup_k u_{jk})) = \emptyset$. Then, $t \preceq_{\mathrm{R}} u$ iff $a(X + \sum_{i=1}^n \sum_{k \in K_i} b_i\mathbf{0}) \preceq_{\mathrm{R}} \sum_j a(Y_j + \sum_{i=1}^n \sum_{k \in K_{ji}} b_i\mathbf{0})$ and $K_i \neq \emptyset$, $K_{ji} \neq \emptyset$ implies $\sum_{k \in K_i} b_i t_{ik} \preceq_{\mathrm{R}} \sum_{k \in K_{ji}} b_i u_{jik}$.*

Proof. ($\Rightarrow$). Since $\mathit{Readies}(a(\sigma(X) + \sum_{K_i \neq \emptyset} b_i\mathbf{0})) = \{(\langle\rangle, \{a\})\} \cup \{(a, I(\sigma(X)) \cup \{b_i\})\} \cup \{(a\alpha, R) | (\alpha, R) \in \mathit{Readies}(\sigma(X))\}$, we have that $\mathit{Readies}(\sigma(t))$ is

$$\mathit{Readies}(a(\sigma(X) + \sum_{K_i \neq \emptyset} b_i\mathbf{0})) \cup \{(ab_i\alpha, R) \mid (\alpha, R) \in \mathit{Readies}(t_i), K_i \neq \emptyset\}\,.$$

There is a similar decomposition for the ready pairs of u. Now, from $t \preceq_{\mathrm{R}} u$ it immediately follows that $X \subseteq \bigcup_j Y_j$ and $a(\sigma(X)+\sum_{K_i\neq\emptyset} b_i\mathbf{0}) \preceq_{\mathrm{R}} \sum_j a(\sigma(Y_j)+\sum_{K_{ji}\neq\emptyset} b_i\mathbf{0})$. To obtain $\sum_{k\in K_i} b_i t_{ik} \preceq_{\mathrm{R}} \sum_{k\in K_{ji}} b_i u_{jik}$ whenever $K_i \neq \emptyset$ and $K_{ji} \neq \emptyset$, let σ be a substitution such that $\sigma(x) = \mathbf{0}$ if $x \in X \cup \bigcup_j Y_j$ and is the identity otherwise. Applying σ to $t \preceq_{\mathrm{R}} u$ we get $t' = a(\sum_{i=1}^n \sum_{k\in K_i} b_i t_{ik}) \preceq_{\mathrm{R}} \sum_j a(\sum_{i=1}^n \sum_{k\in K_{ji}} b_i u_{jik}) = u'$: since $K_i \neq \emptyset$ and $K_{ji} \neq \emptyset$, $\mathit{Readies}(t_{ik}) = \{(\alpha, R) \mid (ab_i\alpha, R) \in \mathit{Readies}(t')\}$ and $\mathit{Readies}(u_{jik}) = \{(\alpha, R) \mid (ab_i\alpha, R) \in \mathit{Readies}(u')\}$, we conclude that $t_{ik} \preceq_{\mathrm{R}} u_{jik}$.

($\Leftarrow$). If we recover the continuations t_{ik} and u_{jik} at each of the places where we have introduced the null process, the new ready pairs created for the lhs are included among those for the rhs since $\sum_{k\in K_j} b_i t_{ik} \preceq_{\mathrm{R}} \sum_{k\in K_{ji}} b_i u_{jik}$. The only problem that could arise is that a removed occurrence of some ready pair $ac_{jk}\mathbf{0}$ turned out to be the only occurrence of that pair in the rhs, while still remaining in the lhs; but this cannot be the case because $X \subseteq \bigcup Y_j$. □

Theorem 8. *Let $\mathit{Var} = \{w_1, \ldots, w_n\}$ be a collection of n different variables and consider the finite set $\mathit{AxReady}_n = \{a(X + \sum_i b_i z_{b_i}) \preceq \sum_j a(Y_j + \sum_k c_{jk} z_{c_{jk}}) \mid b_i, c_{jk} \in A, X \cup \bigcup_j Y_j \cup \{z_b \mid b \in A\} \subseteq \mathit{Var}_n, a(X + \sum_i b_i z_{b_i}) \preceq_{\mathrm{R}} \sum_j a(Y_j + \sum_k c_{jk} z_{c_{jk}})\}$. Then, if $t \preceq_{\mathrm{R}} u$ with $\mathit{Vars}(t) \cup \mathit{Vars}(u) \subseteq \mathit{Var}_n$, it follows that $\mathit{AxReady}_n \vdash t \preceq u$.*

Proof. It is an immediate consequence of Proposition 5 □

Therefore, if we are to find a family of inequations to witness that readiness semantics is not finitely based we need to take inequations whose sets of variables are not bounded. Our decomposition theorem also suggests that it could be enough to consider the simplest inequations wherein t_i and u_{jk} are null; this is, indeed, the case studied in [7]. Hence we will consider the set of inequations $aX \preceq \sum_j aY_j$. We borrow from [7] the following definition of cover set, which will be justified later in the next theorem.

Definition 3. *Given an alphabet of actions A, we say that a set of sets of variables $\{Y_j \mid j \in J\}$ is a cover set of a set of variables X if*

- *for all $Z \subset X$ such that $|Z| < |A|$, there exists j such that $Z \subseteq Y_j \subseteq X$, and*
- *for all $Z \subset X$ such that $|Z| = |A|$, there exists j such that $Z \subseteq Y_j$.*

We denote it by $\{Y_j\} \rhd X$ and say that $aX \preceq \sum_j aY_j$ is a cover inequation.

Theorem 9. *For any nonempty set of variables X, $aX \preceq_{\mathrm{R}} \sum_j aY_j$ if and only if $\{Y_j\} \rhd X$.*

Proof. ($\Rightarrow$). Let us suppose that $\{Y_j\} \not\rhd X$; there are two possible cases:

- There exists $Z = \{x_1, \ldots, x_k\} \subseteq X$, $|Z| < A$, such that there is no Y_j with $Z \subseteq Y_j \subseteq X$. Let then $\sigma(x_i)$ be $a_i\mathbf{0}$ for $x_i \in Z$, $\sigma(x) = \mathbf{0}$ if $x \in X \setminus Z$, and $\sigma(x) = \sum_{i=1}^{|A|} a_i\mathbf{0}$ otherwise. It follows that $I(\sigma(X)) = \{a_1, \ldots, a_k\} \neq A$, but $I(\sigma(Y_j)) \in \{\emptyset, A\}$.

– There exists $Z = \{x_1, \ldots, x_{|A|}\} \subseteq X$, $|Z| = |A|$, such that there is no Y_i with $Z \subseteq Y_i$. Let $\sigma(x_i) = a_i\mathbf{0}$ and $\sigma(x) = \mathbf{0}$ if $x \notin Z$; obviously, $I(\sigma(X)) = A$ but $I(\sigma(Y_j)) \neq A$ for all j.

($\Leftarrow$). Since the first condition guarantees that $X \subseteq \bigcup_j Y_j$, we need only prove that for all σ there is a Y_j such that $I(\sigma(X)) = I(\sigma(Y_j))$.

– If $I(\sigma(X)) = \emptyset$, taking $x \in X$ and $\{x\} \subseteq Y_j \subseteq X$, we obtain $I(\sigma(X)) = \emptyset$.
– If $I(\sigma(X)) \neq \emptyset, A$, taking $I(\sigma(X)) = \{b_1, \ldots, b_k\} \not\subseteq A$ we have a family of variables $\{x_1, \ldots, x_k\} \subseteq X$, $b_i \in I(\sigma(x_i))$, and then taking $\{x_1, \ldots, x_k\} \subseteq Y_j \subseteq X$ we conclude $I(\sigma(Y_j)) = I(\sigma(X))$.
– If $I(\sigma(X)) = A$, we also have a family of variables $\{x_1, \ldots, x_{|A|}\} \subseteq X$ with $b_i \in I(\sigma(x_i))$. Since $|\{x_i\}| \leq |A|$, we have $\{x_i\} \subseteq Y_j$ for some Y_j and then $I(\sigma(Y_j)) = A$.

□

Let us briefly explain the main facts that make this proof work, which will justify the definition of cover set. They are related to the the following axiom, which is in fact satisfied by the ready simulation semantics and thus by all weaker semantics in the ltbt spectrum.

Proposition 6. *Whenever the alphabet of actions $A = \{b_1, \ldots, b_n\}$ is finite, we have*

$$\sum_{i=1}^{n} b_i t_i \preceq_{\mathrm{RS}} \sum_{i=1}^{n} b_i t_i + x\,.$$

Proof. Obvious, since $I(\sigma(x)) \subseteq A$ for every substitution σ. □

Then the main ideas that justify the form of cover sets is that whenever $Z \subseteq X$, with $|Z| < |A|$, we can associate a ifferent action from A to each member of Z and, by associating the full set of actions to any new variable not in X, we impose the existence of Y_j with $Z \subseteq Y_j \subseteq X$ in order to be able to generate the corresponding offering of actions at the term $\sum aY_j$. Instead, when $|Z| = |A|$, using that substitution we obtain $I(\sigma(X)) = A$ and then we can use Proposition 6 to justify the possible existence of new variables not in X in the corresponding Y_j.

Theorem 10. *Let E be any finite set of correct inequations with respect to $\preceq_{\mathrm{R}}$ and $k \geq |\mathit{Vars}(t)| \cdot (|A| + 2)$ for every term t in E. Then we have*

$$E \not\vdash aX \preceq aP + \sum_{i=1}^{|A|-1} aX^{-i} + \sum_{i=|A|}^{n} a(P \cup \{x_i, w_i\})\,,$$

where $X = \{x_1, \ldots, x_n\}$, $P = \{x_1, \ldots, x_{|A|-1}\}$, $X^{-i} = X \setminus \{x_i\}$.

Proof. We have $aX \preceq aP + \sum_{i=1}^{|A|-1} aX^{-i} + \sum_{i=|A|}^{n} a(P \cup \{x_i, w_i\})$ because $\{P\} \cup \{X^{-i} \mid i = 1, \ldots, |A| - 1\} \cup \{P \cup \{x_i, w_i\} \mid i = |A|, \ldots, n\}$ covers X.

It is easy to check that for all X' with $|X'| \geq |A|+2$, $\{P\} \cup \{X^{-i} \mid i = 1,\ldots,|A|-1\} \cup \{P \cup \{x_i, w_i\} \mid i = |A|,\ldots,n\}$ covers X' iff $P \not\subseteq X'$. Since $|X|$ is too large, in order to reduce aX we need to apply some axiom in E with a lhs of the form aZ. Each variable z_j would match a set of variables $X_j \subseteq X$. Then there must be some some z_0 such that $|X_0| \geq |A|+2$, and for each $i = 1,\ldots,|A|-1$ we have z_i such that $x_i \in X_i$. It is possible for several of these z_i to be the same, but in any case $|\{z_0,\ldots,z_{|A|-1}\}| \subseteq |A|$. Therefore, if the axiom applied is $aZ \preceq \sum aY_j$ then $\{Y_j\}$ covers Z so that there exists j such that $\{z_1,\ldots,z_{|A|}\} \subseteq Y_j$. Now, applying the substitution that gives us the value of the z_i in terms of the x_j, we have $|Y_j| \geq |A|+2$, $P \subseteq |Y_j|$. But, if $X \neq Y_j$, this way does not lead to $aP + \sum_{i=1}^{|A|-1} aX^{-i} + \sum_{i=|A|}^{n} a(P \cup \{x_i, w_i\})$ since $\{P\} \cup \{X^{-i} \mid i = 1,\ldots,|A|-1\} \cup \{P \cup \{x_i, w_i\} \mid i = |A|,\ldots,n\}$ cannot cover Y_j and hence $\sum aY_j \not\preceq_{\mathrm{R}} aP + \sum_{i=1}^{|A|-1} aX^{-i} + \sum_{i=|A|}^{n} a(P \cup \{x_i, w_i\})$. Thus, the only remaining possibility is a trivial reduction $aX \preceq aX + \sum au_i$, which keeps aX to be eventually reduced. □

6 Failure semantics

Failure semantics is the strongest one for which we have a finite basis, even in the case of a finite alphabet. Why? It is well known that a denotational model for this semantics is obtained from the model for readiness semantics by a convex and union closure of the sets of offers [10]. This is captured at the level of the corresponding preorder by means of the axiom

$$\text{(F)} \qquad a(x+y) \preceq a(y+z) + ax\,.$$

It is interesting to compare (F) with the conditional presentation of the axiom governing the readiness preorder. Due to their similarities, a large part of the results we obtained for readiness semantics can be transferred back to failure semantics. In particular, this is true for the factorization result in Proposition 5 that can be now rewritten as follows:

Proposition 7. *Let $t = a(X + \sum_{i=1}^{n} \sum_{k \in K_i} b_i t_{ik})$, with K_i possibly empty, and $u = \sum_j a(Y_j + \sum_{i=1}^{n} \sum_{k \in K_{ji}} b_i u_{jik})$ a normal form term, be such that $b_i = b_{i'}$ implies $t_i = t_{i'}$ and $(X \cup \bigcup_j Y_j) \cap (\mathit{Vars}(\bigcup_i t_i) \cup \bigcup_j \mathit{Vars}(\bigcup_k u_{jk})) = \emptyset$. Then, $t \preceq_{\mathrm{F}} u$ iff $a(X + \sum_{i=1}^{n} \sum_{k \in K_i} b_i \mathbf{0}) \preceq_{\mathrm{F}} \sum_j a(Y_j + \sum_{i=1}^{n} \sum_{k \in K_{ji}} b_i \mathbf{0})$ and $K_i \neq \emptyset$, $K_{ji} \neq \emptyset$ implies $\sum_{k \in K_i} b_i t_{ik} \preceq_{\mathrm{F}} \sum_{k \in K_{ji}} b_i u_{jik}$.*

As a consequence, our result on the finite axiomatizability of the part of the theory with a bounded set of variables in Theorem 8 is also valid for failure semantics. The role of (F) in this scenario is simply to extend the set of correct inequalities $a(X + \sum b_i \mathbf{0}) \preceq_{\mathrm{F}} \sum a(Y_j + \sum c_{jk} \mathbf{0})$ and thus the size of the finite axiomatization for each subset of the theory with a bounded set of variables. What about the use of coverings to characterize the set of valid inequations $aX \preceq \sum aY_j$? They can be easily adapted by means of the following definition:

Definition 4. *$\{Y_j\}$ is a lower cover of X iff $X \subseteq \bigcup Y_j$ and there exists $Y_j \subseteq X$.*

Proposition 8. *For all actions a, $aX \preceq_{\mathrm{F}} \sum aY_j$ iff $\{Y_j\}$ is a lower cover of X.*

This characterization can be extended to terms of depth 2. If $|A| = \infty$ we have:

Proposition 9. *$a(X + \sum_B b_i\mathbf{0}) \preceq_{\mathrm{F}} \sum_j a(Y_j + \sum_{c_j} c_{jk}\mathbf{0})$ iff $X \subseteq \bigcup Y_j$, $B \subseteq \bigcup C_j$ and there exists j such that $Y_j \subseteq X$ and $C_j \subseteq B$.*

It is immediate to prove that all these inequations can be derived using (F).

The case for finite alphabets is slightly more complicated because, as we proved in Proposition 6, now we have

$$(\mathrm{F}_n) \qquad \sum_{i=1}^{n} b_i t_i \preceq_{\mathrm{F}} \sum_{i=1}^{n} b_i t_i + x\,.$$

Our new axiom for failure preorder with a finite alphabet can be compared to axiom (F_{3n}) in [8]. It is clear that it is a consequence of ours, which is stronger, simpler, and easier to understand and justify.

Proposition 10. *If $A = \{b_1, \ldots, b_n\}$ we have $a(X + \sum_B b_i\mathbf{0}) \preceq_{\mathrm{F}} \sum_j a(Y_j + \sum_{c_j} c_j k\mathbf{0})$ iff $X \subseteq \bigcup Y_j$, $B \subseteq \bigcup C_j$ and there exists j such that if $B \neq A$ then $C_j \subseteq B$ and $Y_j \subseteq X$.*

Proof. The only difficult part consists in showing that whenever $B = A$ only the first two conditions are needed. Applying (F) we can obtain the union closure:

$$\sum_j a(Y_j + \sum_{c_j} c_{jk}\mathbf{0}) \equiv_{\mathrm{F}} \sum_j a(Y_j + \sum_{c_j} c_{jk}\mathbf{0}) + a(\bigcup_j Y_j + \sum_j \sum_{c_j} c_{jk}\mathbf{0})$$

But then, since $B = A = \bigcup_j C_j$ and $X \subseteq \bigcup_j Y_j$, only ($\mathrm{F}_n$) is needed to get $aX \preceq a(\bigcup_j Y_j + \sum_j \sum_{c_j} c_{jk}\mathbf{0})$. □

By combining our factorization Proposition 7 with the result above we conclude that failure preorder has a finite basis. Certainly, the proof of the factorization theorem that we developed for the readiness semantics is a bit involved, but we have benefited from it when studying two different semantics. This has been possible after taking into account the relationships between the different semantics that are clearly reflected in the corresponding axiomatizations. Besides, we have mainly used the preorders instead of the equivalences because they are simpler and therefore easier to manipulate and reason about, especially when using the axiomatic approach, because the axioms for the preorders reflect in a clearer way the information captured by the semantics.

Each of the results about nonexistence of a finite basis for the preorders can be immediately transferred to the corresponding equivalences. For all the families of inequations $P \preceq_\theta Q$ that we used as counterexamples in the proofs of those results, it is easy to see that we could prove the same result for the families

$P + Q \preceq_\theta Q$. This is a consequence of the fact that all these terms P and Q are prefixed by the same action a, so that

$$E \vdash P + Q \preceq Q \iff E \vdash P \preceq Q\,.$$

But we also have, for all the semantics studied, $P + Q \equiv_\theta Q$. Now, if we had a finite axiomatization E for the equational theory of $\equiv_\theta$, then we could turn E into a finite set E_2 of sound equations for $\preceq_\theta$ by simply taking each axiom of E and its symmetric form. In the inequational calculus there is no symmetric rule, but the theory E_2 would be symmetric and thus it is obvious that if $E_2 \vdash P \preceq Q$, then $E_2 \vdash Q \preceq P$ by reversing the order of inequations throughout the proof, thus showing $E \vdash P \equiv Q$ iff $E_2 \vdash P \preceq Q$. It follows that no equational theory $\equiv_\theta$ has a finite basis whenever $\preceq_\theta$ does not have it.

References

1. Luca Aceto, Wan Fokkink, Anna Ingólfsdóttir, and Bas Luttik. Finite equational bases in process algebra: Results and open questions. In Aart Middeldorp, Vincent van Oostrom, Femke van Raamsdonk, and Roel de Vrijer, editors, *Processes, Terms and Cycles: Steps on the Road to Infinity*, LNCS 3838, pages 338–367. Springer-Verlag, 2005.
2. Taolue Chen and Wan Fokkink. On finite alphabets and infinite bases III: Simulation. In *CONCUR'06*, Lecture Notes in Computer Science. Springer-Verlag, 2006. To appear.
3. Taolue Chen, Wan Fokkink, and Bas Luttik. On finite alphabets and infinite bases IV: Failure traces. Unpublished manuscript.
4. Taolue Chen, Wan Fokkink, and Sumit Nain. On finite alphabets and infinite bases II: Completed and ready simulation. In Luca Aceto and Anna Ingólfsdóttir, editors, *FOSSACS 2006*, LNCS 3921, pages 1–15. Springer-Verlag, 2006.
5. David de Frutos Escrig and Carlos Gregorio-Rodríguez. Bisimulation up-to for the linear time-branching time spectrum. In Martín Abadi and Luca de Alfaro, editors, *CONCUR 2005*, LNCS 3653, pages 278–292. Springer-Verlag, 2005.
6. David de Frutos Escrig and Carlos Gregorio-Rodríguez. Axiomatization of Canonical Preorders via Simulations Up-to. Unpublished Manuscript, 2006.
7. Wan Fokkink and Sumit Nain. On finite alphabets and infinite bases: from ready pairs to possible worlds. In Igor Walukiewicz, editor, *FOSSACS 2004*, pages 182–194. Springer-Verlag, 2004.
8. Wan Fokkink and Sumit Nain. A finite basis for failure semantics. In Luís Caires, Giuseppe F. Italiano, Luís Monteiro, Catuscia Palamidessi, and Moti Yung, editors, *ICALP 2005*, LNCS 3580, pages 755–765. Springer-Verlag, 2005.
9. Jan Friso Groote. A new strategy for proving ω-completeness with applications in process algebra. In J. C. M. Baeten and J. W. Klop, editors, *CONCUR'90*, LNCS 458, pages 314–331. Springer-Verlag, 1990.
10. Matthew Hennessy. *Algebraic theory of processes.* MIT Press, 1988.
11. Faron Moller. *Axioms for concurrency.* PhD thesis, University of Edinburgh, 1989.
12. Rob J. van Glabbeek. The linear time-branching time spectrum I: The semantics of concrete, sequential processes. In J. A. Bergstra, A. Ponse, and S. A. Smolka, editors, *Handbook of process algebra*, pages 3–99. North-Holland, 2001.