Parameterized Metareasoning in Membership Equational Logic*

Manuel Clavel, Narciso Martí-Oliet, and Miguel Palomino

Departamento de Sistemas Informáticos y Computación
Universidad Complutense de Madrid, Spain
{clavel, narciso, miguelpt}@sip.ucm.es

Abstract. Basin, Clavel, and Meseguer showed in [1] that membership equational logic is a good metalogical framework because of its initial models and support of reflective reasoning. A development and an application of those ideas was presented later in [4]. Here we further extend the metalogical reasoning principles proposed there to consider classes of parameterized theories and apply this reflective methodology to the proof of different parameterized versions of the deduction theorem for minimal logic of implication.

1 Motivation

A *reflective* logic is a logic in which important aspects of its metalogic can be represented at the object level in a consistent way, so that the object-level representations correctly simulate the relevant metalogical aspects. As a consequence, in a reflective logic, metatheorems involving families of theories can be represented and logically proved as theorems about its universal theory. Basin, Clavel, and Meseguer showed in [1] that logical frameworks can be good metalogical frameworks when their theories always have initial models and they support reflective and parameterized reasoning; they also showed that membership equational logic is a particular logical framework that satisfies these requirements. In this paper, we extend their ideas and apply them to the (parameterized) deduction theorem.

Basin and Matthews have shown in [2] how metatheories based on inductive definitions can be used to formalize metatheorems that are *parameterized* with their scope of application. As a case study, they formalize different parameterized versions of the deduction theorem in the theory FS_0 [8]; we will use the same case study to motivate the developments of the following sections.

We can use membership equational logic (described in more detail in Section 2) to represent theoremhood in a logic as a sort in a theory. Conditional membership axioms then directly support the representation of rules as schemas, which is typically used in presenting logics and formal systems. Similarly, we can

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represent theoremhood in a parameterized family of logics as a sort in a parameterized theory. A sort in a parameterized membership equational theory can be used to represent theoremhood in a family of logics if and only if there is a correspondence between logics in the family and instances of the parameterized theory. Moreover, this correspondence has to be such that theoremhood in a logic in the family can be represented as membership in this sort in the corresponding instance of the parameterized theory.

We shall now illustrate the above idea using minimal logic (of implication) as a running example. Representing minimal logic in membership equational logic entails defining a theory T that conservatively represents minimal logic's theoremhood. The formulae of minimal logic correspond to members of the set built from the binary connective $\rightarrow$ (written infix, associating to the right) and sentential constants. Theorems correspond to members of a second set, and are either instances of the standard Hilbert axiom schemas K,

$$A \rightarrow B \rightarrow A,$$

```
fth MINIMAL is
kind Symbol[].
kind Expression[SentConstant Formula Theorem].
****************************************************** kinds
*** *** Symbol
op <ASCII-identifiers> : -> Symbol .
*** *** Expression
op <integer> : -> Expression .
op [_,_,_] : Symbol Expression Expression -> Expression .

vars A B C : Expression .
****************************************************** sorts
*** *** SentConstant
mb <integers> : SentConstant .
*** *** Formula
cmb A : Formula if A : SentConstant .
cmb [->, A, B]: Formula if A : Formula /\ B : Formula .
*** *** *** Theorem
cmb [->, A, [->, B, A]] : Theorem
    if A : Formula /\ B : Formula .
cmb [->, [->, A, B], [->, [->, [A, [->, B, C]]], [->, A, C]]] : Theorem
    if A : Formula /\ B : Formula /\ C : Formula .
cmb B : Theorem
    if A : Formula /\ B : Formula
    /\ A : Theorem /\ [->, A, B]: Theorem .
endfth
```

Fig. 1. The theory `MINIMAL`.

and S,

$$(A \to B) \to (A \to B \to C) \to (A \to C),$$

or are generated by applying the *modus ponens* rule,

$$\frac{A \quad A \to B}{B}.$$

Then, the deduction theorem for minimal logic is a metatheorem that states that

$$\text{if } \vdash_A B \text{ then } \vdash A \to B,$$

where $\vdash$ denotes that a formula can be deduced in minimal logic from the rules above and $\vdash_A$ is provability when A is considered to be an additional axiom. Since A is arbitrary, this result is a statement about a family of logics (or theories); actually, the result is also parametric in another sentence since it holds for extensions of minimal logic with additional connectives, like the standard conjunction.

The theory `MINIMAL`—in short, `ML`—in Figure 1 represents minimal logic in membership equational logic using the above idea. The lines starting with `kind` declare the kinds and their associated sorts; for the time being, kinds can be safely ignored. The sort `Formula` represents the well-formed formulae in minimal logic, in the sense that any formula in minimal logic can be represented as a term of this sort and vice versa. For example, if A, B are sentential constants represented respectively by `1` and `2`, then $(A \to B)$ is represented by the term `[->, 1, 2]` of sort `Formula`. Similarly, the sort `Theorem` represents the theorems in minimal logic, so that any theorem in minimal logic can be represented as a term of this sort, and vice versa.

Consider now the task of representing not just minimal logic, but the family of logics that includes any extension of minimal logic with respect to its language—connectives and syntactic rules—and proof system—axioms and inference rules. A solution to this is given by the parameter theory `EXTENDED-MINIMAL`—in short, `EML`—in Figure 2. The parametric sort `@NewSynRule` allows us to capture the extensions of minimal logic's language with new binary connectives. For example, the extension of minimal logic's language with the $\wedge$-operator corresponds to the instantiation of `EML` with the following membership axiom Ax(`@NewSynRule`) associated to `@NewSynRule`:

```
mb [[/\, A, B], A, B]: @NewSynRule .
```

Similarly, the parametric sorts `@NewAxiom` and `@NewInfRule` allow us to capture the extensions of minimal logic's proof system with new axioms and/or new inference rules of two premises. For example, the extension of minimal logic's proof system with the axiom schemas for the binary connective $\wedge$ corresponds to the instantiations of `EML` with the following membership axioms associated to `@NewAxiom`:

```
mb [->, A, [->, B, [/\, A, B]]]: @NewAxiom .
mb [->, [/\, A, B], A]: @NewAxiom .
mb [->, [/\, A, B], B]: @NewAxiom .
```

```
fth EXTENDED-MINIMAL is
including MINIMAL .
kind Expression[@NewAxiom] .
kind Rule[@NewSynRule @NewInfRule] .
****************************************************** kinds
*** *** Rule
op [_,_,_] : Expression Expression Expression -> Rule .

vars A B C : Expression .
****************************************************** sorts
*** *** Formula
cmb A : Formula
    if [A, B, C] : @NewSynRule
    /\ B : Formula /\ C : Formula .
*** *** Theorem
cmb A : Theorem if A : @NewAxiom /\ A : Formula .
cmb A : Theorem
    if [A, B, C] : @NewInfRule
    /\ A : Formula /\ B : Formula /\ C : Formula
    /\ B : Theorem /\ C : Theorem .
****************************************************** parameters
op @A : -> Expression .
mb @A : Formula .
endfth
```

Fig. 2. The theory `EXTENDED-MINIMAL`.

```
fth EXTENDED-MINIMAL-DT[EXTENDED-MINIMAL] is
including EXTENDED-MINIMAL .
mb @A : Theorem .
endfth
```

Fig. 3. The theory `EXTENDED-MINIMAL-DT[EXTENDED-MINIMAL]`.

Now, let `@A` be the parametric constant that appears (as a subscript of $\vdash$) in the deduction theorem. The parameterized theory in Figure 3—in short, `DT[EML]`—can be used to represent any extension of minimal logic with respect to its language and proof system.

With this example in mind, our objectives in this paper move at two different levels. First, we want to design a metareasoning principle over parameterized theories in membership equational logic; a concrete application of this principle would be a proof of the fact that the deduction theorem holds for every possible instantiation of `DT[EML]`. Secondly, and foremost, we intend to reify both parameterized theories and the metareasoning principle in the universal theory U_{MEL} of membership equational logic [6]; that is, our goal is to define representation functions to reify parameterized theories as terms in U_{MEL} and the metareasoning principle as a formula over U_{MEL}. As a concrete application, we will show that the parameterized deduction theorem can be proved by showing that a certain formula holds in U_{MEL}.

2 Membership Equational Logic

Membership equational logic is an expressive version of equational logic. A full account of the syntax and semantics of membership equational logic can be found in [3, 10]. Here we define the basic notions needed in this paper.

A *signature* in membership equational logic is a triple $\Omega = (K, \Sigma, S)$ with K a set of *kinds*, Σ a K-kinded signature $\Sigma = \{\Sigma_{k_1 \ldots k_n, k}\}_{(k_1 \ldots k_n, k) \in K^* \times K}$, and $S = \{S_k\}_{k \in K}$ a pairwise disjoint K-kinded family of sets. We call S_k the set of *sorts* of kind k and write $[s]$ for the kind of a sort s. The pair (K, Σ) is what is usually called a many-sorted signature of function symbols; however we call the elements of K *kinds* because each kind k now has a set S_k of associated *sorts*, which in the models will be interpreted as subsets of the carrier for the kind.

The atomic formulae of membership equational logic are *equations* $t = t'$, where t and t' are Σ-terms of the same kind, and *membership assertions* of the form $t : s$, where the term t has kind k and $s \in S_k$. Sentences are Horn clauses on these atomic formulae, i.e., sentences of the form

$$\forall (x_1, \ldots, x_m).\, A_0 \textbf{ if } A_1 \wedge \ldots \wedge A_n$$

where each A_i is either an equation or a membership assertion, and each x_j is a K-kinded variable. A theory in membership equational logic is a pair (Ω, E), where E is a finite set of sentences in membership equational logic over the signature Ω. We write $(\Omega, E) \vdash \phi$ to denote that (Ω, E) entails the sentence ϕ.

We employ standard semantics concepts from many-sorted logic. Given a signature $\Omega = (K, \Sigma, S)$, an *Ω-algebra* A is a many-kinded Σ-algebra (that is, a K-indexed-set $A = \{A_k\}_{k \in K}$ together with a collection of appropriately kinded functions interpreting the operators in Σ) and an assignment that associates to each sort $s \in S_k$ a subset $A_s \subseteq A_k$. As usual, we denote by T_Ω the K-kinded algebra of ground (K, Σ)-terms, and by $T_\Omega(X)$ the algebra of (K, Σ)-terms on the K-kinded set of variables X. An algebra A and a valuation σ, assigning to variables of kind k values in A_k, satisfy an equation $(\forall X)\, t = t'$ iff $\sigma(t) = \sigma(t')$, where we overload notation by identifying σ with its unique homomorphic extension to terms. We write $A, \sigma \models (\forall X)\, t = t'$ to denote such a satisfaction. Similarly, $A, \sigma \models (\forall X)\, t : s$ holds iff $\sigma(t) \in A_s$.

Note that an Ω-algebra is a K-kinded first-order model with function symbols Σ and a kinded alphabet of unary predicates $\{S_k\}_{k \in K}$. We can then extend the satisfaction relation to Horn and first-order formulae ϕ over the atomic formulae in the standard way. We write $A \models \phi$ when the formula ϕ is satisfied for all valuations σ, and then say that A is a model of ϕ. As usual, we write $(\Omega, E) \models \phi$ when all the models of the set E of sentences are also models of ϕ.

Theories in membership equational logic have initial models [10]. This provides the basis for reasoning by induction. In the initial model of a membership equational theory, sorts are interpreted as the smallest sets satisfying the axioms in the theory, and equality is interpreted as the smallest congruence satisfying those axioms. Given a theory (Ω, E), we denote its initial model by $T_{\Omega/E}$. In particular, when $E = \emptyset$ we obtain the term algebra T_Ω. We write $(\Omega, E) \models \phi$ to denote that the initial model of the membership equational theory (Ω, E) is also a model of ϕ, that is, that the satisfaction relation $T_{\Omega/E} \models \phi$ holds.

2.1 Reflection in Membership Equational Logic

A reflective logic is a logic in which important aspects of its metalogic can be represented at the object level in a consistent way, so that the object-level representation correctly simulates the relevant metalogical aspects. More concretely, a logic is reflective when there exists a *universal* theory in which we can represent and reason about all finitely presentable theories in the logic, including the universal theory itself [5]. As a consequence, in a reflective logic, metatheorems involving families of theories can be represented and proved as theorems about its universal theory [1]. A universal theory U_{MEL} for membership equational logic is described in [6], along with a representation function $(\overline{\_ \vdash \_})$ that encodes pairs, consisting of a finitely presentable membership equational theory with nonempty kinds and a sentence in it, as sentences in U_{MEL}. The signature of U_{MEL} contains constructors to represent operators, variables, terms, kinds, sorts, signatures, axioms, and theories. In particular, the signature of U_{MEL} includes the kinds `[Op]`, `[Var]`, `[Term]`, `[TermList]`, `[Kind]`, `[Sort]`, and `[Theory]` for terms representing, respectively, operators, variables, terms, lists of terms, kinds, sorts, and theories. In addition, it contains three Boolean operators[1]

```
op _::_in_ : [Term] [Kind] [Theory] -> [Bool] .
op _:_in_ : [Term] [Sort] [Theory] -> [Bool] .
op _=_in_ : [Term] [Term] [Theory] -> [Bool] .
```

to represent, respectively, that a term is a ground term of a given kind in a membership equational theory, and that a membership assertion or an equation holds in a membership equational theory.

The representation function $(\overline{\_ \vdash \_})$ is defined in [6] as follows: for all finitely presentable membership equational theories with nonempty kinds R, and atomic formulae ϕ over the signature of R,

$$\overline{R \vdash \phi} \triangleq \begin{cases} (\overline{t} : \overline{s} \texttt{ in } \overline{R}) = \texttt{true} & \text{if } \phi = (t : s) \\ (\overline{t} \texttt{ = } \overline{t'} \texttt{ in } \overline{R}) = \texttt{true} & \text{if } \phi = (t = t'), \end{cases}$$

where $\overline{(\_)}$ is a representation function defined recursively over theories, signatures, axioms, and so on. In particular, to represent terms the signature of U_{MEL} contains the constructors

[1] The operator declarations have been changed slightly from those in [6] to better match their use in this work.

```
op _[_] : [Op] [TermList] -> [Term] .
op nil : -> [TermList] .
op _,_ : [TermList] [TermList] -> [TermList] .
```

and the representation function $\overline{(\_)}$ is defined as follows:

$$\bar{t} \triangleq \begin{cases} \overline{c} & \text{if } t = c \text{ is a constant} \\ \overline{x} & \text{if } t = x \text{ is a variable} \\ \overline{f}\,\texttt{[}\overline{t_1}\texttt{,}\ \ldots\ \texttt{,}\overline{t_n}\texttt{]} & \text{if } t = f(t_1, \ldots, t_n). \end{cases} \tag{1}$$

For example, the term $s(0)$ of kind *Num* is represented in U_{MEL} as the term $\overline{s}\texttt{[}\overline{0}\texttt{]}$ of kind `[Term]`. Constants, operators, variables, kinds, and sorts are represented using strings of ASCII characters preceded by a quote. For example, $s(0)$ can be represented in U_{MEL} as the term `'s['0]`. It is convenient to represent variables along with their kinds using a binary constructor

```
op _|_ : [Var] [Kind] -> [Term] .
```

For example, $s(N)$ is represented in U_{MEL} as the term `'s['N|'Num]`.

The following results state the main properties of U_{MEL} as a universal theory and are proved in [6]. We assume a finitely presentable membership equational theory $R = (\Omega, E)$ with nonempty kinds, and with $\Omega = (K, \Sigma, S)$.

Proposition 1. *For all terms t in T_Ω, and kinds k in K,*

$$t \in (T_\Omega)_k \iff U_{\mathrm{MEL}} \vdash (\bar{t} :: \overline{k}\ \texttt{in}\ \overline{R}) = \texttt{true}.$$

Furthermore, for all ground terms u of kind `[Term]`*, if*

$$U_{\mathrm{MEL}} \vdash (u :: \overline{k}\ \texttt{in}\ \overline{R}) = \texttt{true}\,,$$

then there is a term $t \in (T_\Omega)_k$ such that $\bar{t} = u$.

Proposition 2. *For all terms t, t' in $(T_\Omega)_k$ and sorts s in S_k,*

$$\begin{aligned} R \vdash t : s &\iff U_{\mathrm{MEL}} \vdash (\bar{t} : \overline{s}\ \texttt{in}\ \overline{R}) = \texttt{true} \\ R \vdash t = t' &\iff U_{\mathrm{MEL}} \vdash (\bar{t}\ \texttt{=}\ \overline{t'}\ \texttt{in}\ \overline{R}) = \texttt{true}. \end{aligned}$$

Note that this proposition says that there exists a *logical* proof of $t : s$ (resp. of $t = t'$) in a membership equational theory R if and only if there exists also a *logical* proof of $(\bar{t} : \overline{s}\ \texttt{in}\ \overline{R}) = \texttt{true}$ (resp. of $(\bar{t}\ \texttt{=}\ \overline{t'}\ \texttt{in}\ \overline{R}) = \texttt{true}$) in the universal membership equational theory U_{MEL}.

Finally, not only can the theory U_{MEL} represent and reason about the entailment relation of any other theory but also about their own structure. In particular, we can define an operator

```
op _spec_in_ : [AxiomSet] [Sort] [Signature] -> [Bool] .
```

that distinguishes those axioms that specify a sort in a signature, in the following sense:

Proposition 3. *For any membership equational signature $\Omega = (K, \Sigma, S)$, any set of sentences Ax, and any sort s in some S_k, the following are equivalent:*

- $U_{\mathrm{MEL}} \vdash (\overline{Ax}\ \texttt{spec}\ \overline{s}\ \texttt{in}\ \overline{\Omega}) = \texttt{true}$.
- *Ax is a set of sentences over Ω that specify the sort s.*

Proposition 4. *For any ground terms u, z, and M in U_{MEL}, if*

$$U_{\mathrm{MEL}} \vdash (u\ \texttt{spec}\ z\ \texttt{in}\ M) = \texttt{true},$$

then there is a membership equational signature Ω, a sort s over Ω, and a set of sentences Ax in Ω specifying s, such that $\overline{\Omega} = M$, $\overline{Ax} = u$, and $\overline{s} = z$.

The proofs for these results would follow easily by mimicking the techniques for Propositions 1 and 2.

2.2 Reflecting an Inductive Principle

We need to introduce here some additional notation. For all terms $t \in T_\Omega(X)$, we denote by $\overline{t}^{[X]}$ the reflective representation of t defined in (1), except that now variables $x \in X$ are replaced by variables $\overline{x}^{[X]} = x$ of the kind `[Term]`, and we denote by $\overline{X}^{[X]}$ the set $\overline{X}^{[X]} \triangleq \{\overline{x}^{[X]} \mid x \in X\}$. The key difference between $\overline{t}$ and $\overline{t}^{[X]}$ is that $\overline{t}$ is a *ground term*, whereas $\overline{t}^{[X]}$ is a term of kind `[Term]` with variables of the kind `[Term]`.

In addition, for all membership assertions $t : s$, with t in $T_\Omega(X)$ and s in some S_k,

$$\overline{t : s}^{[R,X]} \triangleq (\overline{t}^{[X]}\ \texttt{:}\ \overline{s}\ \texttt{in}\ \overline{R}) = \texttt{true},$$

and, similarly, for all equations $t = t'$, with t, t' in $T_\Omega(X)$,

$$\overline{t = t'}^{[R,X]} \triangleq (\overline{t}^{[X]}\ \texttt{=}\ \overline{t'}^{[X]}\ \texttt{in}\ \overline{R}) = \texttt{true}.$$

Now we can define a representation function for metalogical statements that satisfies the expected property. Let $\{R_1, \ldots, R_p\}$ be a set of membership equational theories, $\{k_1, \ldots, k_n\}$ a finite multiset of kinds in $\{R_1, \ldots, R_p\}$, $\vec{x} = \{x_1, \ldots, x_n\}$ a finite set of variables, with each x_i of kind k_i, and τ a metalogical statement of the form

$$\forall t_1 \in (T_{\Omega_1})_{k_1}. \ldots \forall t_n \in (T_{\Omega_n})_{k_n}. \mathit{bexp}(R_1 \vdash \phi_1(\vec{t}), \ldots, R_p \vdash \phi_p(\vec{t})), \qquad (2)$$

where each $\phi_l(\vec{x})$ is an atomic Ω_l-formula with free variables in $\vec{x}$ and *bexp* is a Boolean expression. Then,

$$\overline{\tau} \triangleq \forall x_1. \ldots \forall x_n. (((x_1 :: \overline{k_1}\ \texttt{in}\ \overline{R_1}) = \texttt{true} \wedge \cdots \wedge (x_n :: \overline{k_n}\ \texttt{in}\ \overline{R_n}) = \texttt{true}) \\ \Longrightarrow \mathit{bexp}(\overline{\phi_1(\vec{x})}^{[R_1,\vec{x}]}, \ldots, \overline{\phi_p(\vec{x})}^{[R_p,\vec{x}]}))\,,$$

where $\{x_1, \ldots, x_n\}$ are now variables of the kind `[Term]`. Now, the main result in [4] was:

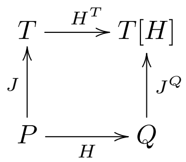
Theorem 1. *Let τ be a metalogical statement of the form (2). Then, τ holds iff $U_{\mathrm{MEL}} \models \overline{\tau}$.*

3 Parameterization

In the previous section we have recalled an inductive principle to reason about terms in a family of theories, which constitutes both an application of the ideas introduced in [1] as well as a generalization.[2] In this section we turn our attention to parameterization, which was already studied in [1] using the deduction theorem as a case study. Here we consider a generalization of the parameter theories and of the corresponding inductive principle, and we use them to formalize two versions of the deduction theorem not expressible in the formalisms presented in [1, 4].

3.1 (Some) Parameterized Membership Equational Theories

As pointed out by Goguen and Burstall [9], a *parameterized theory* can be defined for logics in general as a pair of theories: the *parameter* P and the *body* T, that are related by a theory map $J : P \to T$ which is typically a theory inclusion. To *instantiate* such a parameterized theory, the key data needed is a theory morphism $H : P \to Q$ from the parameter theory to another theory Q. The *instantiation* by H is then defined as the pushout commutative diagram



in the category *Th* of theories and theory maps [9], when such a pushout exists.

Now we employ an instance of the previous construction to define, for each appropriate parameter theory P, a class $\mathcal{P}_P$ of membership equational theories parameterized by P and a class $\mathcal{V}_P$ of theory morphisms that instantiate parameterized theories in $\mathcal{P}_P$. For that, given two membership equational signatures Ω and Ω', we will write $\Omega \cup \Omega'$ for the signature whose set of kinds is the set-theoretic union of those of Ω and Ω', and whose operators and sorts are those of Ω and Ω'.

Then, we consider parameter theories P of the form

$$P = (\Omega \cup V \cup Z, E \cup Mb(V));$$

that is, P's signature is built from

- a finite signature $\Omega = (K, \Sigma, S)$,
- a finite signature of parameters $V = (K, \{V_{\lambda,k}\}_{k \in K}, \emptyset)$, consisting of a pairwise disjoint K-kinded family of constants which satisfies that, for all $k \in K$, $\Sigma_{\lambda,k} \cap V_{\lambda,k} = \emptyset$, and

[2] The result proved in [4] also allowed to reason about equivalence classes of terms, which were not considered in [1].

– a finite signature of parameters $Z = (K, \emptyset, \{Z_k\}_{k \in K})$, consisting of a pairwise disjoint K-kinded family of sets which satisfies that, for all $k \in K$, $S_k \cap Z_k = \emptyset$;

and P's axioms consist of

– a finite set of sentences E on terms in $T_\Omega(X)$, and
– a finite set of membership assertions $Mb(V)$ that specify a sort (possibly in Z) for each v in V.

Moreover, we consider theory maps $P \longrightarrow T$ which are *theory inclusions* and, for this reason, we usually denote parameterized theories by $T[P]$. Specifically, we define $\mathcal{P}_P$ as the class of parameterized theories $T[P]$ of the form

$$T[P] = (\Omega' \cup V \cup Z, E \cup G \cup Mb(V)),$$

where $\Omega \subseteq \Omega'$ and G is a finite set of additional axioms (which extend P's axioms). Note that for all parameter theories P there is a trivial extension $P[P]$ of P, namely, $P[P] = P$.

Now, let $Inst(P)$ be the class of theories

$$Q = (\Omega \cup V \cup Z, E \cup Eq(V) \cup Ax(Z)),$$

where

– $Eq(V)$ is a finite set of equations of the form

$$v = t \qquad (v \in V),$$

assigning to each constant $v \in V$ a *ground* term $t \in T_\Omega$ such that $Q \vdash t : s$, where s is the sort assigned to v in $Mb(V)$, and

– $Ax(Z)$ is a finite set of membership axioms of the form

$$\forall(x_1, \ldots, x_m).\, t : z \textbf{ if } A_1 \wedge \ldots \wedge A_n,$$

where $z \in Z_k$ for some kind $k \in K$, t is a term over the signature Ω, and A_i is an atomic formula over the same signature, for $i = 1, \ldots, n$. We collect all the axioms specifying a sort $z \in Z_k$ in a set $Ax(z)$.

We define $\mathcal{V}_P$ as the class of theory morphisms $\beta : P \longrightarrow Q$ such that $Q \in Inst(P)$ and β is the identity signature morphism. Note that the set $\mathcal{V}_P$ is in bijective correspondence with the set $Inst(P)$.

The above defines a notion of instantiation for parameterized theories that, for any $T[P] \in \mathcal{P}_P$ and $\beta \in \mathcal{V}_P$, specializes the pushout construction to

$$\begin{array}{ccc} T[P] & \longrightarrow & T[\beta] \\ \uparrow & & \uparrow \\ P & \xrightarrow[\beta]{} & Q \end{array}$$

where $T[\beta] = (\Omega' \cup V \cup Z, E \cup G \cup Eq(V) \cup Ax(Z))$.

One of the key ideas behind our use of theory morphisms is the following. Although β is the identity morphism on signatures, it identifies terms in Q, and hence in $T[\beta]$, by adding equations of the form $v = t$. This has an effect equivalent to mapping constants to terms. More formally, suppose $T[P] \in \mathcal{P}_P$ and $\beta \in \mathcal{V}_P$. For all terms $t \in T_{\Omega \cup V}(X)$, we denote by t_β the term in $T_\Omega(X)$ that results from replacing all parameters v in t by their instantiations in $Eq(V)$. We can extend this notion of term replacement to atomic formulae in the standard way: $(t : s)_\beta \triangleq t_\beta : s$ and $(t = t')_\beta \triangleq t_\beta = t'_\beta$. Note then that for all atomic formulae ϕ over the signature of $T[\beta]$, and due to the equations in $Eq(V)$, it holds that

$$T[\beta] \vdash \phi \iff T[\beta] \vdash \phi_\beta. \tag{3}$$

4 Induction Principles for Parameterized Theories

We next introduce an inductive metareasoning principle over parameterized theories. First, we need the following definition.

Definition 1. *Let $P = (\Omega \cup V \cup Z, E \cup Mb(V))$ be a parameter theory with $\Omega = (K, \Sigma, S)$, let $\mathcal{P} = \{R_1[P], \ldots, R_p[P]\}$ be a finite multiset of parameterized theories in $\mathcal{P}_P$, and $e \in [1..p]$. We say that $\mathcal{P}$ is* coherent modulo $R_e[P]$ *if*

1-a. every term t of kind $k \in K$ in $R_e[P]$ is also a term of kind k in $R_l[P]$ for $1 \leq l \leq p$, and

1-b. for all theory morphisms $\beta : P \longrightarrow Q$ in $\mathcal{V}_P$, all terms t and t' of kind $k \in K$ in $R_e[P]$, and all $1 \leq l \leq p$, it holds that

$$R_e[\beta] \vdash t = t' \implies R_l[\beta] \vdash t = t'.$$

That is, we assume that among the parameterized theories in $\mathcal{P}$ there is one that is "equationally generic" in the sense that, if an equation holds in any of its instances, then it also holds in the corresponding instance of any of the rest of the parameterized theories in $\mathcal{P}$. We can then use this distinguished theory to reason inductively about the whole family.

Proposition 5. *Let $\mathcal{P} = \{R_1[P], \ldots, R_p[P]\}$ be a finite multiset of parameterized theories in $\mathcal{P}_P$ that is coherent modulo $R_e[P]$. Let $R_e[P] = (\Omega'_e \cup V \cup Z, E \cup G_e \cup Mb(V))$, let s be a sort in some S_k, and let $C_{[R_e[P],s]} = \{C_1, \ldots, C_n\}$ be those sentences in $E \cup G_e$ that specify the sort s, i.e., those C_i of the form*

$$\forall (x_1, \ldots, x_{r_i}).\, A_0 \textbf{ if } A_1 \wedge \ldots \wedge A_{q_i},$$

where, for some term w of kind k, A_0 is $w\,{:}\,s$.

Then, for all finite multisets of atomic formulae $\{\phi_l(x)\}_{l \in [1..p]}$ with free variable x of kind k, and Boolean expressions bexp, the following metalogical statement holds:

$$\forall \beta \in \mathcal{V}_P.(\psi_1 \wedge \cdots \wedge \psi_n)$$
$$\Longrightarrow$$
$$\forall \beta \in \mathcal{V}_P.\big(\forall t \in T_\Omega.(R_e[\beta] \vdash t : s \Longrightarrow bexp(R_1[\beta] \vdash \phi_1(t)_\beta, \ldots, R_p[\beta] \vdash \phi_p(t)_\beta))\big)$$

where, for $1 \le i \le n$ *and* C_i *in* $C_{[R_e[P],s]}$, ψ_i *is*

$$\forall t_1 \in (T_\Omega)_{k_{i_1}} \ldots \forall t_{r_i} \in (T_\Omega)_{k_{i_{r_i}}}.[A_1]^\# \wedge \cdots \wedge [A_{q_i}]^\# \Longrightarrow [A_0]^\#$$

and, for $0 \le j \le q_i$,

$$[A_j]^\sharp \triangleq \begin{cases} \mathit{bexp}\left(R_1[\beta] \vdash \phi_1(u(\vec{t})), \ldots, R_p[\beta] \vdash \phi_p(u(\vec{t}))\right) & \text{if } A_j = u\!:\!s \\ R_e[\beta] \vdash A_j(\vec{t}) & \text{otherwise.} \end{cases}$$

Actually, this proposition is a particular case of the following, more general one, that will be needed for the deduction theorem.

Proposition 6. *Let* $\mathcal{P} = \{R_1[P], \ldots, R_p[P]\}$ *be a finite multiset of parameterized theories in* $\mathcal{P}_P$ *that is coherent modulo* $R_e[P]$. *Let* $R_e[P] = (\Omega'_e \cup V \cup Z, E \cup G_e \cup Mb(V))$, *let* s *be a sort in some* S_k, *and let* $C_{[R_e[P],s]} = \{C_1, \ldots, C_n\}$ *be those sentences in* $E \cup G_e$ *that specify the sort* s, *i.e., those* C_i *of the form*

$$\forall (x_1, \ldots, x_{r_i}).\, A_0 \textbf{ if } A_1 \wedge \ldots \wedge A_{q_i},$$

where, for some term w *of kind* k, A_0 *is* $w\!:\!s$.

Then, for all finite multisets of atomic formulae $\{\phi_l(x)\}_{l \in [1..p]}$ *with free variable* x *of kind* k, *and Boolean expressions bexp, the following metalogical statement holds:*

$$\begin{array}{l} \forall \beta \in \mathcal{V}_P.(U_{\text{MEL}} \models\!\simeq \overline{\beta}(\gamma) \Longrightarrow \psi_1) \wedge \cdots \wedge \forall \beta \in \mathcal{V}_P.(U_{\text{MEL}} \models\!\simeq \overline{\beta}(\gamma) \Longrightarrow \psi_n) \\ \Longrightarrow \\ \forall \beta \in \mathcal{V}_P.\big(U_{\text{MEL}} \models\!\simeq \overline{\beta}(\gamma) \Longrightarrow \\ \qquad \forall t \in T_\Omega.(R_e[\beta] \vdash t : s \Longrightarrow \mathit{bexp}(R_1[\beta] \vdash \phi_1(t)_\beta, \ldots, R_p[\beta] \vdash \phi_p(t)_\beta))\big) \end{array}$$

where, for $1 \le i \le n$ *and* C_i *in* $C_{[R_e[P],s]}$, ψ_i *is*

$$\forall t_1 \in (T_\Omega)_{k_{i_1}} \ldots \forall t_{r_i} \in (T_\Omega)_{k_{i_{r_i}}}.[A_1]^\# \wedge \cdots \wedge [A_{q_i}]^\# \Longrightarrow [A_0]^\#$$

and, for $0 \le j \le q_i$,

$$[A_j]^\sharp \triangleq \begin{cases} \mathit{bexp}\left(R_1[\beta] \vdash \phi_1(u(\vec{t})), \ldots, R_p[\beta] \vdash \phi_p(u(\vec{t}))\right) & \text{if } A_j = u\!:\!s \\ R_e[\beta] \vdash A_j(\vec{t}) & \text{otherwise.} \end{cases}$$

Proof. Let $\beta \in \mathcal{V}_P$ be such that $U_{\text{MEL}} \models\!\simeq \overline{\beta}(\gamma)$ and let t be such that $R_e[\beta] \vdash t : s$. Then, we have to show that $\mathit{bexp}(R_1[\beta] \vdash \phi_1(t)_\beta, \ldots, R_p[\beta] \vdash \phi_p(t)_\beta)$ is true.

We proceed by structural induction on the derivation of $R_e[\beta] \vdash t : s$. There exists a sentence C_i in $C_{[R_e[P],s]}$ and a substitution $\sigma : \{x_1, \ldots, x_{r_i}\} \longrightarrow T_{\Omega_e}$, such that

- $R_e[\beta] \vdash t = \sigma(w)$, and
- $R_e[\beta] \vdash \sigma(A_j)$, for $1 \le j \le q_i$.

By hypothesis, $U_{\text{MEL}} \models \overline{\beta}(\gamma) \implies \psi_i$, and since we are assuming $U_{\text{MEL}} \models \overline{\beta}(\gamma)$, ψ_i must hold. But then, in particular, it also holds $[A_1]^\sharp_\sigma \wedge \ldots \wedge [A_{q_i}]^\sharp_\sigma \implies [A_0]^\sharp_\sigma$, where, for $0 \leq j \leq q_i$,

$$[A_j]^\sharp_\sigma \triangleq \begin{cases} \mathit{bexp}\,(R_1[\beta] \vdash \phi_1([\sigma(u)]_{R_e}), \ldots, R_p[\beta] \vdash \phi_p([\sigma(u)]_{R_e})) & \text{if } A_j = u\!:\!s \\ R_e \vdash \sigma(A_j) & \text{otherwise.} \end{cases}$$

Note now that, for $1 \leq j \leq q_i$,

- If $A_j = (u\!:\!s)$, then $[A_j]^\sharp_\sigma$ holds by induction hypothesis, since $R_e[\beta] \vdash \sigma(u)\!:\! s$.
- If $A_j \neq (u\!:\!s)$, then $[A_j]^\sharp_\sigma$ holds by assumption.

Hence, $[A_0]^\sharp_\sigma$, that is, $\mathit{bexp}(R_1[\beta] \vdash \phi_1(\sigma(w)), \ldots, R_p[\beta] \vdash \phi_p(\sigma(w))$, also holds. Finally, since $R_e[\beta] \vdash t = \sigma(w)$ and $\mathcal{P}$ is coherent modulo $R_e[P]$, we have that $\mathit{bexp}(R_1[\beta] \vdash \phi_1(t), \ldots, R_p[\beta] \vdash \phi_p(t))$ as required. □

We will be mainly interested in those γ such that $U_{\text{MEL}} \models \overline{\beta}(\gamma)$ is equivalent to imposing some restrictions on the instances β at the object level. This will be illustrated in Section 6.

5 Reflected Parameterized Induction

In this section we explain how the inductive principle for reasoning about parameterized theories introduced in Section 4 can be reflected. To accomplish this, the key ideas are the following.

- Parameterization is reflected as quantification over (meta)variables representing the parameters. In particular, parameterized atomic formulae are represented as atomic formulae which contain free (meta)variables representing the parameters.
- Instantiation requirements are reflected as a formula (γ), which contains also free (meta)variables representing the parameters. The idea is that all substitutions of the (meta)variables representing the parameters must satisfy this formula.

5.1 Representing Parameterized Theories

We first need to further extend the notation introduced in Section 2.2 to deal with parameters. Let $P = (\Omega \cup V \cup Z, E \cup Mb(V))$ be a parameter theory with $\Omega = (K, \Sigma, S)$. For all terms $t \in T_{\Omega \cup V}(X)$, we will denote by $\overline{t}^{[V,X]}$ its reflective $\overline{(\_)}$-representation except that now parameters $v \in V$ and variables $x \in X$ are replaced by (meta)variables v and x of the kind `[Term]`. For t a ground term, we shall simply write $\overline{t}^{[V]}$. Similarly, if $t \in T_{\Omega \cup Z}(X)$ we shall write $\overline{t}^{[X]}$ as we did in Section 2.2. Also, for any sort z in Z_k, $k \in K$, we will denote by $\overline{z}^{[Z]}$ a (meta)variable of the kind `[AxiomSet]`. In addition, we will denote by $\overline{V}^{[V]}$ the set $\overline{V}^{[V]} \triangleq \{\overline{v}^{[V]} \mid v \in V\}$, and by $\overline{Z}^{[Z]}$ the set $\overline{Z}^{[Z]} \triangleq \{\overline{z}^{[Z]} \mid z \in Z_k, k \in K\}$, and assume that they are disjoint.

Finally, for any theory morphism $\beta : P \longrightarrow Q$ in $\mathcal{V}_P$, with $Q = (\Omega \cup V \cup Z, E \cup Eq(V) \cup Ax(Z))$, we will denote by $\overline{\beta}$ the ground substitution $\overline{\beta} : \overline{V}^{[V]} \cup \overline{Z}^{[Z]} \longrightarrow T_{U_{\mathrm{MEL}}}$, defined as follows: $\overline{\beta}(\overline{v}^{[V]}) \triangleq \overline{t}$, if $(v = t) \in Eq(V)$, and $\overline{\beta}(\overline{z}^{[Z]}) \triangleq \overline{Ax(z)}$, if $z \in Z_k$.

Proposition 7. *For all theory morphisms $\beta : P \longrightarrow Q$ in $\mathcal{V}_P$ and all terms $t \in T_{\Omega \cup V}(X)$,*

$$\overline{\beta}(\overline{t}^{[V,X]}) = \overline{t_\beta}^{[X]}.$$

Proof. By structural induction on t. □

We now define a *generic* representation function $\overline{(\_)}^{\mathcal{P}}$ for parameterized membership equational theories. Let $P = (\Omega \cup V \cup Z, E \cup Mb(V))$ be a parameter theory with $\Omega = (K, \Sigma, S)$, $K = \{k_1, \ldots, k_m\}$ and $Mb(V) = \{v_1 : s_1, \ldots, v_n : s_n\}$. Then, for any parameterized theory $T[P] = (\Omega' \cup V \cup Z, E \cup G \cup Mb(V))$ in $\mathcal{P}_P$,

$$\overline{T[P]}^{\mathcal{P}} \triangleq (\overline{\Omega' \cup V \cup Z}, \overline{E}\ \overline{G}\ \overline{Mb(V)}^{\mathcal{P}}\ \overline{Z}^{\mathcal{P}}),$$

where

- $\overline{Mb(V)}^{\mathcal{P}}$ is the term

$$\overline{Mb(V)}^{\mathcal{P}} \triangleq (\mathtt{eq}\ \overline{v_1}\ \mathtt{=}\ \overline{v_1}^{[V]}\ \mathtt{.} \cdots \mathtt{eq}\ \overline{v_n}\ \mathtt{=}\ \overline{v_n}^{[V]}\ \mathtt{.}),$$

  and
- $\overline{Z}^{\mathcal{P}}$ is the term

$$\overline{Z}^{\mathcal{P}} \triangleq (\overline{Z_{k_1}}^{\mathcal{P}} \ldots \overline{Z_{k_m}}^{\mathcal{P}})$$

  where, for any $k_i \in K$, if $Z_{k_i} = \{z_{i1}, \ldots, z_{iq_{k_i}}\}$, then $\overline{Z_{k_i}}^{\mathcal{P}}$ is the term

$$\overline{Z_{k_i}}^{\mathcal{P}} \triangleq (\overline{z_{i1}}^{[Z]} \ldots \overline{z_{iq_{k_i}}}^{[Z]}).$$

  Intuitively, $\overline{Z}^{\mathcal{P}}$ is a term representing all possible instantiations of the set of axioms defining the sorts in Z.

Proposition 8. *For any parameterized membership equational theory $T[P] \in \mathcal{P}_P$, $T[P] = (\Omega' \cup V \cup Z, E \cup G \cup Mb(V))$, and any theory morphism $\beta : P \longrightarrow Q$ in $\mathcal{V}_P$, it holds that*

$$\overline{\beta}(\overline{T[P]}^{\mathcal{P}}) = \overline{T[\beta]}.$$

Proof. By definition of substitution application and $\overline{\beta}$ we have

$$\begin{aligned}\overline{\beta}(\overline{T[P]}^{\mathcal{P}}) &= (\overline{\Omega' \cup V \cup Z}, \overline{E}\ \overline{G}\ \overline{\beta}(\overline{Mb(V)}^{\mathcal{P}})\ \overline{\beta}(\overline{Z}^{\mathcal{P}})) \\ &= (\overline{\Omega' \cup V \cup Z}, \overline{E}\ \overline{G}\ (\mathtt{eq}\ \overline{v_1}\ \mathtt{=}\ \overline{t_1}\ \mathtt{.} \cdots \mathtt{eq}\ \overline{v_n}\ \mathtt{=}\ \overline{t_n}\ \mathtt{.}) \\ &\qquad (\overline{Ax(z_{11})} \cdots \overline{Ax(z_{mq_{k_m}})}))\end{aligned}$$

which, by the definition of $T[\beta]$, yields the desired result. □

5.2 Representing Parameterized Atomic Formulae

We now define a *generic* representation function $\overline{(\_)}^{[\_,\_]}$ for atomic formulae over parameterized membership equational theories. Note that we use the same notation as in Section 2.2.

For $P = (\Omega \cup V \cup Z, E \cup Mb(V))$ with $\Omega = (K, \Sigma, S)$, any parameterized theory $T[P] \in \mathcal{P}_P$, and any membership assertion $t : s$,

$$\overline{t : s}^{[T[P],X]} \triangleq (\overline{t}^{[V,X]} : \overline{s} \texttt{ in } \overline{T[P]}^{\mathcal{P}}) = \texttt{true}.$$

Similarly, for any equation $t = t'$,

$$\overline{t = t'}^{[T[P],X]} \triangleq (\overline{t}^{[V,X]} \texttt{ = } \overline{t'}^{[V,X]} \texttt{ in } \overline{T[P]}^{\mathcal{P}}) = \texttt{true}.$$

Proposition 9. *For all ground atomic formulae ϕ over the signature of the parameterized theory $T[P]$ and all theory morphisms $\beta : P \longrightarrow Q$ in $\mathcal{V}_P$,*

$$U_{\mathrm{MEL}} \models\!\!\simeq \overline{\beta}(\overline{\phi}^{[T[P],\emptyset]}) \iff T[\beta] \vdash \phi_\beta.$$

Proof. Let $\phi = t : s$ (the proof is analogous for $\phi = (t = t')$). Notice that by the definition of substitution application and Propositions 7 and 8,

$$\begin{aligned} \overline{\beta}(\overline{t : s}^{[T[P],\emptyset]}) &= (\overline{\beta}(\overline{t}^{[V]} : \overline{s} \texttt{ in } \overline{T[P]}^{\mathcal{P}}) = \texttt{true}) \\ &= (\overline{\beta}(\overline{t}^{[V]}) : \overline{s} \texttt{ in } \overline{\beta}(\overline{T[P]}^{\mathcal{P}}) = \texttt{true}) \\ &= (\overline{t_\beta} : \overline{s} \texttt{ in } \overline{T[\beta]} = \texttt{true}). \end{aligned}$$

Thus, since $(\overline{t_\beta} : \overline{s} \texttt{ in } \overline{T[\beta]} = \texttt{true})$ is a ground atomic formula, due to the soundness and completeness of membership equational logic we can reduce the problem to proving that

$$U_{\mathrm{MEL}} \vdash (\overline{t_\beta} : \overline{s} \texttt{ in } \overline{T[\beta]}) = \texttt{true} \Longleftrightarrow T[\beta] \vdash \phi_\beta,$$

which holds by Proposition 2. □

Corollary 1. *For P a parameter theory with $Mb(V) = \{v_1 : s_1, \ldots, v_n : s_n\}$, and $\beta : P \longrightarrow Q$ in $\mathcal{V}_P$,*

$$U_{\mathrm{MEL}} \models\!\!\simeq \overline{\beta}(\overline{v_i : s_i}^{[P,\emptyset]}) \qquad (1 \le i \le n).$$

Proof. Notice that in this case the parameterized theory $T[P]$ is $P[P] = P$, and hence $T[\beta]$ is Q. Then, by Proposition 9,

$$U_{\mathrm{MEL}} \models\!\!\simeq \overline{\beta}(\overline{v_i : s_i}^{[P,\emptyset]}) \Longleftrightarrow Q \vdash (v_i : s_i)_\beta, \qquad (1 \le i \le n)$$

and the righthand side entailments hold by definition of Q. □

5.3 Representing Requirements

We will need to impose, at the metalevel, that the parameters in the theory P are correctly instantiated. For that, if $Mb(V) = \{v_1 : s_1, \ldots, v_n : s_n\}$, we define

$$\begin{aligned}\overline{Mb(V)}^{\mathcal{C}(P)} \triangleq ((&\overline{v_1}^{[V]} :: \overline{k_1} \texttt{ in } \overline{P}) = \texttt{true} \wedge \cdots \wedge (\overline{v_n}^{[V]} :: \overline{k_n} \texttt{ in } \overline{P}) = \texttt{true} \wedge \\ &\overline{v_1 : s_1}^{[P,\emptyset]} \wedge \cdots \wedge \overline{v_n : s_n}^{[P,\emptyset]}),\end{aligned}$$

where k_i is the kind of s_i for $i = 1, \ldots, n$. It immediately follows from Proposition 1 and Corollary 1 that

Proposition 10. *For $P = (\Omega \cup V \cup Z, E \cup Mb(V))$ a parameter theory, and $\beta : P \longrightarrow Q$ in $\mathcal{V}_P$,*

$$U_{\text{MEL}} \models \overline{\beta}(\overline{Mb(V)}^{\mathcal{C}(P)}).$$

The formula $\overline{Mb(V)}^{\mathcal{C}(P)}$ will be used to impose that the parameters in V are instantiated with ground terms of the appropriate sort. Analogously, we will also require that the variables in $\overline{Z}^{[Z]}$ are correctly instantiated (that is to say, with membership axioms specifying the sorts in Z), and for that we will use a new representation function $\overline{(\_)}^{\mathcal{D}(P)}$, defined over sorts in Z as follows:

$$\overline{z}^{\mathcal{D}(P)} \triangleq (\overline{z}^{[Z]} \texttt{ spec } \overline{z} \texttt{ in } \overline{\Omega \cup Z}) = \texttt{true} \qquad (z \in Z_k).$$

Proposition 11. *For $P = (\Omega \cup V \cup Z, E \cup Mb(V))$, any sort $z \in Z_k$ for some kind k, and $\beta : P \longrightarrow Q$ in $\mathcal{V}_P$,*

$$U_{\text{MEL}} \models \overline{\beta}(\overline{z}^{\mathcal{D}(P)}).$$

Proof. By definition of substitution application and $\overline{\beta}$,

$$\begin{aligned}\overline{\beta}(\overline{z}^{\mathcal{D}(P)}) &= (\overline{\beta}(\overline{z}^{[Z]}) \texttt{ spec } \overline{z} \texttt{ in } \overline{\Omega \cup Z} = \texttt{true}) \\ &= (\overline{Ax(z)} \texttt{ spec } \overline{z} \texttt{ in } \overline{\Omega \cup Z} = \texttt{true}),\end{aligned}$$

and hence the result follows from Proposition 3 by soundness of membership equational logic. □

The representation function $\overline{(\_)}^{\mathcal{D}(P)}$ is extended to Z in the obvious way by

$$\overline{Z}^{\mathcal{D}(P)} \triangleq \bigwedge_{z \in Z_k} \overline{z}^{\mathcal{D}(P)}.$$

Corollary 2. *For $P = (\Omega \cup V \cup Z, E \cup Mb(V))$ a parameter theory and $\beta : P \longrightarrow Q$ in $\mathcal{V}_P$,*

$$U_{\text{MEL}} \models \overline{\beta}(\overline{Z}^{\mathcal{D}(P)}).$$

5.4 Reflecting Parameterized Induction Principles

We now define a representation function for metalogical statements. Let $\mathcal{P} = \{R_1[P], \ldots, R_p[P]\}$ be a finite multiset of parameterized theories in $\mathcal{P}_P$ that is coherent modulo $R_e[P]$, $\{k_1, \ldots, k_n\}$ a finite multiset of kinds, and τ a metalogical statement of the form

$$\begin{array}{l}\forall \beta \in \mathcal{V}_P.\big(U_{\text{MEL}} \models \overline{\beta}(\gamma) \Longrightarrow \\ \forall t_1 \in (T_{\Omega_1})_{k_1}.\ldots\forall t_n \in (T_{\Omega_n})_{k_n}.\mathit{bexp}(R_1[\beta] \vdash (\phi_1(\vec{t}))_\beta, \ldots, R_p[\beta] \vdash (\phi_p(\vec{t}))_\beta)\big)\end{array}$$

where each $\phi_l(\vec{x})$ is an atomic formula with free variables in $\vec{x}$, each x_i of kind k_i. Then, $\overline{\tau}$ is defined as

$$\begin{array}{l}\forall \overline{V}^{[V]}.\forall \overline{Z}^{[Z]}.\ \big((\overline{Mb(V)}^{\mathcal{C}(P)} \wedge \overline{Z}^{\mathcal{D}(P)} \wedge \gamma) \Longrightarrow \\ \forall x_1.\ldots\forall x_n.((x_1 :: \overline{k_1}\ \texttt{in}\ \overline{R_1[P]}) = \texttt{true} \wedge \cdots \wedge (x_n :: \overline{k_n}\ \texttt{in}\ \overline{R_n[P]}) = \texttt{true}) \\ \Longrightarrow \qquad \mathit{bexp}(\overline{\phi_1(\vec{x})}^{[R_1[P],\vec{x}]}, \ldots, \overline{\phi_p(\vec{x})}^{[R_p[P],\vec{x}]})\big)\end{array}$$

where $\{x_1, \ldots, x_n\}$ are now variables of the kind `[Term]`.

The following auxiliary result is needed in the proof of our main theorem.

Proposition 12. *For $P = (\Omega \cup V \cup Z, E \cup Mb(V))$ a parameter theory with $Mb(V) = \{v_1 : s_1, \ldots, v_n : s_n\}$, and any ground substitution $h : \overline{V}^{[V]} \cup \overline{Z}^{[Z]} \longrightarrow T_{U_{\text{MEL}}}$ such that*

$$U_{\text{MEL}} \models h(\overline{Mb(V)}^{\mathcal{C}(P)} \wedge \overline{Z}^{\mathcal{D}(P)}),$$

there is a theory morphism $\beta : P \longrightarrow Q$ in $\mathcal{V}_P$, with $Q = (\Omega \cup V \cup Z, E \cup Eq(V) \cup Ax(Z))$, such that $\overline{\beta}$ is the ground substitution h.

Proof. By definition of substitution application, for $1 \leq i \leq n$,

$$h(\overline{v_i}^{[V]} :: \overline{k_i}\ \texttt{in}\ \overline{P} = \texttt{true}) = (h(\overline{v_i}^{[V]}) :: \overline{k_i}\ \texttt{in}\ \overline{P} = \texttt{true}).$$

The hypothesis implies $U_{\text{MEL}} \models h(\overline{v_i}^{[V]} :: \overline{k_i}\ \texttt{in}\ \overline{P} = \texttt{true})$ for each v_i, $1 \leq i \leq n$, and by Proposition 1, using the completeness of membership equational logic and the fact that $h(\overline{v_i}^{[V]} :: \overline{k_i}\ \texttt{in}\ \overline{P} = \texttt{true})$ is a ground atomic formula, it follows that there are ground terms $t_i \in (T_\Omega)_{k_i}$ such that $\overline{t_i} = h(\overline{v_i}^{[V]})$.

Similarly, from $U_{\text{MEL}} \models h(\overline{Z}^{\mathcal{D}(P)})$, by Proposition 4, it follows that there are sets of axioms $Ax(z)$ specifying z in $\overline{\Omega \cup Z}$ for each $z \in Z_k$, $k \in K$, such that $h(\overline{z}^{[Z]}) = \overline{Ax(z)}$.

Let $Q = (\Omega \cup V \cup Z, E \cup \{v_1 = t_1, \ldots, v_n = t_n\} \cup \bigcup_{z \in Z_k} Ax(z))$. By definition of substitution application,

$$\begin{array}{l}h(\overline{v_i : s_i}^{[P,\emptyset]}) \\ = (h(\overline{v_i}^{[V,\emptyset]} : \overline{s_i}\ \texttt{in}\ \overline{P}^{\mathcal{P}}) = \texttt{true}) \\ = (h(\overline{v_i}^{[V]}) : \overline{s_i}\ \texttt{in}\ (\overline{\Omega \cup V \cup Z}, \overline{E}\ \texttt{eq}\ \overline{v}_1\ \texttt{=}\ h(\overline{v_1}^{[V]})\ \texttt{.} \cdots \texttt{eq}\ \overline{v}_n\ \texttt{=}\ h(\overline{v_n}^{[V]})\ \texttt{.} \\ \qquad\qquad\qquad\qquad\qquad\qquad h(\overline{z_{11}}^{[Z]}) \cdots h(\overline{z_{mq_{k_m}}}^{[Z]}))) = \texttt{true}) \\ = (\overline{t_i} : \overline{s_i}\ \texttt{in}\ \overline{Q} = \texttt{true}).\end{array}$$

Since $h(\overline{v_i : s_i}^{[P,\emptyset]})$ is an atomic ground formula and $U_{\mathrm{MEL}} \models h(\overline{v_i : s_i}^{[P,\emptyset]})$, by Proposition 2 and completeness of membership equational logic we have $Q \vdash t_i : s_i$, $1 \leq i \leq n$. Then, the identity signature morphism $\beta : P \longrightarrow Q$ satisfies the requirements to be in $\mathcal{V}_P$. □

Theorem 2. *Let τ be a metalogical statement of the above form. Then, τ holds iff $U_{\mathrm{MEL}} \models \overline{\tau}$.*

Proof. Assume that τ holds and let $h : \overline{V}^{[V]} \cup \overline{Z}^{[Z]} \longrightarrow T_{U_{\mathrm{MEL}}}$ be such that $U_{\mathrm{MEL}} \models h(\overline{Mb(V)}^{\mathcal{C}(P)} \wedge \overline{Z}^{\mathcal{D}(P)} \wedge \gamma)$. By Proposition 12 there is $\beta \in \mathcal{V}_P$ with $\overline{\beta} = h$, so our task is reduced to proving that

$$\begin{aligned}&\forall x_1 \ldots \forall x_n.((x_1 :: \overline{k_1}\ \texttt{in}\ \overline{R_1[P]}) = \texttt{true} \wedge \cdots \wedge (x_n :: \overline{k_n}\ \texttt{in}\ \overline{R_n[P]}) = \texttt{true})\\ &\quad \Longrightarrow \overline{\beta}(\mathit{bexp}(\overline{\phi_1(\vec{x})}^{[R_1[P],\vec{x}]}, \ldots, \overline{\phi_p(\vec{x})}^{[R_p[P],\vec{x}]}))\end{aligned}$$

holds in the initial model of U_{MEL}. So let $\sigma : \{x_1, \ldots, x_n\} \longrightarrow T_{U_{\mathrm{MEL}}}$ be a substitution such that

$$(\sigma(x_1) :: \overline{k_1}\ \texttt{in}\ \overline{R_1[P]}) = \texttt{true} \wedge \cdots \wedge (\sigma(x_n) :: \overline{k_n}\ \texttt{in}\ \overline{R_n[P]}) = \texttt{true}$$

holds in $T_{U_{\mathrm{MEL}}}$; by Proposition 1 we know that, for $i = 1, \ldots, n$, $\sigma(x_i) = \overline{w_i}$ for some $w_i \in (T_{\Omega_i})_{k_i}$. By the definition of substitution application and Propositions 7 and 8, for $1 \leq l \leq p$ and $\phi_l = (t_l : s_l)$ (similarly for $\phi_l = (t_l = t'_l)$),

$$\begin{aligned}\sigma(\overline{\beta}(\overline{\phi_l(\vec{x})}^{[R_l[P],\vec{x}]})) &= \sigma(\overline{\beta}(\overline{t_l(\vec{x}) : s_l}^{[R_l[P],\vec{x}]}))\\ &= \sigma(\overline{\beta}(\overline{t_l(\vec{x})}^{[V,\vec{x}]} : \overline{s_l}\ \texttt{in}\ \overline{R_l[P]}^{\mathcal{P}} = \texttt{true}))\\ &= (\sigma(\overline{\beta}(\overline{t_l(\vec{x})}^{[V,\vec{x}]})) : \overline{s_l}\ \texttt{in}\ \overline{\beta}(\overline{R_l[P]}^{\mathcal{P}}) = \texttt{true})\\ &= (\sigma(\overline{(t_l(\vec{x}))_\beta}^{[\vec{x}]}) : \overline{s_l}\ \texttt{in}\ \overline{R_l[\beta]} = \texttt{true})\\ &= (\overline{\sigma((t_l(\vec{x}))_\beta)} : \overline{s_l}\ \texttt{in}\ \overline{R_l[\beta]} = \texttt{true})\\ &= (\overline{(t_l(\sigma(\vec{x})))_\beta} : \overline{s_l}\ \texttt{in}\ \overline{R_l[\beta]} = \texttt{true})\\ &= (\overline{\beta}(\overline{t_l(\vec{w})}^{[V]}) : \overline{s_l}\ \texttt{in}\ \overline{\beta}(\overline{R_l[P]}^{\mathcal{P}}) = \texttt{true})\\ &= \overline{\beta}(\overline{t_l(\vec{w}) : s_l}^{[R_l[P],\emptyset]})\\ &= \overline{\beta}(\overline{\phi_l(\vec{w})}^{[R_l[P],\emptyset]}).\end{aligned}$$

Hence, by Proposition 9, $U_{\mathrm{MEL}} \models \sigma(\overline{\beta}(\overline{\phi_l(\vec{x})}^{[R_l[P],\vec{x}]}))$ iff $R_l[\beta] \vdash \phi_l(\vec{w})_\beta$. But then, since τ holds and we are assuming that $U_{\mathrm{MEL}} \models \overline{\beta}(\gamma)$, we have $\mathit{bexp}(R_1[\beta] \vdash (\phi_1(\vec{t}))_\beta, \ldots, R_p[\beta] \vdash (\phi_p(\vec{t}))_\beta)$ for all $\vec{t}$, in particular for $\vec{w}$, and the result follows.

We have just shown the implication from left to right. A careful examination reveals that all the implications are in fact equivalences and hence this proves the theorem. □

In particular, Theorem 2 can be applied to the inductive principle

$$\begin{aligned}&\forall \beta \in \mathcal{V}_P.(U_{\mathrm{MEL}} \models \overline{\beta}(\gamma) \Longrightarrow \psi_1) \wedge \cdots \wedge \forall \beta \in \mathcal{V}_P(U_{\mathrm{MEL}} \models \overline{\beta}(\gamma) \Longrightarrow \psi_n)\\ &\Longrightarrow\\ &\forall \beta \in \mathcal{V}_P.\big(U_{\mathrm{MEL}} \models \overline{\beta}(\gamma) \Longrightarrow\\ &\qquad \forall t \in T_\Omega.(R_0[\beta] \vdash t : s \Longrightarrow \mathit{bexp}(R_1[\beta] \vdash \phi_1(t)_\beta, \ldots, R_p[\beta] \vdash \phi_p(t)_\beta))\big)\end{aligned}$$

by replacing each metalogical statement ϕ by its logical representation $\overline{\phi}$ to get an inductive principle for U_{MEL}.

6 The Deduction Theorem Revisited

6.1 Formalizing the Deduction Theorem

The parameterized versions of the deduction theorem can now be expressed as metatheoretic statements relating the initial models of all the different instantiations of DT[EML] and EML that satisfy certain requirements. In its standard form, the deduction theorem can be formalized as follows:

$$\forall\beta \in \mathcal{V}^1_{\texttt{EML}}.\,\forall t \in T_{\texttt{EML}}.\,(\texttt{DT[}\beta\texttt{]} \vdash t\texttt{:Theorem} \Rightarrow \texttt{EML[}\beta\texttt{]} \vdash \texttt{[->,@A,}t\texttt{]}_\beta\texttt{:Theorem}), \tag{4}$$

where

$$\mathcal{V}^1_{\texttt{EML}} = \{\beta \in \mathcal{V}_{\texttt{EML}} \mid Ax(\texttt{@NewAxiom}) \cup Ax(\texttt{@NewSynRule}) \cup Ax(\texttt{@NewInfRule}) = \emptyset\}\,.$$

Note that $\{\texttt{DT[EML]},\texttt{EML[EML]}\}$ is coherent module DT[EML].

However, the deduction theorem also holds for all extensions of minimal logic's language and minimal logic's axioms, which can be formalized as follows:

$$\forall\beta \in \mathcal{V}^2_{\texttt{EML}}.\,\forall t \in T_{\texttt{EML}}.\,(\texttt{DT[}\beta\texttt{]} \vdash t\texttt{:Theorem} \Rightarrow \texttt{EML[}\beta\texttt{]} \vdash \texttt{[->,@A,}t\texttt{]}_\beta\texttt{:Theorem}), \tag{5}$$

where $\mathcal{V}^2_{\texttt{EML}} = \{\beta \in \mathcal{V}_{\texttt{EML}} \mid Ax(\texttt{@NewInfRule}) = \emptyset\}$.

Furthermore, the deduction theorem can also be verified for all extensions of minimal logic's language, axioms, and two-premise rules (this can be generalized to finitely many assumptions), provided that all new rules of the form

$$\frac{B \quad C}{D}$$

are such that, for all formulae A, if $(A \to B)$ and $(A \to C)$ are theorems in the corresponding extension of minimal logic, then $(A \to D)$ is also a theorem [2]. This version of the deduction theorem can be formalized as follows:

$$\forall\beta \in \mathcal{V}^3_{\texttt{EML}}.\,\forall t \in T_{\texttt{EML}}.\,(\texttt{DT[}\beta\texttt{]} \vdash t\texttt{:Theorem} \Rightarrow \texttt{EML[}\beta\texttt{]} \vdash \texttt{[->,@A,}t\texttt{]}_\beta\texttt{:Theorem}), \tag{6}$$

where

$$\begin{aligned}\mathcal{V}^3_{\texttt{EML}} = \{\beta \in \mathcal{V}_{\texttt{EML}} \mid \forall x_1.\forall x_2.\forall x_3.\forall x_4.\,(&\texttt{DT[}\beta\texttt{]} \vdash \texttt{[}x_4\texttt{,}x_2\texttt{,}x_3\texttt{]:@NewInfRule}\\ &\Rightarrow \texttt{EML[}\beta\texttt{]} \vdash \texttt{[->,}x_1\texttt{,}x_2\texttt{]:Theorem} \Rightarrow \texttt{EML[}\beta\texttt{]} \vdash \texttt{[->,}x_1\texttt{,}x_3\texttt{]:Theorem}\\ &\Rightarrow \texttt{EML[}\beta\texttt{]} \vdash \texttt{[->,}x_1\texttt{,}x_4\texttt{]:Theorem})\}.\end{aligned}$$

6.2 Proving the Deduction Theorem

In what follows, we denote by V-EM and Z-EM, respectively, the sets of parameters {@A} and {@NewAxiom, @NewSynRule, @NewInfRule}, and by MB-V-EM the set of axioms {mb @A : Formula .}. Using the results of Section 5 we can formalize the different versions of the deduction theorem, (4), (5), and (6), as theorems about U_{MEL}. All these theorems have a common structure

$$\begin{aligned}&\forall \overline{\texttt{V-EM}}^{[\texttt{V-EM}]}.\forall \overline{\texttt{Z-EM}}^{[\texttt{Z-EM}]}.\,((\overline{\texttt{MB-V-EM}}^{\mathcal{C}(\texttt{EML})} \wedge \overline{\texttt{Z-EM}}^{\mathcal{D}(\texttt{EML})} \wedge \gamma) \Longrightarrow \\ &\quad \forall x.\,((x :: \overline{\texttt{Expression}} \; \texttt{in} \; \overline{\texttt{EML}} = \texttt{true}) \Longrightarrow \\ &\qquad (\overline{x\texttt{:Theorem}}^{[\texttt{DT[EML]},x]} \Rightarrow \overline{\texttt{[->,@A,x]:Theorem}}^{[\texttt{EML},x]}))),\end{aligned} \tag{7}$$

but differ in the definition of γ. Note that this is in direct correspondence with the fact that the metatheoretic statements (4), (5), and (6) only differ in the requirements imposed over the instantiations $\beta \in \mathcal{V}_{\texttt{EML}}$. Concretely, for (4), (5), and (6) the formula γ is defined, respectively, as:

$$\begin{aligned}\gamma^1 &\triangleq (\overline{\texttt{@NewAxiom}}^{[\texttt{Z-EM}]} = \texttt{none} \;\wedge\; \overline{\texttt{@NewSynRule}}^{[\texttt{Z-EM}]} = \texttt{none} \;\wedge \\ &\qquad \overline{\texttt{@NewInfRule}}^{[\texttt{Z-EM}]} = \texttt{none}),\\ \gamma^2 &\triangleq (\overline{\texttt{@NewInfRule}}^{[\texttt{Z-EM}]} = \texttt{none}),\\ \gamma^3 &\triangleq (\forall x_1.\forall x_2.\forall x_3.\forall x_4.\,\overline{[x_4, x_2, x_3]\texttt{:@NewInfRule}}^{[\texttt{DT[EML]},\vec{x}]}\\ &\qquad \Rightarrow \overline{\texttt{[->,}x_1\texttt{,}x_2\texttt{]:Theorem}}^{[\texttt{EML},\vec{x}]} \;\Rightarrow\; \overline{\texttt{[->,}x_1\texttt{,}x_3\texttt{]:Theorem}}^{[\texttt{EML},\vec{x}]}\\ &\qquad \Rightarrow \overline{\texttt{[->,}x_1\texttt{,}x_4\texttt{]:Theorem}}^{[\texttt{EML},\vec{x}]}).\end{aligned}$$

By Theorem 2, (7) implies, for each definition of γ, the corresponding parameterized version of the deduction theorem. The correctness of the above formalizations follows from the following remark: for all theory morphisms $\beta \in \mathcal{V}_{\texttt{EML}}$,

$$\beta \in \mathcal{V}^i_{\texttt{EML}} \iff U_{\mathrm{MEL}} \models\!\!\simeq \overline{\beta}(\gamma^i) \qquad i = 1, 2, 3.$$

Finally, to prove each version of (7) in U_{MEL} we apply the reflected version of the induction principle for the sort Theorem in the parameterized theory DT[EML]. The proofs mirror the standard proof of the deduction theorem: we show $A \to B$ by induction on the structure of possible derivations of B when A is assumed as an axiom. Note, however, that to prove the deduction theorem for all extensions of minimal logic's language and minimal logic's axioms we have to consider as an additional base case of the inductive proof when B is one of the new axioms. Moreover, to prove the deduction theorem for all extensions of minimal logic's language, axioms, and two-premise rules satisfying the above mentioned requirement, we also have to consider as an additional step case of the inductive proof when B follows by an application of one of the new rules. By using the reflected version of the induction principle for the sort Theorem in the parameterized theory DT[EML], all these considerations are appropriately mirrored in our proofs.

7 Conclusion

Based on the ideas introduced in [1] by Basin, Clavel, and Meseguer about reflective metalogical frameworks, and about membership equational logic as one of them, we have further explored the capabilities of membership equational logic as a logic to reason *about* logics and about relationships *between* logics.

In this paper we have extended the notion of parameterized membership equational theories and of reflected parameterized induction introduced in [1]. By doing this, we are able to formalize and prove a wider class of metatheorems: for example, the parameterized versions (5) and (6) of the deduction theorem cannot be formalized in [1, 4]. Our experiments show that one can prove metatheorems similar to those provable in logical frameworks based on parameterized inductive definitions [2]. In essence, we can do this because the requirements that such metatheorems pose on the metatheory—namely, that one can build families of sets using parameterized inductive definitions and that one can reason about their elements by induction—are realizable in membership equational logic using parameterization and reflection.

This work can be extended in a number of directions, both theoretical and practical. From the theoretical side, a research line would be to investigate how to reflect induction principles other than structural induction, e.g., induction over an arbitrary, user-definable well-founded order; also, our notion of parameterized membership equational theories and of their instantiations could be further generalized. From the practical side, the obvious application would be to extend the ITP theorem prover [7] with reflected parameterized induction principles so as to carry out inductive proofs of metatheorems; however, the development of the tool has changed hands and gone undercover, so it is not clear how it will evolve.

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References

1. D. Basin, M. Clavel, and J. Meseguer. Reflective metalogical frameworks. *ACM Transactions on Computational Logic*, 5(3):528–576, 2004.
2. D. Basin and S. Matthews. Structuring metatheory on inductive definitions. *Information and Computation*, 162(1/2):80–95, 2000.
3. A. Bouhoula, J.-P. Jouannaud, and J. Meseguer. Specification and proof in membership equational logic. *Theoretical Computer Science*, 236:35–132, 2000.
4. M. Clavel, N. Martí-Oliet, and M. Palomino. Formalizing and proving semantic relations between specifications by reflection. In C. Rattray, S. Maharaj, and C. Shankland, editors, *Algebraic Methodology and Software Technology. 10th International Conference, AMAST 2004, Stirling, Scotland, UK, July 12-16, 2004, Proceedings*, volume 3116 of *Lecture Notes in Computer Science*, pages 72–86. Springer, 2004.

5. M. Clavel and J. Meseguer. Axiomatizing reflective logics and languages. In G. Kiczales, editor, *Proceedings of Reflection'96, San Francisco, California, April 1996*, pages 263–288, 1996.
6. M. Clavel, J. Meseguer, and M. Palomino. Reflection in membership equational logic, many-sorted equational logic, Horn-logic with equality, and rewriting logic. *Theoretical Computer Science*, 373(1-2):70–91, 2007.
7. M. Clavel, M. Palomino, and A. Riesco. Introducing the ITP tool: A tutorial. *Journal of Universal Computer Science*, 12(11):1618–1650, 2006. Special issue with extended versions of selected papers from PROLE 2005: The Fifth Spanish Conference on Programming and Languages.
8. S. Feferman. Finitary inductively presented logics. In R. Ferro, C. Bonotto, S. Valentini, and A. Zanardo, editors, *Logic Colloquium'88*, pages 191–220. North-Holland, 1989.
9. J. Goguen and R. Burstall. Institutions: Abstract model theory for specification and programming. *Journal of the Association for Computing Machinery*, 39(1):95–146, 1992.
10. J. Meseguer. Membership algebra as a logical framework for equational specification. In F. Parisi-Presicce, editor, *Recent Trends in Algebraic Development Techniques, 12th International Workshop, WADT'97, Tarquinia, Italy, June 3 - 7, 1997, Selected Papers*, volume 1376 of *Lecture Notes in Computer Science*, pages 18–61. Springer, 1998.