Ready to Preorder: an Algebraic and General Proof

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Abstract

There have been quite a few proposals for behavioural equivalences for concurrent processes, and many of them are presented in Van Glabbeek's linear time-branching time spectrum. Since their original definitions are based on rather different ideas, proving general properties of them all would seem to require a case-by-case study. However, the use of their axiomatizations allows a uniform treatment that might produce general proofs of those properties. Recently Aceto, Fokkink and Ingólfsdóttir have presented a very interesting result: for any process preorder coarser than the ready simulation in the linear time-branching time spectrum they show how to get an axiomatization of the induced equivalence. Unfortunately, their proof is not uniform and requires a case-by-case analysis. Following the algebraic approach suggested above, in this paper we present a much simpler proof of that result which, in addition, is more general and totally uniform, so that it does not need to consider one by one the different semantics in the spectrum.

Key words: Process algebra, semantic equivalence, semantic preorder, axiomatization, linear-time branching-time spectrum.

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1 Introduction

Most of the popular semantics for concurrent processes appear in Van Glabbeek's linear time-branching time spectrum [Gla01] (see Figure 1). In his famous paper all the semantics in the spectrum are characterized by means of adequate testing scenarios. Each such scenario generates not only an equivalence relation between processes but, even more importantly, a natural preorder relation that has this equivalence as its kernel. Bisimulation semantics [Par81] is the finest of the semantics in the spectrum. It is known to be finer than simulation equivalence due to its symmetric pure coinductive definition. Ready simulation semantics [LS91] appears as a good compromise between bisimilarity and similarity, and has been shown to be the finest semantics having a large collection of desirable properties [BIM95]. In particular, all the semantics in the spectrum that are coarser than ready simulation are finitely axiomatizable, [3] and this is true both for the equivalences and the preorders that define them.

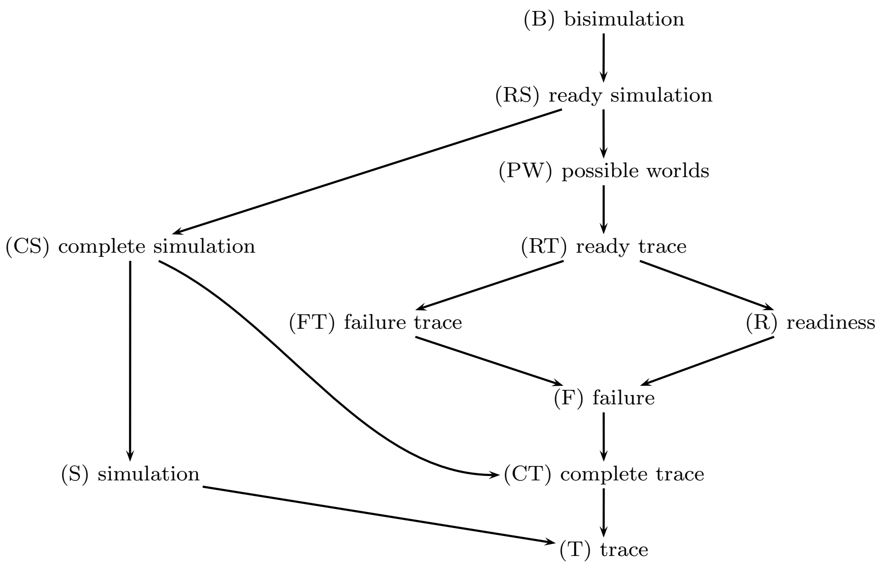

Fig. 1. Axiomatic semantics in the linear time-branching time spectrum.

Besides, we have recently proved that semantic equivalences and preorders coarser, respectively, than the ready simulation equivalence and preorder, also admit coinductive characterizations based on either bisimulations up-to [dFG05] or I-simulations up-to [dFG07]. These two kinds of characterizations have as their main property that of providing uniform definitions instead of the quite different notions that are needed to define the models of each of the semantics; take for instance traces, refusals, ready sets or the quite complex models for the bisimulation semantics [Abr91]. It is true that in some

[3] The one notable exception being ready trace semantics, as shown in [BFN03], although it can still be axiomatized with a simple conditional axiom [Gla01].

cases these explicit models allow one to foresee general properties satisfied either by all of the semantics, or by a suitable class, but considering one by one all these models to obtain those results can become quite boring, and even a bit frustrating, since the fundamental facts that underlie the results cannot be inferred.

Recently Aceto, Fokkink and Ingólfsdóttir [AFI07] have provided a good example to support our claims. They have shown how to obtain an axiomatization for the semantic equivalence defined as the kernel of a given preorder in the linear-time branching-time spectrum, from the axiomatization of that preorder, thus showing a natural connection between the two. But even if their construction seems to be quite general, their proof is based on several results that are proved by considering the semantics in the spectrum one by one. In more detail, they use the concept of *cover equation* [FN04] in order to obtain a kind of basis for the sound equations with respect to a given semantics. The restriction to cover equations allows a (relatively) general proof, by means of rule induction, for the main theorem in [AFI07]. In order to prove that cover equations are indeed sufficient to prove the desired completeness of the defined axiomatization they need a technical result, Lemma 3 in [AFI07], that says that whenever an inequality $t + x \leq u + x$ is sound and x is not a summand of $t + u$, then the inequality $t \leq u$ holds too. The proof of this lemma for the ten different semantics in the spectrum coarser than ready simulation takes up to twenty pages of lengthy reasonings, which are based on the definitions of the semantics.

Our study of the coinductive characterizations of the semantics has shed light on some very simple properties of the reasonable semantics, such as initials preservation and action factorisation, which are the only ones we need in our proofs as hypotheses and are satisfied by all the semantics in the spectrum.

The algebraic characterizations of the semantics have also proved to be very useful when studying the details of the proof of Theorem 1 in [AFI07]. There the application of an inequational axiom in the derivation of any sound inequality can be made under an arbitrary context C, and this gives rise to structural induction in the form of the context. This is a natural downwards induction where the involved context is simplified by removing its root, and then the induction hypothesis is applied. The problem is that under the root of the context there are several children, but its hole is contained in one of them only, and the remaining have to be adequately taken into account. This is why in the original proof cover equations were considered, because it was expected that thanks to their simplicity the desired result could be proved avoiding those technical difficulties. Unfortunately, there is a serious flaw in the arguments supporting the induction step: to be precise, when they claim that the derivation of $E \vdash a\sigma(q_j) \preccurlyeq \sum au_i$ is not longer than the derivation of $E \vdash \sum a\sigma(q_j) \precsim \sum au_i$, which cannot be the case.[4] We tried to use the algebraic characterization of the semantics to obtain a new simpler proof of the theorem and we found that the key idea was to reverse that structural induction, which means to enlarge the hole of the context to reduce the depth at which it appears. This is performed by including into the hole the node over it and all of its descendants. This produces a more elegant proof, which avoids the necessity to restrict ourselves to cover equations, and whose main fundamental fact is the closure property expressed by the simple conditional axiom $t \precsim u \Rightarrow b(t+x) \precsim b(u+x)$ that generates the axiom $b(t+x) + b(u+x) \approx b(u+x)$. As a matter of fact, Aceto, Fokkink and Ingólfsdóttir already cleverly discovered this was the key to constructing the axiomatization of the equivalence induced by a behaviour preorder starting from the axiomatization of this preorder, but it seems they did not notice that it could be exploited to get a direct proof of the completeness of the axiomatization so obtained.

The rest of the paper is structured as follows. Section 2 contains basic notions and results taken from previous works, and introduces the notation. Next, in Section 3 we present the algorithm by Aceto, Fokkink, and Ingólfsdóttir in [AFI07] that has inspired this paper. Section 4 is devoted to our own presentation of that algorithm and the algebraic results that lead us to a simpler proof of its correctness. An enhanced version of the algorithm producing a simpler axiomatization is studied in Section 5. We show that the enhanced algorithm is not valid for all the semantics under consideration and introduce slight variants that cover the different classes of semantics in the spectrum. We conclude the paper with a brief conclusion and some hints on future and (very recent) related work.

Acknowledgments We would like to thank Aceto, Fokkink, and Ingólfsdóttir for very interesting discussions, both through email and during a brief meeting in Paris. They also helped us polish previous drafts of this paper and encouraged us to present our results as soon as possible. We also thank the two anonymous referees, who wrote very detailed reports pointing out some mistakes and providing comments that have allowed a better presentation of some of the results; in particular, they stressed the necessity of presenting in a more accurate way the results about ω-completeness in Section 5.

[4] As a matter of fact, we only discovered this mistake when preparing the full version of this paper; we contacted the authors of [AFI07] and they realized it too. And it seems difficult to fix it except by rearranging somehow the ideas in our new proof. But even so, it is only fair to acknowledge the merit of the discovery of the correct algorithm for the construction of the axiomatization of the induced equivalence relation. Without their insight, we probably would have never found the correct proof and the improved algorithms we present in Section 4.

2 Preliminaries

It is well known that finite processes can be described by means of ground terms of the basic process algebra BCCSP, as introduced for instance in [Gla01].

Definition 1 *Given a set of actions Act, the set of BCCSP processes is defined by the following BNF-grammar:*

$$p ::= \mathbf{0} \mid ap \mid p + q$$

where $a \in Act$. $\mathbf{0}$ represents the process that performs no action; for every action in Act, there is a prefix operator; and + is a choice operator.

Adding variables representing unknown or arbitrary processes we get as usual the corresponding class of open terms.

$$ap \xrightarrow{a} p \qquad \frac{p \xrightarrow{a} p'}{p + q \xrightarrow{a} p'} \qquad \frac{q \xrightarrow{a} q'}{p + q \xrightarrow{a} q'}$$

Fig. 2. Operational Semantics for BCCSP Terms

Since the operations of the language BCCSP naturally describe the branching structure of transition systems, BCCSP terms are arguably the simplest way to represent (finite) non-deterministic processes.

Many different semantics for these non-deterministic processes have been defined in the literature. The most important and popular semantics appear in Van Glabbeek's spectrum [Gla01] where we refer the reader for the original definitions, either in terms of simulations or decorated traces. One indirect way to capture any semantics is by means of the equivalence relation induced by it: given a formal semantics $[\![\cdot]\!]$, we say that processes p and q are equivalent iff they have the same semantics, that is, $p \equiv q \Leftrightarrow [\![p]\!] = [\![q]\!]$. We are interested in compositional semantics that can be defined in a modular way. Compositionality corresponds to the fact that the equivalence is a congruence with respect to the operators in the syntax (prefix and choice in our case).

Any semantics can be characterized in many other alternative ways. Possibly, the most popular ones are the testing or observational scenarios and the logical characterizations. In the first case we have a family of tests and an adequate definition for the passing of tests, which depends on the underlying semantics. We say that two terms are equivalent iff they pass the same tests. To define a logical semantics we have a family of formulas and the corresponding satisfaction relation. Then, two terms are equivalent iff they satisfy exactly the same set of formulas. But whenever we have these two kinds of characterizations, we obtain for free another indirect way to characterize the semantics by means of a preorder. We write $p \leq q$ iff q passes each test that is passed by p or, in the second case, when q satisfies any formula satisfied by p.

Another way to characterize a semantics is by means of the (in)equational axiomatization of the equivalence or preorder [5] that defines it. Finite axiomatizations for most of the popular semantics over BCCSP are well known: see Table 1 borrowed from [Gla01]. It has also been proved that some less popular semantics, such as the nested semantics or possible futures, do not have such a finite axiomatization [AFI01,AFGI04].

	B	RS	PW	RT	FT	R	F	CS	CT	S	T
$(x+y)+z = x+(y+z)$	+	+	+	+	+	+	+	+	+	+	+
$x+y = y+x$	+	+	+	+	+	+	+	+	+	+	+
$x+0 = x$	+	+	+	+	+	+	+	+	+	+	+
$x+x = x$	+	+	+	+	+	+	+	+	+	+	+
$ax \sqsubseteq ax+ay$		+	+	+	+	+	+				
$a(bx+by+z) = a(bx+z)+a(by+z)$			+								
$I(x) = I(y) \Rightarrow ax+ay = a(x+y)$				+							
$ax+ay \sqsupseteq a(x+y)$					+						
$a(bx+u)+a(by+v) \sqsupseteq a(bx+by+u)$						+					
$ax+a(y+z) \sqsupseteq a(x+y)$							+				
$ax \sqsubseteq ax+y$								+	+		
$a(bx+u)+a(cy+v) = a(bx+cy+u+v)$									+		
$x \sqsubseteq x+y$										+	+
$ax+ay = a(x+y)$											+

Table 1
Axiomatizations for some of the preorders in Van Glabbeek's spectrum.

Once we have recalled the axiomatizations of the preorders that define the semantics that are coarser than ready simulation in the linear time-branching time spectrum, we have preferred to omit the original definitions of such semantics. The reason is to stress the fact that the results in this paper are valid for a wide class of semantics, characterized by some simple properties to be detailed below, and of which those in [Gla01] are but a sample.

When we have said that BCCSP terms correspond to finite tree-like transitions systems we have already assumed a non-trivial semantics, namely that defined by bisimilarity, which can be axiomatically defined by the four axioms [6] in Figure 3, which are satisfied by any reasonable preorder. These axioms also justify the use of the notation $\sum_a \sum_i ap_a^i$ for processes, where the commutativity and associativity of the choice operator is used to group together the summands whose initial action is a, and we avoid the explicit notation of the finite set

[5] As for equivalences, the interesting preorders should also be preserved by the operators in the signature, so that they become precongruences.
[6] To be precise we are assuming the commutativity, associativity and null element of choice, while we could still distinguish P from $P + P$.

of offered actions and the finite sets I_a where i ranges for each offered action a. Besides, we interpret as usual that an empty sum stands for the neutral element $\mathbf{0}$. We will also write $p|_a$ for the (sub)process we get by adding all the a-summands of p. That is, if $p = \sum_a \sum_i ap_a^i$, then $p|_a = \sum_i ap_a^i$. And we write $I(p) = \{a \mid \exists p \xrightarrow{a} p_a^i\}$ for the set of actions offered by p.

$$
\begin{array}{ll}
(B_1) & x + y \approx y + x \\
(B_2) & (x + y) + z \approx x + (y + z) \\
(B_3) & x + x \approx x \\
(B_4) & x + \mathbf{0} \approx x
\end{array}
$$

Fig. 3. Axiomatisation for the (Strong) Bisimulation Equivalence

The fact that an equivalence (resp. preorder) is a congruence (resp. precongruence) can be equivalently captured by means of contexts. A context $\mathcal{C}[\cdot]$ is an extended BCCSP term where we allow one occurrence of the new constant $[\cdot]$ that represents a hole. For every context $\mathcal{C}[\cdot]$ and term p we write $\mathcal{C}[p]$ for the term that results by placing p in the hole in $\mathcal{C}[\cdot]$, which can be formally defined by structural induction on the form of the context $\mathcal{C}[\cdot]$.

For the sake of clarity we follow the same notation as in [AFI07] for preorders and equivalences. We use $\precsim$ to denote a semantic preorder and $\simeq$ to denote the corresponding equivalence (that is, $\precsim \cap \precsim^{-1}$). To refer to a specific preorder in the linear time-branching time spectrum we shall append the initials of the intended semantics as subscripts to symbol $\precsim$ ($\precsim_{RS}$ for ready simulation, $\precsim_F$ for failures and so on). A similar convention applies to the kernels of the preorders ($\simeq_{RS}$, $\simeq_F$, ...) and to the bisimulation equivalence $\simeq_B$. For the inequations and equations in the axiomatizations we use, respectively, the symbols $\preccurlyeq$ and $\approx$. We write $E \vdash t \preccurlyeq u$ or $E \vdash t \approx u$ for the (in)equations that can be derived from the (in)equations in E using the standard rules of (in)equational logic, where the symmetry rule can be applied in the equational derivations, but not in the inequational ones.

An axiomatization E is sound modulo $\precsim$ (resp. $\simeq$) if, for all open terms t and u, $E \vdash t \preccurlyeq u$ (resp. $E \vdash t \approx u$) implies that $t \precsim u$ (resp. $t \simeq u$). An axiomatization E is ground complete modulo $\precsim$ (resp. $\simeq$) if $p \precsim q$ (resp. $p \simeq q$) implies $E \vdash p \preccurlyeq q$ (resp. $E \vdash p \approx q$) for all ground terms p and q. We say that E is ω-complete if, whenever $E \vdash \rho(t) \preccurlyeq \rho(u)$ (resp. $E \vdash \rho(t) \approx \rho(u)$) for all ground substitutions ρ, then $E \vdash t \preccurlyeq u$ (resp. $E \vdash t \approx u$).

As said above, we are mainly interested in preorders that are coarser than bisimilarity (i.e. satisfying the axioms of bisimulation equivalence) and that are precongruences with respect to the prefix and choice operators. We will call them behaviour preorders.

Definition 2 *A preorder relation* $\precsim$ *over BCCSP processes is a* behaviour preorder *if*

- *it is weaker than bisimilarity, i.e.* $p \simeq_B q \Rightarrow p \precsim q$*, and*
- *it is a precongruence with respect to the prefix and choice operators, i.e. if* $p \precsim q$ *then* $ap \precsim aq$ *and* $p + r \precsim q + r$*, for each process* r*.*

There are also two quite simple properties that will be fulfilled by most of the interesting behaviour preorders. We have already used them in previous works (for instance [dFG05]) and they will be considered in the following developments.

Definition 3 *A behaviour preorder* $\precsim$ *is* initials preserving *when* $p \precsim q$ *implies* $I(p) \subseteq I(q)$*. It is* action factorised *(or just* factorised*) when* $p \precsim q$ *implies* $p|_a \precsim q|_a$*, for all* $a \in I(p)$*.*

To be precise, all the preorders in the linear time-branching time spectrum are action factorised, while exactly any congruence finer than the trace preorder (including all those in the linear time-branching time spectrum) is initials preserving.

3 The Original Algorithm and its Proof

Recently, Aceto, Fokkink and Ingólfsdóttir [AFI07] have studied the problem of relating the axiomatization of a preorder to that of its induced equivalence. In more detail, given an axiomatization for a preorder they have defined an algorithm to obtain an axiomatization for its kernel.

Algorithm ([AFI07])

Consider a preorder $\precsim$ in the linear time-branching time spectrum coarser than the ready simulation preorder. Let E be a sound and complete inequational axiomatization for BCCSP modulo $\precsim$ that contains the ready simulation axiom:

$$ax \preccurlyeq ax + ay$$

for each $a \in A$. Then the axiomatization $\mathcal{A}(E)$ is constructed as follows. The axioms B_1–B_4 are included in $\mathcal{A}(E)$. Furthermore, for each inequational axiom $t \preccurlyeq u$ we add to $\mathcal{A}(E)$:

A. $t + u \approx u$; and
B. $b(t+x) + b(u+x) \approx b(u+x)$ (for all $b \in A$, and some x that does not occur in $t + u$).

The following theorem states that the axiomatization $\mathcal{A}(E)$ obtained by the application of the algorithm is indeed sound and (ω-)complete for the equivalence relation $\simeq$.

Theorem 1 ([AFI07]) *Let $\precsim$ be a preorder in the linear time-branching time spectrum that satisfies $\precsim_{RS} \subseteq \precsim$. Let E be a sound and ground complete inequational axiomatization for BCCSP terms modulo $\precsim$. Then the equational axiomatization $\mathcal{A}(E)$ is sound and ground complete for BCCSP(A) modulo $\simeq$. Moreover, if E is ω-complete, then so is $\mathcal{A}(E)$.*

Let us comment on the original proof of this theorem. The authors in [AFI07] start by claiming that the soundness of $\mathcal{A}(E)$ can be easily, but tediously, checked for each of the axiomatizations E of the preorders in the linear time-branching time spectrum coarser than the ready simulation. This is the first place where our algebraic approach produces a shorter and much more general proof as we are going to see in Section 4. The key idea is that instead of proving a statement individually for each preorder in the spectrum, we find the adequate general properties of these preorders that allow one to obtain a single general proof of the result.

As for the completeness of the axiomatization produced by the algorithm ($t \simeq u \Rightarrow \mathcal{A}(E) \vdash t \approx u$) the authors claim that they can restrict themselves to a very particular kind of equalities $t \simeq u$: those in which $u = \sum au_i$ and $t = at' + u$. Besides, they assume that *"the inequational axiomatization E that we start with can be pre-processed so that there are no multiple a-summands on the left-hand sides of the inequational axioms in E"*. Then the proof proceeds using this (new!) axiomatization, which is certainly equivalent to the original one, but it is far from trivial to prove that the application of the algorithm to both of them produces equivalent axiomatizations. So what is actually proved in [AFI07] is only that for any axiomatization E there exists an equivalent computable[7] axiomatization E' such that $\mathcal{A}(E')$ is a complete axiomatization of the kernel of E.

Moreover, a further simplification of the given set of axioms is assumed: *"the inequational axiomatization E that we start with only contains inequational axioms of the form $ap \preccurlyeq \sum_{i=1}^{n} aq_i$ (with $n \geq 1$) or $\mathbf{0} \preccurlyeq q$"*. This simplification is based on their Lemma 3 that we are going to comment on in detail below and that allows to reduce the kind of equalities they need to prove to be derivable.

The proof of Theorem 1 concludes by showing the derivability of the equations $at + \sum_{i=1}^{n} au_i \approx \sum_{i=1}^{n} au_i$. To do it, first each of the semantics is considered separately again, to conclude that in each case $at \precsim \sum_{i=1}^{n} au_i$. Then the com-

[7] Computable in the sense that there is a finitary procedure to obtain for each axiom in E a finite set of axioms that, considered together, constitute E'.

pleteness of E is used to obtain $E \vdash at \preccurlyeq \sum_{i=1}^{n} au_i$, and the rest of the proof proceeds by induction on the derivation of this inequality.

Lemma 1 (Lemma 3 in [AFI07]) *Let $\precsim$ be a preorder in the linear time-branching time spectrum. If $t + x \precsim u + x$ and x is not a summand of $t + u$, then $t \precsim u$.*

Let us note that they need this lemma even for proving ground completeness. Although in this case no variables appear in the equations to be derived, variables could still appear in the axiomatization and they should be removed in the preprocessing of the set of axioms when appearing as summands in both sides of an inequality.

The proof of this lemma takes a full appendix with eighteen pages of lengthy reasonings based on the concrete characteristics of each of the nine semantics considered, plus two more pages for the case (failures semantics) that was already proved in [FN05].

As a summary, the main weaknesses of the results presented and proved in [AFI07] are the following ones:

- The correctness of the axioms produced by the algorithm is proved separately for each of the semantics in the spectrum.
- Lemma 1, that plays an essential role in the proof of Theorem 1, is proved in a separate and ad hoc manner for each one of the ten semantics in the linear time-branching time spectrum considered.
- As a consequence, Theorem 1 only applies to the semantics in the linear time-branching time spectrum.
- Besides, the proof of Theorem 1 presented in [AFI07] requires the given axiomatization to have the adequate form and therefore Theorem 1 is not totally proved.
- Finally, the inductive argument in the proof of that theorem is not correct and cannot to be easily turned into a correct one.

In the next section we present our alternative approach which has the following advantages:

- Our Proposition 1 to follow allows to prove in a general way the correctness of the axiomatization generated by the algorithm for all the semantics that fulfill some quite natural and simple properties, including all those in the linear time-branching time spectrum.
- Theorem 2 generalizes in a nice way Theorem 1, and its general proof only uses the simple properties mentioned above, combining them by means of algebraic arguments that are valid for all the semantics fulfilling those properties.

Moreover, we expect that the algebraic arguments in our proofs and the simple basic properties that we use in them will be applicable to prove some other general properties of the semantics, without having to develop a different proof for each one of them.

4 Our General Theorem and its Proof

Let us try to explain the ideas and intuitions that make the algorithm above work. The first part, (A), introduces the simple axiom $t+u \approx u$ for every $t \preccurlyeq u$ in E. We can justify this clause of the algorithm by means of the following result that we proved in [dFG07].

Proposition 1 *For every behaviour preorder $\precsim$ that satisfies $\precsim_{RS} \subseteq \precsim$ and is initials preserving, we have that $p \precsim q \Rightarrow q \simeq q + p$.*

Proof. We prove the result for ground terms p and q; the result immediately follows for open terms.

If $p \precsim q$ then, since $\precsim$ is a precongruence with respect to the choice operator, $p + q \precsim q + q$ and thus $p + q \precsim q$.

To prove $q \precsim p + q$ it is enough to show that $q|_a \precsim q|_a + p|_a$, since $\precsim$ is a precongruence wrt the choice operator, and if $p \precsim q$ then $I(p) \subseteq I(q)$. But for all $a \in I(q)$ we have $q|_a \precsim q|_a + p|_a$, because $\precsim$ satisfies the (RS) axiom that characterises the ready simulation preorder. □

This result not only justifies the correctness of the axiom for the preorders in the linear time-branching time spectrum that are coarser than ready simulation, but in general for any initials preserving behaviour preorder coarser than $\precsim_{RS}$.

But, why is the second axiom $b(t + x) + b(u + x) \approx b(u + x)$ also needed? We are going to argue below that this is mainly because of a *technical* reason, although certainly we need the inclusion of the axiom into $\mathcal{A}(E)$ to make the algorithm work properly in all cases.

One could be tempted to conclude that this second axiom is just a particular case of the first one, that we write now as $t' + u' \approx u'$ to avoid name collision, just by substituting t' by $b(t+x)$ and u' by $b(u+x)$. But this is not really the case, because the identifiers t' and u' were not free variables but metavariables representing any two terms for which we have $t' \preccurlyeq u' \in E$.

Anyway, the interesting point is that the correctness of the second axiom can be indeed obtained as an immediate corollary of our Proposition 1, because if $t \preccurlyeq u \in E$ then we can infer $t + x \precsim u + x$ and $b(t + x) \precsim b(u + x)$, since E is a sound axiomatisation of the behaviour preorder $\precsim$. Then we can apply Proposition 1 to obtain the correctness of the axiom $b(t + x) + b(u + x) \approx b(u + x)$.

After the above discussion, we consider more adequate to present the algorithm for constructing the axiomatization for the induced equivalence in a slightly different way. The new presentation stresses the importance of the first axiom and better shows the role of the construction involved in the second one. We will benefit from these two facts in the general proof that we present later in this section.

Definition 4 *Let E be an inequational axiomatization for BCCSP(A) terms modulo $\precsim$. We define its* BCCSP-context closure $\overline{E}$ *as*

$$\overline{E} = E \cup \{b(t + x) \preccurlyeq b(u + x) \mid t \preccurlyeq u \in E\}$$

where b represents a generic action in A, and x is a process variable not appearing in E.

Since we will only consider finite inequational axiomatizations, such a new variable x always exists. Whenever E is finite and non-empty we have that $\overline{\overline{E}} \neq \overline{E}$. Even so, the result in Proposition 2 below justifies that we call this construction a closure.

Our algorithm

Given an inequational system of axioms E defining a preorder $\preccurlyeq$ on BCCSP(A), we define the axiomatization $\mathcal{A}(E)$ as follows:

- Axioms B_1–B_4 are in $\mathcal{A}(E)$.
- For each axiom $t \preccurlyeq u \in \overline{E}$ we have $u \approx t + u \in \mathcal{A}(E)$.

It is clear that for any inequational axiomatization E this algorithm produces exactly the same set of axioms as the one proposed in [AFI07] that we recalled above.

The correctness of the axiomatization generated by the algorithm follows immediately from Proposition 1. As for the (ω-)completeness of $\mathcal{A}(E)$, it is enough to prove that $\mathcal{A}(E) \vdash u' \approx u' + t'$, whenever we have $t' \preccurlyeq u'$. The key idea is that the set of derivable equations is essentially formed by the equations $\mathcal{C}[u] \approx \mathcal{C}[u] + \mathcal{C}[t]$ generated from the inequalities $t \preccurlyeq u \in E$ and any arbitrary context $\mathcal{C}[\cdot]$. Then the role of the construction $b(\cdot + x)$ used to define $\overline{E}$ is that of a nearly universal context. As we will show below, this is because it combines the two operators in the syntax of BCCSP, so that by iterating that context and instantiating the introduced variables in an arbitrary way, we could generate any context $\mathcal{C}[\cdot]$ that contains some prefix operator above its hole.

We foresaw that if we could close any construction under this particular context, preserving the property in which we were interested, then a simple reasoning by induction would prove that the property was preserved by arbitrary contexts. This indeed works to prove the following extension of Theorem 1 that reads now as follows.

Theorem 2 *Let $\precsim$ be an initials preserving behaviour preorder that satisfies $\precsim_{RS} \subseteq \precsim$. Let E be a sound and (ω-)complete inequational axiomatization for BCCSP terms modulo $\precsim$ containing the axiom (RS). Then the equational axiomatization $\mathcal{A}(E)$ is sound and (ω-)complete for BCCSP modulo $\simeq$.*

In order to prove this theorem we first present the following proposition that formalizes the closure character of the construction $\overline{E}$ with respect to the transformation $\mathcal{A}$ defined by the algorithm above.

Proposition 2 *For any inequational axiomatization that contains the axiom (RS) we have that $\mathcal{A}(E) \vdash \mathcal{A}(\overline{E})$.*

Proof. $\mathcal{A}(\overline{E})$ contains the set of equations generated by the inequations in $\overline{\overline{E}} = \overline{E} \cup \{c(b(t+x)+y) \preccurlyeq c(b(u+x)+y) \mid t \preccurlyeq u \in E\}$. Therefore, we need to prove that for any $t \preccurlyeq u \in E$ we have $\mathcal{A}(E) \vdash c(b(u+x)+y) \approx c(b(u+x)+y)+c(b(t+x)+y)$. This is indeed the case because $b(u+x) \approx b(u+x)+b(t+x) \in \mathcal{A}(E)$ and then $\mathcal{A}(E) \vdash c(b(u+x)+y) \approx c(b(u+x)+b(t+x)+y)$. Now, by applying the ready simulation equivalence axiom generated by (RS), we have $\mathcal{A}(E) \vdash c(b(u+x)+b(t+x)+y) \approx c(b(u+x)+b(t+x)+y)+c(b(t+x)+y)$ and thus $\mathcal{A}(E) \vdash c(b(u+x)+y) \approx c(b(u+x)+y)+c(b(t+x)+y)$. □

We can then conclude that $\mathcal{A}(E) \vdash \mathcal{A}(\overline{E}^n)$, where $\overline{E}^n$ represents the n-iterated application of our closure operator, thus justifying its name. This is sufficient to extend the result to any arbitrary context.

Corollary 1 *Let E be an axiomatization that contains the axiom (RS): for any $t \preccurlyeq u \in E$ and any context $\mathcal{C}[\cdot]$ we have $\mathcal{A}(E) \vdash \mathcal{C}[u] \approx \mathcal{C}[u] + \mathcal{C}[t]$.*

Proof. By induction on the number n of prefix operators above the hole in the context $\mathcal{C}[\cdot]$, proving the result for all the axiomatizations that verify the provisos at the same time.

If $n = 0$ then we have $\mathcal{C}[\cdot] = \cdot + v$, where v is an arbitrary term, possibly $\mathbf{0}$. We have to prove that $\mathcal{A}(E) \vdash u + v \approx (u + v) + (t + v)$. Since $t \precsim u \in E$, we have $\mathcal{A}(E) \vdash u \approx u + t$, and therefore $\mathcal{A}(E) \vdash u + v \approx (u + v) + (t + v)$.

If $n > 0$ we can write $\mathcal{C}[\cdot]$ as $\mathcal{C}'[b(\cdot + v)]$ where, again, v could be the term $\mathbf{0}$. Since $b(t + x) \precsim b(u + x) \in \overline{E}$, by applying the induction hypothesis to $\overline{E}$ and $\mathcal{C}'[\cdot]$ we have $\mathcal{A}(\overline{E}) \vdash \mathcal{C}'[b(u + x)] \approx \mathcal{C}'[b(u + x)] + \mathcal{C}'[b(t + x)]$; then instantiating x with v and applying Proposition 2, we conclude that $\mathcal{A}(E) \vdash \mathcal{C}[u] \approx \mathcal{C}[u] + \mathcal{C}[t]$. $\square$

We have stated this result as a corollary because we want to stress the fact that the essential point in the proof of the result is the closure property of $\overline{E}$, captured by the result in Proposition 2. However, it is true that, technically, the rest of the proof of Corollary 1 is far from trivial for two reasons: we do not use structural (bottom-up) induction over the context, but instead a kind of top-down induction: In order to apply the induction hypothesis the context is reduced by removing one of it subtrees (that starting at the b-node corresponding to the decomposition $\mathcal{C}[\cdot] = \mathcal{C}'[b(\cdot + v)]$). Besides, we need to prove the result for all the possible axiomatizations at the same time, which allows one to use the result for $\mathcal{A}(\overline{E})$ when needed.

We can now prove the following proposition, that is the core of the (ω-)completeness part of the proof of Theorem 2.

Proposition 3 *Let E be a set of inequational axioms containing the axiom (RS), that defines the preorder $\precsim$. Then, for any derivable inequation $E \vdash t \precsim u$, we also have $\mathcal{A}(E) \vdash u \approx u + t$.*

We could prove this result by induction on the derivation of $E \vdash t \precsim u$, but we prefer to present it in a coinductive way, introducing the following invariant result.

Lemma 2 *Let E be a set of inequational axioms containing the axiom (RS), that defines the preorder $\precsim$. Let E' with $E \subseteq E'$ that satisfies*

$$\mathcal{A}(E) \vdash \mathcal{C}[u] \approx \mathcal{C}[u] + \mathcal{C}[t] \quad \forall\, t \precsim u \in E' \;\; \forall\, \mathcal{C}[\cdot].$$

Then, whenever we have $E' \vdash_1 t' \precsim u'$, where $\vdash_1$ defines the set of inequations that can be derived from E' by the application of a single derivation rule, we also have $\mathcal{A}(E) \vdash \mathcal{C}[u'] \approx \mathcal{C}[u'] + \mathcal{C}[t']$ for any context, which implies that the set $E' \cup \{t' \precsim u'\}$ satisfies the same property we required of E'.

Proof. Let us consider the different rules of the inequational calculus:

Reflexivity: For each derivation $E' \vdash_1 u' \precsim u'$ we have to prove that $\mathcal{A}(E) \vdash \mathcal{C}[u'] \approx \mathcal{C}[u'] + \mathcal{C}[u']$ for all $\mathcal{C}[\cdot]$, but this is an immediate application of axiom

(B_3).

Instantiation: $t \preccurlyeq u \in E'$, $t' = \sigma(t)$, $u' = \sigma(u)$, for a given substitution σ. By hypothesis, $\mathcal{A}(E) \vdash \mathcal{C}[u] \approx \mathcal{C}[u] + \mathcal{C}[t]$ for any context $\mathcal{C}[\cdot]$ and we have to check that $\mathcal{A}(E) \vdash \mathcal{C}[\sigma(u)] \approx \mathcal{C}[\sigma(u)] + \mathcal{C}[\sigma(t)]$; this is not obvious since a variable could appear both in u and $\mathcal{C}[\cdot]$ and, in general, $\mathcal{C}[\sigma(u)] \neq \sigma(\mathcal{C}[u])$.

But, if we replace each variable x in $\mathcal{C}[\cdot]$ by a new variable x' that does not appear either in u or in t, we obtain a new context $\mathcal{C}'[\cdot]$ for which we also have $\mathcal{A}(E) \vdash \mathcal{C}'[u] \approx \mathcal{C}'[u] + \mathcal{C}'[t]$. Taking as σ' the substitution that extends σ with $\sigma'(x') = x$, for each new variable x', we obtain $\mathcal{A}(E) \vdash \sigma'(\mathcal{C}'[u]) \approx \sigma'(\mathcal{C}'[u]) + \sigma'(\mathcal{C}'[t])$, and $\sigma'(\mathcal{C}'[u]) = \mathcal{C}[\sigma(u)]$ and $\sigma'(\mathcal{C}'[t]) = \mathcal{C}[\sigma(t)]$, so that we conclude $\mathcal{A}(E) \vdash \mathcal{C}[u'] \approx \mathcal{C}[u'] + \mathcal{C}[t']$, as desired.

Substitution: $t \preccurlyeq u \in E'$, $\mathcal{C}'[\cdot]$ is an arbitrary context. We need to prove that for any context $\mathcal{C}[\cdot]$ we have $\mathcal{A}(E) \vdash \mathcal{C}[\mathcal{C}'[u]] \approx \mathcal{C}[\mathcal{C}'[u]] + \mathcal{C}[\mathcal{C}'[t]]$, but this is obvious since the composition of contexts is a new context.

Transitivity: $t' \preccurlyeq v'$ and $v' \preccurlyeq u'$ are both in E'. Then we have $\mathcal{A}(E) \vdash \mathcal{C}[v'] \approx \mathcal{C}[v'] + \mathcal{C}[t']$ and $\mathcal{A}(E) \vdash \mathcal{C}[u'] \approx \mathcal{C}[u'] + \mathcal{C}[v']$, and therefore $\mathcal{A}(E) \vdash \mathcal{C}[u'] \approx \mathcal{C}[u'] + \mathcal{C}[v'] + \mathcal{C}[t']$, and finally $\mathcal{A}(E) \vdash \mathcal{C}[u'] \approx \mathcal{C}[u'] + \mathcal{C}[t']$. □

Proof. [Proposition 3] For each natural number k we consider the set E_k of inequations that can be derived from E by means of a derivation tree of depth k. We prove by induction on k that all the sets E_k satisfy the invariant in Lemma 2. For $k = 0$ we have $E_0 = E$, and then we only need to apply Corollary 1 to obtain the desired result. Given $k + 1$, if $t \preccurlyeq u \in E_{k+1}$ we can infer it from E_k by the application of a single rule. By application of the induction hypothesis we know that E_k satisfies the invariant in Lemma 2, and then the application of the lemma shows that the set E_{k+1} also satisfies it. Now, if $E \vdash t \preccurlyeq u$ we have $t \preccurlyeq u \in E_k$ for some natural number k. So that we have $\mathcal{A}(E) \vdash \mathcal{C}[u] \approx \mathcal{C}[u] + \mathcal{C}[t]$ for all $\mathcal{C}[\cdot]$. And taking the trivial identity context, we obtain in particular $\mathcal{A}(E) \vdash u \approx u + t$. □

It is quite interesting to observe that in order to conclude the thesis of Proposition 3 we needed to prove a much more general result, namely the derivability of $\mathcal{C}[u] \approx \mathcal{C}[u] + \mathcal{C}[t]$. This gave us much more power when developing our proofs, and is obviously related with the modularity of the algebraic arguments, which provide nice symmetric and scalable rules. But at the same time the reader should notice that this general result is far from trivial, since in general $\mathcal{C}[u + t] \neq \mathcal{C}[u] + \mathcal{C}[t]$. However, one nice consequence of the proved results is that under the hypothesis of Proposition 3 this is indeed the case: whenever $t \preccurlyeq u$ we also have $\mathcal{C}[u + t] \approx \mathcal{C}[u] + \mathcal{C}[t]$.

By using Proposition 1 and Proposition 3 we have now an immediate proof of Theorem 2.

Proof. [Theorem 2]

Correctness. This is just our Proposition 1.

Completeness. Let $t \simeq u$, with t and u ground (resp. open) terms, which means that $t \precsim u$ and $u \precsim t$. Since E is (ω-)complete for $\precsim$ we have both $E \vdash t \preccurlyeq u$ and $E \vdash u \preccurlyeq t$. Then, by applying Proposition 3 we have both $\mathcal{A}(E) \vdash u \approx u + t$ and $\mathcal{A}(E) \vdash t \approx t + u$ and therefore $\mathcal{A}(E) \vdash u \approx t$. □

Theorem 1, which was the main result in [AFI07], can now be obtained as a particular case of our Theorem 2 since all the preorders in the linear time-branching time are initials preserving.

5 An Enhanced version of the Algorithm

In Section 6 of [AFI07] several examples were presented comparing the axiomatizations resulting from their algorithm with those already known from the literature. In particular, for the failures semantics it was proved that the axiom

$$b(a(x+y)+w) + b(ax + a(y+z) + w) \approx b(ax + a(y+z) + w)$$

generated by closing the characteristic axiom for the failures preorder [Gla01]

$$(F) \qquad a(x+y) \preccurlyeq ax + a(y+z)$$

with respect to the universal context, can be inferred from the ready similarity axiom

$$(RS_{\equiv}) \qquad b(ax+ay+z) \approx b(ax+ay+z) + b(ax+z)$$

obtained by the application of the universal context to the ready simulation axiom (RS) and the axiom resulting from applying step (A) of the algorithm to the failures preorder axiom (F). Inspired by this result, and particularly by its proof in [AFI07], we have proved that the following simplified algorithm also produces a complete axiomatization in many cases.

The simplified algorithm

Given an inequational system of axioms E defining a preorder $\preccurlyeq$ on BCCSP(A), we define the axiomatization $\mathcal{A}_{RS}(E)$ as follows:

- Axioms B_1–B_4 are in $\mathcal{A}_{RS}(E)$.
- For each axiom $t \preccurlyeq u \in E$ we have $u \approx u + t \in \mathcal{A}_{RS}(E)$.

- The ready similarity axiom $(RS_{\equiv})$ is in $\mathcal{A}_{RS}(E)$.

It is quite surprising to discover that, after all, our closure $\overline{E}$ does not appear in this construction. However, as pointed out above, the ready similarity axiom arises from the application of our closure construction to the singleton $\{(RS)\}$. As we will see below, whenever we fall under the quite general hypotheses of the following Theorem 3 we have that from $\mathcal{A}_{RS}(E)$ we can derive any ground instance of an equation in $\mathcal{A}(E)$, so that the rest of the equations generated by the inequalities in $\overline{E}$ become redundant. Therefore, our new algorithm produces an axiomatization $\mathcal{A}_{RS}(E)$ strictly contained in $\mathcal{A}(E)$ whenever $E \neq \{(RS)\}$, and that is why we say that it is a simplified algorithm.

Theorem 3 *Let $\precsim$ be a behaviour preorder that satisfies $\precsim_{RS} \subseteq \precsim \subseteq I$, where I is the equivalence relation defined by $p\ I\ q \Leftrightarrow I(p) = I(q)$. Let E be a sound and ground complete inequational axiomatization for BCCSP terms modulo $\precsim$. Then the equational axiomatization $\mathcal{A}_{RS}(E)$ is sound and ground complete for BCCSP modulo $\simeq$.*

Proof. We only need to prove that for all $t \preccurlyeq u \in E$ and any ground substitution ρ, we have to show that $\mathcal{A}_{RS}(E) \vdash b\rho(u+z) \approx b\rho(u+z)+b\rho(t+z)$. We have the following chain of equivalences:

$$
\begin{aligned}
b\rho(u+z) &\approx b\rho(u+t+z) && u \approx u+t \in \mathcal{A}_{RS}(E) \\
&\approx b\rho(u+t+z) + b\rho(t+z) && (RS_{\equiv}), I(\rho(u+t+z)) = I(\rho(t+z)) \\
&\approx b\rho(u+z) + b\rho(t+z) && u+t \approx u \in \mathcal{A}_{RS}(E)
\end{aligned}
$$

where we have used the alternative presentation of axiom $(RS_{\equiv})$

$$(I(x) \subseteq I(y)) \Rightarrow b(x+y) \approx b(x+y) + b(y)\,. \qquad \square$$

Corollary 2 *When applied to the classic axiomatizations of their preorders, algorithm $\mathcal{A}_{RS}(E)$ provides a sound and complete axiomatization for BCCSP modulo $\simeq_X$ for all the semantics in the spectrum coarser than ready simulation and finer than failures semantics: that is, for all $X \in \{RS, PW, RT, FT, R, F\}$.*

Proof. All the preorders $\precsim_X$ with $X \in \{RS, PW, RT, FT, R, F\}$ are finer than the relation I. $\square$

To get a nice application of the result above, we could easily check that the obtained axiomatizations and the classic ones in [Gla01] are indeed equivalent, thus getting an immediate indiret proof of the soundness and completeness of the latter.

If the alphabet of actions A is infinite, we will show that our simplified algorithm also preserves ω-completeness. Otherwise, taking $A = \{a_1, \ldots, a_n\}$, in order to have that property we need to add to $\mathcal{A}_{RS}(E)$ the axiom $(F3_n)$ in [FN05]:

$$(F3_n) \qquad a(x + \sum_{i=1}^{n} a_i z_i) + a(x + y + \sum_{i=1}^{n} a_i z_i) \approx a(x + y + \sum_{i=1}^{n} a_i z_i)$$

Definition 5 *If $A = \{a_1, \ldots, a_n\}$ then $\mathcal{A}^{\omega}_{RS}(E) = \mathcal{A}_{RS}(E) \cup \{(F3_n)\}$.*

Proposition 4 *If $A = \{a_1, \ldots, a_n\}$ then $\mathcal{A}^{\omega}_{RS}(E)$ is sound modulo $\precsim_{RS}$.*

Proof. We just need to check that, for any ground substitution ρ, a ready simulation can be constructed that proves that

$$a(\rho(x) + \sum_{i=1}^{n} a_i \rho(z_i)) + a(\rho(x + y) + \sum_{i=1}^{n} a_i \rho(z_i)) \precsim_{RS} a(\rho(x + y) + \sum a_i \rho(z_i))$$

This is immediate by noticing that for any such ρ we have

$$I(\rho(x) + \sum_{i=1}^{n} a_i \rho(z_i)) = A = I(\rho(x + y) + \sum_{i=1}^{n} a_i \rho(z_i)) \qquad \square$$

Theorem 4 *Let $\precsim$ be a behaviour preorder that satisfies $\precsim_{RS} \subseteq \precsim \subseteq I$, where I is the equivalence relation defined by $p\ I\ q \Leftrightarrow I(p) = I(q)$. Let E be a sound and ω-complete inequational axiomatization for BCCSP terms modulo $\precsim$. If A is infinite then the equational axiomatization $\mathcal{A}_{RS}(E)$ is sound and ω-complete for BCCSP modulo $\simeq$ whereas, for A finite, $\mathcal{A}^{\omega}_{RS}(E)$ is sound and ω-complete modulo $\simeq$.*

Proof. We mimic the argument in Theorem 3. Now we have to consider open terms and thus we have to be more precise about the meaning of I when applied to them.

Let us first consider the case in which A is infinite. For $t \preccurlyeq u \in E$ we have to show that $\mathcal{A}_{RS}(E) \vdash b(u + z) \approx b(u + z) + b(t + z)$. We can decompose t as $\sum_{i \in I_t} \sum_j a_i t_{ij} + V_t$ and u as $\sum_{i \in I_u} \sum_j a_i u_{ij} + V_u$, for V_t and V_u summands of variables. Then, considering the null substitution $\rho(x) = \mathbf{0}$ for all x and using the fact that $\precsim \subseteq I$ we obtain $I_t = I_u$. Besides, by taking $\rho(v) = a_v \notin I_t$ where we select a different action for each $v \in V_t \cup V_u$, we conclude that $V_t = V_u$. It is easy then to see that $b(u + t + z) \approx b(u + t + z) + b(t + z)$ can be derived by repeated application of $(RS_{\equiv})$, concluding the proof as in Theorem 3.

If $A = \{a_1, \ldots, a_n\}$, the equality $I_t = I_u$ can be obtained in a similar manner to the case above and the proof would proceed analogously if $I_t \neq A$. But for

$I_t = A$ we cannot infer any relation between V_t and V_u because for any ground substitution we would always have $I(\rho(t)) = A = I(\rho(u))$. However, in this case we can use $(F3_n)$ to derive $\mathcal{A}^{\omega}_{RS}(E) \vdash b(u+t+z) \simeq b(u+t+z)+b(t+z)$ and we conclude as above. $\square$

Since the classic preorders defining the rest of the semantics in the spectrum are not finer than I, we cannot apply Theorem 3 to prove that algorithm $\mathcal{A}_{RS}(E)$ will provide a complete axiomatization for the corresponding equivalences. As a matter of fact, this is not the case as we show below for the simulation semantics.

Proposition 5 *If we consider the classic axiomatization defining the simulation preorder, that is, $E = \{B_1\text{–}B_4\,, (S)\}$, the application of the algorithm $\mathcal{A}_{RS}$ to it does not provide a complete axiomatization of the simulation equivalence $\simeq_S$.*

Proof. Since in this case we have $\mathcal{A}_{RS}(E) = \{B_1\text{–}B_4\,, (RS_\equiv), x+y \approx x+x+y\}$, it is clear that it defines ready similarity, which is strictly stronger than plain similarity. $\square$

Therefore, the fact that $\precsim_S \not\subseteq I$ makes the algorithm fail in this case. Fortunately, we have seen in [dFG07] that all the semantics in the spectrum can be classified in slices, each of them governed by a different kind of simulation. Moreover, in [dFG08] we have developed the full theory of constrained simulations. Although we will refrain from repeating here all the results concerning them, let us recall that given an (appropriate) constraint N relating pairs of processes, N-simulations are just plain simulations included in the set defining N, and that N-similarity can be axiomatized by means of the axioms B_1–B_4 together with (NS) $N(x,y) \Rightarrow x \preccurlyeq x+y$. Inspired by the results in that paper we have conceived and proved the following generalized alternative algorithm which we next show to be adequate, in particular, for the preorders in the spectrum that are not finer than I (for instance, simulation preorder).

The generalized simplified algorithm.

Given an inequational system of axioms E defining a preorder $\preccurlyeq$ on BCCSP(A) and a constraint N relating pairs of processes, we define the axiomatization $\mathcal{A}_{NS}(E)$ as follows:

- Axioms B_1–B_4 are in $\mathcal{A}_{NS}(E)$.
- For each axiom $t \preccurlyeq u \in E$ we have $u \approx u + t \in \mathcal{A}_{NS}(E)$.
- The constrained similarity axiom

$$(NS_\equiv) \qquad N(x,y) \Rightarrow b(x+y) \approx b(x+y) + by$$

is in $\mathcal{A}_{NS}(E)$.

Certainly, the last axiom in the set $\mathcal{A}_{NS}(E)$ is a conditional axiom and in order to obtain an equational axiomatization we should look for an equivalent equational presentation of it, which may not always exist. We will see below that, for the two remaining kinds of simulation in the spectrum, such a presentation does exist. As a matter of fact, this is also the case for ready simulation, which corresponds to the constraint I. In that case the application of the generalized algorithm to the constraint I produces the conditional axiom $I(x) = I(y) \Rightarrow b(x+y) \approx b(x+y) + by$, which is known to be equivalent to the equational axiom $(RS_{\equiv})$, so that we can conclude that the axiomatizations produced by the application of $\mathcal{A}_{RS}(E)$ and the generalized algorithm, $\mathcal{A}_{IS}(E)$, taking $N = I$ above, are equivalent.

The next theorem specifies the conditions under which the generalized simplified algorithm produces a complete axiomatization of the corresponding equivalences.

Theorem 5 *Let $\precsim$ be an initials preserving behaviour preorder and N be a behaviour equivalence that satisfies that $N(x,y)$ implies $x \precsim x+y$, and $\precsim \subseteq N \supseteq I$. Let E be a sound and ground complete axiomatization for BCCSP terms modulo $\precsim$. Then the conditional axiomatization $\mathcal{A}_{NS}(E)$ is sound and ground complete for BCCSP modulo $\simeq$.*

Proof. For correctness we first observe that $I \subseteq N$ implies that $\precsim$ satisfies the axiom (RS), so that all the equivalences $u \approx u+t$ corresponding to axioms $t \preccurlyeq u \in E$ are indeed sound. Next we show that the axiom $N(x,y) \Rightarrow b(x+y) \approx b(x+y)+by$ holds by proving that we have both $b(x+y) \precsim b(x+y)+by$ and $b(x+y)+by \precsim b(x+y)$. The first follows because $\precsim$ satisfies the axiom (RS), while for the second it is enough to show that $N(x,y)$ implies $y \precsim x+y$, which is obvious because N is an equivalence relation, so that $N(x,y)$ implies $N(y,x)$.

For ground completeness we show that for all $t \preccurlyeq u \in E$ and any ground substitution ρ we need to show that $\mathcal{A}_{NS}(E) \vdash b\rho(u+z) \approx b\rho(u+z)+b\rho(t+z)$. We have

$$
\begin{aligned}
b\rho(u+z) &\approx b\rho(u+t+z) && u \approx u+t \in \mathcal{A}_{NS}(E)\\
&\approx b\rho(u+t+z)+b\rho(t+z) && \preccurlyeq \subseteq N, (NS_{\equiv}) \in \mathcal{A}_{NS}(E)\\
&\approx b\rho(u+z)+b\rho(t+z) && u \approx u+t \in \mathcal{A}_{NS}(E)
\end{aligned}
$$

where in the second step we used the fact that $t \preccurlyeq u$ implies $N(\rho(t), \rho(u))$: since N is a behaviour equivalence then we also have $N(\rho(t+z), \rho(u+z))$ and, by applying $(NS_{\equiv})$, we finally obtain $b\rho(u+t+z) \approx b\rho(u+t+z)+b\rho(t+z)$. □

We will not consider ω-completeness in depth in this case. In order to do it we would need to assume that there exists a finite set of equational axioms equivalent to $(NS_{\equiv})$ and then use them as we did with $(RS_{\equiv})$ for the case $N = I$ above. Instead, we show below that for the particular cases $N = C, U$ corresponding to the rest of the semantics in the linear time-branching time spectrum, the application of the general construction $\mathcal{A}_{NS}$ preserves the ω-completeness of axiomatizations.

Let us now see that we can indeed apply this algorithm to the complete and plain simulations that appear in the spectrum. We start with the following immediate results.

Proposition 6 *(1) Plain simulations are just constrained simulations for the trivial universal relation U relating any pair of processes.*

(2) Both the simulation preorder $\precsim_S$ and the trace preorder $\precsim_T$ satisfy the universally constrained simulations axiom

$$U(x, y) \Rightarrow x \preccurlyeq x + y\,,$$

and, trivially, $\precsim\, \subseteq U \supseteq I$, so that Theorem 5 guarantees that $\mathcal{A}_{US}(E_S)$ and $\mathcal{A}_{US}(E_T)$ are, respectively, sound and ground complete for BCCSP modulo $\simeq_S$ and $\simeq_T$.

(3) For each $t \preccurlyeq u \in E$, the conditional axiom $(US_{\equiv})$ is just the equational axiom $(S_{\equiv})$ $b(x+y) \approx b(x+y)+by$, so that by substituting the former by the latter in $\mathcal{A}_{US}(E_S)$ and $\mathcal{A}_{US}(E_T)$ we get two sound and (ω-)complete equational axiomatizations for these two equivalences.

Proof. The only nontrivial result is the one corresponding to ω-completeness but, since U is a trivial constraint, by using $(S_{\equiv})$ in place of $(US_{\equiv})$ we obtain an equational axiomatization for which we can reason as in the proof of Theorem 4. In this case we can always apply $(S_{\equiv})$ to obtain $b(u + t + z) \approx b(u + t + z) + b(t + z)$ and then $\mathcal{A}_{US}(E)$ is ω-complete whenever E is ω-complete. $\square$

Complete simulations are treated similarly. We consider the predicate C defined by

$$C(p, q) ::= (p = \mathbf{0} \iff q = \mathbf{0})\,.$$

Proposition 7 *(1) Both the complete simulation preorder $\precsim_{CS}$ and the complete traces preorder $\precsim_{CT}$ satisfy the complete constrained simulation axiom $C(x, y) \Rightarrow x \preccurlyeq x + y$. Besides, $\precsim_{CS}, \precsim_{CT} \subseteq C \supseteq I$ and thus Theorem 5 guarantees that $\mathcal{A}_{CS}(E_{CS})$ and $\mathcal{A}_{CS}(E_{CT})$ are, respectively, sound and complete for BCCSP terms modulo $\simeq_{CS}$ and $\simeq_{CT}$.*

(2) The conditional axiom $(CS_{\equiv})$ is equivalent to the classical equational ax-

iom characterizing complete similarity,

$$a(x + by + z) \approx a(x + by + z) + a(by + z)\,,$$

so that by substituting the former by the latter in $\mathcal{A}_{CS}(E_{CS})$ and $\mathcal{A}_{CS}(E_{CT})$ we obtain two sound and ω-complete equational axiomatizations for these two equivalences.

Proof.

(1) Let x and y be such that $C(x, y)$. Then we either have $x = \mathbf{0} = y$ and the result is obvious, or $x \neq \mathbf{0} \neq y$. In this last case, $x = \sum_{a\in A}\sum_{i\in I_a} ax_a^i$ with $|\bigcup I_a| > 0$ and we prove the result by induction on $|\bigcup I_a|$:
- $|\bigcup I_a| = 1$. In this case $x = ax_a$ and by applying the axiom that defines the complete simulation preorder we have $ax_a \precsim ax_a + y$.
- $|\bigcup I_a| > 1$. Let us distinguish one of the summands of x to get $x = ax_a + x'$. By the induction hypothesis we have $x' \precsim x' + y$ and then $x' + ax_a \precsim x' + ax_a + y$.

The fact that $I \subseteq C$ is obvious and since $\precsim_{CS} \subseteq \precsim_{CT}$ we only have to show $\precsim_{CT} \subseteq C$. Since $\precsim_{CT}$ is initials preserving, if $p \precsim_{CT} \mathbf{0}$ then $p = \mathbf{0}$. Now, assuming $\mathbf{0} \precsim_{CS} q$, the empty trace is a complete trace of process $\mathbf{0}$ but it is not of any $q \neq \mathbf{0}$, so that q is $\mathbf{0}$ because $\precsim_{CS}$ defines complete traces containment. Therefore, if $p \precsim q$, p is $\mathbf{0}$ if and only if q is $\mathbf{0}$, as desired.

(2) Since neither of $by+z$ and $x+by+z$ is null, we have $C(x+by+z, by+z)$ and therefore $(CS_{\equiv})$ produces as a particular case the axiom $a(x + by + z) \approx a(x + by + z) + a(by + z)$. Let us now consider x and y with $C(x, y)$. If $x = \mathbf{0} = y$ then we obtain $b\mathbf{0} \approx b\mathbf{0} + b\mathbf{0}$, which is obvious. Otherwise, $y \neq \mathbf{0}$ and we can write $y = ay_a + y'$ so that the application of the axiom defining $\precsim_{CS}$ produces $b(x + ay_a + y') \approx b(x + ay_a + y') + b(ay_a + y')$, which means $b(x + y) \approx b(x + y) + by$, as desired.

To prove that $\mathcal{A}_{CS}(E)$ is also ω-complete whenever E is so, we need to check that for $t \preccurlyeq u \in E$ we have $\mathcal{A}_{CS}(E) \vdash b(u + t + z) \approx b(u + t + z) + b(t + z)$. We consider the decomposition $t = \sum_{i\in I_t}\sum_j a_i t_{ij} + V_j$ and, if its first summand is not $\mathbf{0}$, it is clear that we can use $(CS_{\equiv})$ to infer the required equation. For the other case, since $\precsim \subseteq C$, from $t = V_t$ we obtain $u = V_u$ and can conclude that $V_u = V_t$, so that we would have the trivial case $t = u$. $\square$

6 Conclusion and Future Work

In this paper we have given a simple proof of the correctness of the algorithm presented by Aceto, Fokkink and Ingólfsdóttir in [AFI07] that, from the axiomatization of a preorder, constructs an axiomatization for the semantic equivalence defined as the kernel of such a preorder. Our proof is valid not only for those semantics in the linear time-branching time spectrum that are coarser than ready simulation, as established in [AFI07], but for any initials preserving semantics coarser than ready simulation.

Our proof is based on the algebraic characterizations of the semantics but it is a completely general one, so that we never have to distinguish cases depending on the concrete collection of axioms defining each of the semantics. Besides, we want to stress the fact that the elegance of the proof has provided us with a general technique to prove other general properties of the semantics. This is based on the isolation of $b(t + \cdot)$ as a universal context, so that we can build inductive arguments to cover arbitrary contexts, not in the usual top-down way, by decomposing a context into subcontexts, but in a bottom-up manner, which implies to reduce the size of the context by enlarging its hole.

After proving the correctness of the original algorithm we have observed that we can simplify it, obtaining an enhanced algorithm which also provides a complete axiomatization of the corresponding equivalence. Even though it only works for strong enough semantics, we have presented the necessary changes to obtain the complete axiomatizations for coarser semantics such as trace semantics or plain simulation.

Using our general results on constrained simulations to be presented in [dFG08], valid for constraints stronger than I such as that defined by trace equivalence, we have been able (see [dFGP08]) to further extend the results in this paper to also cover semantics that are not finer than ready simulation. These new results are obtained by following a quite different approach, combining in an elegant way our results on the coalgebraic characterization of the semantics and other algebraic techniques.

Moreover, by exploiting the knowledge that we have accumulated studying common properties of the semantics for concurrency, we are developing a general theory where we will point out which are the common characteristics of them all, and how the different semantics are obtained by parameterizing those common elements in various ways. As a consequence, we will obtain a clearer picture of the spectrum and powerful techniques to prove general properties of the semantics, as we have done in this paper.

Finally, let us also cite a quite recent paper by Chen, Fokkink, and Van Glabbeek [CFG08] where the original algorithm in [AFI07] is extended to weak process semantics. In their proofs, they adapt the concepts and methods used in this paper to treat τ actions in an adequate manner.

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