Reflection and preservation of properties in coalgebraic (bi)simulations Extended

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Abstract. Our objective is to extend the standard results of preservation and reflection of properties by bisimulations to the coalgebraic setting, as well as to study under what conditions these results hold for simulations. The notion of bisimulation is the classical one, while for simulations we use that proposed by Hughes and Jacobs. As for properties, we start by using a generalization of linear temporal logic to arbitrary coalgebras suggested by Jacobs, and then an extension by Kurtz which includes atomic propositions too.

1 Introduction

To reason about computational systems it is customary to mathematically formalize them by means of state-based structures such as labelled transitions systems or Kripke structures. This is a fruitful approach since it allows to study the properties of a system by relating it to some other, possibly better-known system, by means of simulations and bisimulations (see e.g., [14, 13, 11, 3]).

The range of structures used to formalize computational systems is quite wide. In this context, coalgebras have emerged with a unifying aim [17]. A coalgebra is simply a function $c : X \longrightarrow FX$, where X is the set of states and FX is some expression on X (a functor) that describes the possible outcomes of a transition from a given state. Choosing different expressions for F one can obtain coalgebras that correspond to transition systems, Kripke structures, automata, ...

Coalgebras can also be related by means of (bi)simulations. Our goal in this paper is to prove that, like their concrete instantiations, (bi)simulations between arbitrary coalgebras preserve some interesting properties. A first step in this direction consists in choosing an appropriate notion for both bisimulation and simulation, as well as a logic in which to express these properties.

Bisimulations were originally introduced by Aczel and Mendler [1], who showed that the general definition coincided with the standard ones when particularized; it is an established notion. Simulations, on the other hand, were defined by Hughes and Jacobs [7] and lack such canonicity. Their notion of simulation depends on the use of orders that allow (perhaps too) much flexibility in what it can be considered as a simulation; in order to show that simulations preserve properties, we will have to impose certain restrictions on such orders. As for the logic used for the properties, there is likewise no canonical choice at the moment. Jacobs proposes a temporal logic (see [8]) that generalizes linear temporal logic (LTL), though without atomic propositions; a clever insight of Pattinson [16] provides us with a way to endow Jacobs' logic with atomic propositions.

Since our original motivation was the generalization of the results about simulations and preservation of LTL properties, we will focus on Jacobs' logic and its extension with atomic propositions. Actually, modal logic seems to be the right logic to express properties of coalgebras and several proposals have been made in this direction, among them those in [9, 12, 16], which are invariant under behavioral equivalence. The reason for studying preservation/reflection of properties by bisimulations here is twofold: on the one hand, some of the operators in Jacobs' logic do not seem to fall under the framework of those general proposals; on the other hand, some of the ideas and insights developed for that study are needed when tackling simulations. As far as we know, reflection of properties by simulations in coalgebras has not been considered before in the literature.

2 Preliminaries

In this section we summarize definitions and concepts from [7, 10, 8], and introduce the notation we are going to use.

Given a category $\mathbb{C}$ and an endofunctor F in $\mathbb{C}$, an F-coalgebra, or just a coalgebra, consists of an object $X \in \mathbb{C}$ together with a morphism $c : X \longrightarrow FX$. We often call X the state space and c the transition or coalgebra structure.

Example 1. We show how two well-known structures can be seen as coalgebras:

– Labelled transition systems are coalgebras for the functor $F = \mathcal{P}(id)^A$, where A is the set of labels.
– Kripke structures are coalgebras for the functor $F = \mathcal{P}(AP) \times \mathcal{P}(id)$, where AP is a set of atomic propositions.

It is well-known that an arbitrary endofunctor F on **Sets** can be lifted to a functor in the category **Rel** of relations, that is, $\mathrm{Rel}(F) : \mathbf{Rel} \longrightarrow \mathbf{Rel}$. Given a relation $R \subseteq X \times Y$, its lifting is defined by

$$\mathrm{Rel}(F)(R) = \{\langle u, v\rangle \in FX_1 \times FX_2 \mid \exists w \in F(R).\, F(r_1)(w) = u, F(r_2)(w) = v\} \ ,$$

where $r_i : R \longrightarrow X_i$ are the projection morphisms.

A predicate P of a coalgebra $c : X \longrightarrow FX$ is just a subset of the state space. Also, a predicate $P \subseteq X$ can be lifted to a functor structure using the relation lifting:

$$\mathrm{Pred}(F)(P) = \textstyle\coprod_{\pi_1}(\mathrm{Rel}(F)(\coprod_{\delta}(P))) = \coprod_{\pi_2}(\mathrm{Rel}(F)(\coprod_{\delta}(P))),$$

where $\delta = \langle id, id \rangle$ and $\coprod_f(X)$ is the image of X under f, so $\coprod_{\delta_x}(P) = \{(x,x) \mid x \in P\}$, $\coprod_{\pi_1}(R) = \{x_1 \mid \exists x_2 . x_1 R x_2\}$ is the domain of the relation R, and $\coprod_{\pi_2}(R) = \{x_2 \mid \exists x_1 . x_1 R x_2\}$ is its codomain.

The class of polynomial endofunctors is defined as the least class of endofunctors on **Sets** such that it contains the identity and constant functors, and is closed under product, coproduct, constant exponentiation, powerset and finite sequences. For polynomial endofunctors, $\mathrm{Rel}(F)$ and $\mathrm{Pred}(F)$ can be defined by induction on the structure of F. For further details on these definitions see [8]; we will introduce some of those when needed. For example, for the cases of labelled transition systems and Kripke structures we have:

$$\mathrm{Rel}(\mathcal{P}(id)^A)(R) = \{(f,g) \mid \forall a \in A.\, (f(a), g(a)) \in \{(U,V) \mid \forall u \in U.\, \exists v \in V.\, uRv \wedge \\ \forall v \in V.\, \exists u \in U.\, uRv\}\}$$

$$\mathrm{Pred}(\mathcal{P}(id)^A)(P) = \{f \mid \forall a \in A.\, f(a) \in \{U \mid \forall u \in U.\, Pu\}\}$$

$$\begin{aligned}\mathrm{Rel}(\mathcal{P}(AP) \times \mathcal{P}(id))(R) = \ & \{((u_1,u_2),(v_1,v_2)) \ \mid (u_1 = v_1.\, u_1, v_1 \in \mathcal{P}(AP)) \wedge \\ & (u_2, v_2) \in \{(U,V) \mid \forall u \in U.\, \exists v \in V.\, uRv \wedge \\ & \forall v \in V.\, \exists u \in U.\, uRv\}\}\end{aligned}$$

$$\mathrm{Pred}(\mathcal{P}(AP) \times \mathcal{P}(id))(P) = \{(u,v) \mid \ (u \subseteq \mathcal{P}(AP)) \wedge (v \in \{U \mid \forall u \in U.\, Pu)\}$$

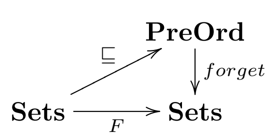
A bisimulation for coalgebras $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$ is a relation $R \subseteq X \times Y$ which is "closed under c and d":

$$\text{if } (x,y) \in R \text{ then } (c(x), d(y)) \in \mathrm{Rel}(F)(R)\,.$$

In the same way, an invariant for a coalgebra $c : X \longrightarrow FX$ is a predicate $P \subseteq X$ such that it is "closed under c", that is,

$$\text{if } x \in P \text{ then } c(x) \in \mathrm{Pred}(F)(P)\,.$$

We will use the definition of simulation introduced by Hughes and Jacobs in [7] which uses an order $\sqsubseteq$ for functors F that makes the following diagram commute



Given an order $\sqsubseteq$ on F, a simulation for the coalgebras $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$ is a relation $R \subseteq X \times Y$ such that

$$\text{if } (x,y) \in R \text{ then } (c(x), d(y)) \in \mathrm{Rel}(F)_{\sqsubseteq}(R)\,,$$

where $\mathrm{Rel}(F)_{\sqsubseteq}(R)$ is defined as

$$\mathrm{Rel}(F)_{\sqsubseteq}(R) = \{(u, v) \mid \exists w \in F(R).\ u \sqsubseteq Fr_1(w) \wedge Fr_2(w) \sqsubseteq v\}\,.$$

To express properties we will use a generalization of LTL proposed by Jacobs (see [8]) that applies to arbitrary coalgebras, whose formulas are given by the following BNF expression:

$$\varphi = P \subseteq X \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \Rightarrow \varphi \mid \bigcirc\varphi \mid \Diamond\varphi \mid \Box\varphi \mid \varphi\,\mathcal{U}\,\varphi$$

$\bigcirc$ is the *nexttime* operator and its semantics (abusing notation) is defined as $\bigcirc P = c^{-1}(\mathrm{Pred}(F)(P)) = \{x \in X \mid c(x) \in \mathrm{Pred}(F)(P)\}$; $\Box$ is the *henceforth* operator defined as $\Box P = \exists Q \subseteq X.\ Q$ is an invariant for c, and $Q \subseteq P$ with $x \in Q$ or, equivalently by means of the greatest fixed point ν, $\Box P = \nu S.(P \wedge \bigcirc S)$; $\Diamond$ is the *eventually* operator defined as $\Diamond P = \neg\Box\neg P$; and $\mathcal{U}$ is the *until* operator defined as $P\,\mathcal{U}\,Q = \mu S.(Q \vee (P \wedge \neg \bigcirc \neg S))$, where μ is the least fixed point.

We denote the set of states in X that satisfies φ as $[\![\varphi]\!]_X$. That is, if $P \subseteq X$ is a predicate, then $[\![P]\!]_X = P$; if $\alpha \in \{\neg, \bigcirc, \Box, \Diamond\}$ then $[\![\alpha\varphi]\!]_X = \alpha[\![\varphi]\!]_X$, and if $\beta \in \{\wedge, \vee, \Rightarrow, \mathcal{U}\}$ then $[\![\varphi_1\beta\varphi_2]\!]_X = [\![\varphi_1]\!]_X\beta[\![\varphi_2]\!]_X$. We will usually omit the reference to the set X when it is clear from the context. We say that an element x satisfies a formula φ, and we denote it by $c, x \models \varphi$, when $x \in [\![\varphi]\!]$. Again, we will usually omit the reference to the coalgebra c.

3 Reflection and preservation in bisimulations

These definitions of reflection and preservation are slightly more involved than for classical LTL because the logic proposed by Jacobs does not use atomic propositions, but predicates (subsets of the set of states). Later, we will see how atomic propositions can be introduced in the logic.

Given a predicate P on X and a binary relation $R \subseteq X \times Y$, we will say that an element $y \in Y$ is in the direct image of P, and we will denote it by $y \in RP$, if there exists $x \in X$ with $x \in P$ and xRy. The inverse image of R is just the direct image for the relation R^{-1}.

Definition 1. *Given two formulas φ on X and ψ on Y, built over predicates $P_1, \ldots P_n$ and $Q_1, \ldots Q_n$, respectively, and a binary relation $R \subseteq X \times Y$, we define the image of φ as a formula φ^* on Y, obtained by substituting in φ RP_i for P_i. Likewise, we build ψ^{-1}, the inverse of ψ, substituting $R^{-1}Q_i$ for Q_i in ψ.*

Remark 1. It is important to notice that φ^* coincides with φ^{-1} when we consider R^{-1} instead of R. Analogously, φ^{-1} is just φ^* when we consider R^{-1} instead of R.

Now we can define when a relation preserves or reflects properties.

Definition 2. *Let $R \subseteq X \times Y$ be a binary relation and a and b elements such that aRb. We say that R* preserves *the property φ on X if, whenever $a \models \varphi$, $b \models \varphi^*$. We say that R* reflects *the property φ on Y if $b \models \varphi$ implies $a \models \varphi^{-1}$.*

For the sake of the clarity of the proofs we will present the results on preservation and reflection of properties with one proposition for each temporal operator instead of one main theorem.

Let us first prove a couple of technical lemmas.

Lemma 1. *Let F be a polynomial functor, $R \subseteq X \times Y$ a bisimulation between coalgebras $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$, $P \subseteq Y$ and xRy. If $d(y) \in \mathrm{Pred}(F)(P)$, then $c(x) \in \mathrm{Pred}(F)(R^{-1}P)$.*

Proof. We are going to prove this result by structural induction on the functor F.

1. $F = cnst$. In this case $\mathrm{Pred}(F)(P) = \top$ and also $\mathrm{Pred}(F)(R^{-1}P) = \top$, and $c(x) \in \mathrm{Pred}(F)(R^{-1}P)$ trivially.
2. $F = id$. In this case $\mathrm{Pred}(F)(P) = P$ for all P and we have to prove that $c(x) \in R^{-1}P$, that is, that there exists some $b \in P$ with $c(x)Rb$. If we take $b = d(y)$, then we have the result because R is a bisimulation.
3. $F = F_1 \times F_2$. In this case we have

   $$\mathrm{Pred}(F)(P) = \{(u, v) \mid \mathrm{Pred}(F_1)(P)(u) \wedge \mathrm{Pred}(F_2)(P)(v)\}\,.$$

   Let us suppose that $d(y) = (d_1(y), d_2(y))$ and analogously $c(x) = (c_1(x), c_2(x))$. Therefore, if $d(y) \in \mathrm{Pred}(F)(P)$ then we also have $d_1(y) \in \mathrm{Pred}(F_1)(P)$ and $d_2(y) \in \mathrm{Pred}(F_2)(P)$. However, since R is a bisimulation between c and d and xRy, we have $c(x)\mathrm{Rel}(F)(R)d(y)$, where

   $$\mathrm{Rel}(F)(R) = \{((u_1, u_2), (v_1, v_2)) \mid \mathrm{Rel}(F_1)(R)(u_1, v_1) \wedge \mathrm{Rel}(F_2)(R)(u_2, v_2)\}\,.$$

   So, in particular, we have $c_1(x)\mathrm{Rel}(F_1)(R)d_1(y)$ as well as $c_2(x)\mathrm{Rel}(F_2)(R)d_2(y)$. That is, R is also a bisimulation for c_1 and d_1 and c_2 and d_2. Now we can apply our induction hypothesis and since $d_1(y) \in \mathrm{Pred}(F_1)(P)$, we get $c_1(x) \in \mathrm{Pred}(F_1)(R^{-1}P)$ and, analogously $c_2(x) \in \mathrm{Pred}(F_2)(R^{-1}P)$, so $c(x) \in \mathrm{Pred}(F)(R^{-1}P)$, as we wanted to prove.
4. $F = F_1 + F_2$. In this case we have

   $$\mathrm{Pred}(F)(P) = \{\kappa_1(u) \mid \mathrm{Pred}(F_1)(P)(u)\} \cup \{\kappa_2(v) \mid \mathrm{Pred}(F_2)(P)(v)\}\,.$$

   Let us suppose that $d(y) = \kappa_1(d_1(y)) = (d_1(y), 1)$; we must have that $d_1(y) \in \mathrm{Pred}(F_1)(P)$. Let us consider the constant coalgebras:

   $$\begin{aligned} c_X : X &\to F_1X \\ z &\mapsto c_1(x) \end{aligned} \qquad \begin{aligned} d_Y : Y &\to F_1Y \\ z &\mapsto d_1(y) \end{aligned}$$

   Trivially, R is a bisimulation between c_X and d_Y; then, if we apply the induction hypothesis we get $c_1(x) \in \mathrm{Pred}(F_1)(R^{-1}P)$ and hence $c(x) \in \mathrm{Pred}(F)(R^{-1}P)$. Reasoning in an analogous way we get that if $d(y) = \kappa_2(d_1(y))$, also $c(x) \in \mathrm{Pred}(F)(R^{-1}P)$.

5. $F = G^A$. In this case,

$$\text{Pred}(F)(P) = \{f \mid \forall a \in A.\ \text{Pred}(G)(P)(f(a))\}\,.$$

Now, for each $a \in A$ and F-coalgebra $d : Y \longrightarrow F(Y)$ we can define a coalgebra in G: $d^a : Y \longrightarrow G(Y)$ where, for each $y \in Y$, $d^a(y) = d(y)(a)$; analogously we define $c^a(x) = c(x)(a)$ for all $x \in X$. In this way we have that xRy and $d^a(y) = d(y)(a) \in \text{Pred}(G)(P)$. Now, using the definition of the relation lifting for the exponent functor and the fact that R is a bisimulation between c and d it follows that R is also a bisimulation between c^a and d^a. Applying the induction hypothesis we get $c^a(x) \in \text{Pred}(G)(R^{-1}P)$. Since this argument is valid for all $a \in A$, we get $c(x) \in \text{Pred}(F)(R^{-1}P)$, as we wanted to prove.

6. $F = \mathcal{P}(G)$. In this case

$$\text{Pred}(F)(P) = \{U \mid \forall u \in U.\ \text{Pred}(G)(P)(u)\}\,.$$

We have $d(y) \in \text{Pred}(F)(P)$, so for all $b \in d(y)$ it is true that $b \in \text{Pred}(G)(P)$ and we want to prove that $c(x) \in \text{Pred}(F)(R^{-1}P)$ or equivalently that for all $a \in c(x)$ we have $a \in \text{Pred}(G)(R^{-1}P)$. Let us take $a \in c(x)$ and define a constant coalgebra:

$$\begin{aligned} c^a_X : X &\to GX \\ z &\mapsto a \end{aligned}$$

Now, from our hypothesis that xRy and R is a bisimulation, we have $c(x)\text{Rel}(F)(R)d(y)$. By the definition of relation lifting it follows that there exists some $b \in d(y)$ such that $a\text{Rel}(G)(R)b$.
Now we consider another constant coalgebra

$$\begin{aligned} d^b_Y : Y &\to GY \\ z &\mapsto b \end{aligned}$$

Trivially, R is a bisimulation between the coalgebras c^a_X and d^b_Y because $a\text{Rel}(G)(R)b$; also $d^b_Y = b \in \text{Pred}(G)(P)$, so by induction hypothesis it follows that $c^a_X = a \in \text{Pred}(G)(R^{-1}P)$. This argument is valid for all $a \in c(x)$, therefore, as we wanted to prove, $c(x) \in \text{Pred}(F)(R^{-1}P)$.

7. $F = G^*$. In this case,

$$\text{Pred}(F)(P) = \{\langle u_1, \ldots, u_n\rangle \mid \forall i \leq n.\ \text{Pred}(G)(P)(u_i)\}\,.$$

Let us suppose that $d(y) = \langle d_1(y), \ldots, d_n(y)\rangle$ and $c(x) = \langle c_1(x), \ldots, c_n(x)\rangle$. Then, for each $i \leq n$ we have $d_i(y) \in \text{Pred}(G)(P)$. Now, we define these two families of constant coalgebras:

$$\begin{aligned} c^x_i : X &\to GX \\ z &\mapsto c_i(x) \end{aligned} \qquad\qquad \begin{aligned} d^y_i : Y &\to GY \\ z &\mapsto d_i(y) \end{aligned}$$

Trivially, R is a bisimulation between c^x_i and d^y_i for each $i \leq n$. Therefore, using the induction hypothesis we get $c_i(x) \in \text{Pred}(G)(R^{-1}P)$ for each $i \leq n$, that is, $c(x) \in \text{Pred}(F)(R^{-1}P)$.

□

As a direct consequence of this lemma and the fact that the inverse of a bisimulation, is still a bisimulation we also have the following result.

Lemma 2. *Let F be a polynomial functor, $R \subseteq X \times Y$ a bisimulation between coalgebras $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$, $P \subseteq X$ and xRy. If $c(x) \in \mathrm{Pred}(F)(P)$, then $d(y) \in \mathrm{Pred}(F)(RP)$.*

Another auxiliary lemma we need to prove the main result of this section is the following:

Lemma 3. *The direct and inverse images of an invariant are also invariants.*

Proof. Let R be a bisimulation between $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$. Let us suppose that $P \subseteq X$ is an invariant and let us prove that RP is too; that is, for all $y \in RP$ it must be the case that $d(y) \in \mathrm{Pred}(F)(RP)$. If $y \in RP$, then there exists $x \in P$ such that xRy. Since P is an invariant, we also have $c(x) \in \mathrm{Pred}(F)(P)$ and by Lemma 1 we get $d(y) \in \mathrm{Pred}(F)(RP)$.

On the other hand, since R^{-1} is also a bisimulation, the inverse image of an invariant is an invariant too.

At this point it is interesting to recall that our objective is to prove that bisimulations preserve and reflect properties of a temporal logic, that is, if we have xRy and $y \models \varphi$ then we must also have $x \models \varphi^{-1}$; and, analogously, if $x \models \varphi$ then $y \models \varphi^*$. We will show this result for all temporal operators except for the negation; it is well-known that negation is reflected and preserved by standard bisimulations, but not here because of the lack of atomic propositions in the coalgebraic temporal logic.

To prove the result for the rest of temporal operators, we will see that if $y \in [\![\varphi]\!]$ then we also have $x \in R^{-1}[\![\varphi]\!]$ and, analogously, if $x \in [\![\varphi]\!]$ then $y \in R[\![\varphi]\!]$. Ideally, we would like to have both $R^{-1}[\![\varphi]\!] = [\![\varphi^{-1}]\!]$ and $R[\![\varphi]\!] = [\![\varphi^*]\!]$ but, in general, only the inclusion $\subseteq$ is true. Fortunately this is enough to prove our propositions, since the temporal operators are all monotonic except for the negation. In fact, here is where the problem with negation appears.

Before continuing, it is interesting to show that the other inclusion is not true.

Let us take $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2\}$, the functor $F = id$ and the coalgebras $c : X \longrightarrow X$ and $d : Y \longrightarrow Y$ defined as $c(x_1) = x_2$, $c(x_2) = x_3$, $c(x_3) = x_3$, $d(y_1) = y_2$ and $d(y_2) = y_2$. Then, we take R as the bisimulation between c and d defined as $R = \{(x_3, y_1), (x_3, y_2), (x_1, y_2), (x_2, y_2), (x_3, y_2)\}$. We also consider the predicate $P = \{y_1\}$ and the formula $\varphi = \bigcirc\varphi_0$, where $\varphi_0 = \bigcirc P$. Let us show that $x_1 \in [\![\varphi^{-1}]\!]$ but $x_1 \notin R^{-1}[\![\varphi]\!]$:

- $x_1 \in [\![\varphi^{-1}]\!]$. By definition of inverse of a formula we have $\varphi^{-1} = \bigcirc\bigcirc R^{-1}P$, who also, by definition of R, is equal to $\bigcirc\bigcirc\{x_3\}$. Therefore, $x_1 \in [\![\varphi^{-1}]\!]$ if and only if $x_1 \in \bigcirc\bigcirc\{x_1\}$, in other words, if $c(x_1) = x_2 \in \mathrm{Pred}(F)(\bigcirc\{x_3\}) = \bigcirc\{x_3\}$ that, in turn, is equivalent to $c(x_2) = x_3 \in \mathrm{Pred}(F)(\{x_3\}) = \{x_3\}$. That way we conclude that $x_1 \in [\![\varphi^{-1}]\!]$.

- $x_1 \notin R^{-1}[\![\varphi]\!]$. Let us suppose that it is not true, in other words, $x_1 \in R^{-1}[\![\varphi]\!]$. Then, by definition of inverse predicate it would exist some $y \in Y$ such that $x_1 R y$ with $y \in [\![\varphi]\!]$. By definition of R the only possible candidate is y_2. So we get $y_2 \in [\![\varphi]\!] = \bigcirc \bigcirc \{y_1\}$, which is equivalent to $d(y_2) = y_2 \in \mathrm{Pred}(F)(\bigcirc\{y_1\}) = \bigcirc\{y_1\}$, that is, $d(y_2) = y_2 \in \mathrm{Pred}(F)(\{y_1\}) = \{y_1\}$. But the latter is a contradiction, so we have proved that $x_1 \notin R^{-1}[\![\varphi]\!]$.

he proof of the next is not specially difficult except for the case of the *until* operator, that will need an auxiliary result that we state now but will be proved later.

Lemma 4. *Let φ_1, φ_2 be temporal formulas which do not contain the negation operator and R a bisimulation between coalgebras $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$ such that xRy. If $y \in [\![\varphi_1]\!] \,\mathcal{U}\, [\![\varphi_2]\!]$, then $x \in R^{-1}[\![\varphi_1]\!] \,\mathcal{U}\, R^{-1}[\![\varphi_2]\!]$.*

Lemma 5. *Let R be a bisimulation between coalgebras $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$. For all temporal formulas φ which do not contain the negation operator, it follows that*

$$R^{-1}[\![\varphi]\!]_Y \subseteq [\![\varphi^{-1}]\!]_X \,.$$

Proof. We will proceed by structural induction on φ.

1. $\varphi = P$, where $P \subset Y$ is a predicate. In this case $R^{-1}[\![\varphi]\!] = R^{-1}P = [\![\varphi^{-1}]\!]$, by definition.
2. $\varphi = \varphi_1 \vee \varphi_2$, or $\varphi = \varphi_1 \wedge \varphi_2$. In both cases we trivially get the result.
3. $\varphi = \bigcirc\varphi_1$. Let us suppose $x \in R^{-1}[\![\bigcirc\varphi_1]\!]$; then it exists y such that xRy with $y \in [\![\bigcirc\varphi_1]\!] = \bigcirc[\![\varphi_1]\!]$. Equivalently, we have xRy with $d(y) \in \mathrm{Pred}(F)([\![\varphi_1]\!])$ so by Lemma 2 we have $c(x) \in \mathrm{Pred}(F)(R^{-1}[\![\varphi_1]\!])$. Now, by induction hypothesis we know that $R^{-1}[\![\varphi_1]\!] \subseteq [\![\varphi_1^{-1}]\!]$ and by monotony of operator $\mathrm{Pred}(F)$ we obtain $c(x) \in \mathrm{Pred}(F)([\![\varphi_1^{-1}]\!])$, that is, $x \in [\![\varphi^{-1}]\!]$.
4. $\varphi = \Box\varphi_1$. By definition $\Box[\![\varphi_1]\!]$ is the greatest invariant contained in $[\![\varphi_1]\!]$ and, henceforth by Lemma 3, $R^{-1}\Box[\![\varphi_1]\!]$ is also an invariant. Also, as $\Box[\![\varphi_1]\!] \subseteq [\![\varphi_1]\!]$, is $R^{-1}\Box[\![\varphi_1]\!] \subseteq R^{-1}[\![\varphi_1]\!]$. By induction hypothesis $R^{-1}[\![\varphi_1]\!] \subseteq [\![\varphi_1^{-1}]\!]$, so $R^{-1}\Box[\![\varphi_1]\!]$ is an invariant contained in $[\![\varphi_1^{-1}]\!]$ and thus contained in $\Box[\![\varphi_1^{-1}]\!]$, as we wanted to prove.
5. $\varphi = \Diamond\varphi_1$. Let us suppose $x \in R^{-1}\Diamond[\![\varphi_1]\!]$; then it exists some y such that xRy with $y \in \Diamond[\![\varphi_1]\!]$, that is, $y \in \neg\Box\neg[\![\varphi_1]\!]$. We recall that we must prove $x \in \neg\Box\neg[\![\varphi_1^{-1}]\!]$, assuming xRy and $y \in \neg\Box\neg[\![\varphi_1]\!]$; using the counter-reciprocal, we will see that xRy and $x \in \Box\neg[\![\varphi_1^{-1}]\!]$ implies $y \in \Box\neg[\![\varphi_1]\!]$.
   Let us take $x \in \Box\neg[\![\varphi_1^{-1}]\!]$. By definition, we know that it must exist an invariant S such that $S \subseteq \neg[\![\varphi_1^{-1}]\!]$ with $x \in S$. Let us see that RS is an invariant such that $RS \subseteq \neg[\![\varphi_1]\!]$ with $y \in RS$ (if it is so, then we will get $y \in \Box\neg[\![\varphi_1]\!]$). Indeed, since S is an invariant, then RS is also an invariant and, trivially, $y \in RS$. Let us see now that $RS \subseteq \neg[\![\varphi_1]\!]$. To prove this, let us suppose that it is false, that is, there exists an element $b \in RS$ such that $b \in [\![\varphi_1]\!]$. But since $b \in RS$, then it must exist $a \in S \subseteq \neg[\![\varphi_1^{-1}]\!]$, that is, $a \notin [\![\varphi_1^{-1}]\!]$. On the other hand, since $b \in [\![\varphi_1]\!]$ and aRb then $a \in R^{-1}[\![\varphi_1]\!]$, that by induction hypothesis is contained in $[\![\varphi_1^{-1}]\!]$, that is, $a \in [\![\varphi_1^{-1}]\!]$, but this is a contradiction so $RS \subseteq \neg[\![\varphi_1]\!]$.

6. $\varphi = \varphi_1 \; \mathcal{U} \; \varphi_2$. Let us suppose that $x \in R^{-1}[\![\varphi_1 \; \mathcal{U} \; \varphi_2]\!]$, then there exists y such that xRy with $y \in [\![\varphi_1 \; \mathcal{U} \; \varphi_2]\!] = [\![\varphi_1]\!] \; \mathcal{U} \; [\![\varphi_2]\!]$. By Lemma 4 we get $x \in R^{-1}[\![\varphi_1]\!] \; \mathcal{U} \; R^{-1}[\![\varphi_2]\!]$ that is, for all S we must have $x \in R^{-1}[\![\varphi_2]\!] \cup (R^{-1}[\![\varphi_1]\!] \cap \neg \bigcirc \neg S)$. By induction hypothesis, we have $R^{-1}[\![\varphi_i]\!] \subseteq [\![\varphi_i^{-1}]\!]$ for $i = 1, 2$, so, for all S we have $x \in [\![\varphi_2^{-1}]\!] \cup ([\![\varphi_1^{-1}]\!] \cap \neg \bigcirc \neg S)$, that is $x \in [\![\varphi_1^{-1}]\!] \; \mathcal{U} \; [\![\varphi_2^{-1}]\!]$.

□

Once again, due to the fact that R^{-1} is a bisimulation we also have:

Lemma 6. *Let R be a bisimulation between coalgebras $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$. For all temporal formulas φ which do not contain the negation operator, it follows that*

$$R[\![\varphi]\!]_X \subseteq [\![\varphi^*]\!]_Y \, .$$

Finally we can show that bisimulations reflect and preserve properties given by any temporal operator except for the negation.

The firsts two cases correspond to the reflection of the predicates and the reflection of formulas with elemental operators, whose proofs are trivial.

Proposition 1. *Let ψ be a predicate $P \subseteq Y$ and R a bisimulation between the coalgebras $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$. Then the property ψ is reflected by R.*

Proposition 2. *Let $\psi = \varphi_1 \vee \varphi_2$ or $\psi = \varphi_1 \wedge \varphi_2$ where φ_1 and φ_2 are formulas reflected by bisimulations, and let R be a bisimulation between the coalgebras $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$. Then the property ψ is reflected by R.*

Now we can show more interesting cases, like the reflection with the *nexttime* operator.

Proposition 3. *Let $\psi = \bigcirc\varphi$ be a formula on a set Y such that φ is reflected by bisimulations, and let R be a bisimulation between coalgebras $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$. Then the property ψ is reflected by R.*

Proof. Let aRb and suppose that $b \models \psi$, that is, $b \in \bigcirc[\![\varphi]\!]$ which, by definition, means that $d(b) \in \mathrm{Pred}(F)([\![\varphi]\!])$. We are going to show that $a \in \bigcirc R^{-1}[\![\varphi]\!]$ or, equivalently, $c(a) \in \mathrm{Pred}(F)(R^{-1}[\![\varphi]\!])$. But the latter is straightforward from Lemma 2.

Now, from Lemma 5 and the monotony of the nexttime operator we get $a \in \bigcirc[\![\varphi^{-1}]\!]$. □

The same result is true for the *henceforth* operator:

Proposition 4. *Let $\psi = \Box\varphi$ be a formula on Y such that φ is reflected by bisimulations and let R be a bisimulation between the coalgebras $c : X \longrightarrow F(X)$, $d : Y \longrightarrow F(Y)$. Then, the property ψ is reflected by R.*

Proof. Let aRb and let's suppose that $b \models \Box\varphi$, that is, there exists $S \subseteq Y$ such that S is an invariant, $S \subseteq [\![\varphi]\!]$ with $b \in S$. Once again, we will show that $a \in \Box R^{-1}[\![\varphi]\!]$ and then use Lemma 5 to get $a \in \Box[\![\varphi^{-1}]\!]$.

In fact, we only need to show that there exists an invariant $T \subseteq X$ such that $T \subseteq R^{-1}[\![\varphi]\!]$ and $a \in T$. If we take the invariant $T = R^{-1}S$, then $a \in T$ since aRb and $b \in S$; also $R^{-1}S \subseteq R^{-1}[\![\varphi]\!]$ because $S \subseteq [\![\varphi]\!]$. $\Box$

Now we can state and prove the corresponding proposition for the operator *eventually*.

Proposition 5. *Let $\psi = \Diamond\varphi$ be a formula on a set Y such that φ is reflected by bisimulations and let R be a bisimulation between coalgebras $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$. Then the property ψ is reflected by R.*

Proof. Let aRb and let us suppose $b \models \psi$; by definition, $b \in \neg\Box\neg[\![\varphi]\!]$. Once again, it will be enough to prove that $a \in \neg\Box\neg R^{-1}[\![\varphi]\!]$. So, let us assume that $a \in \Box\neg R^{-1}[\![\varphi]\!]$ and let us show that $b \in \Box\neg[\![\varphi]\!]$. Indeed, if $a \in \Box\neg R^{-1}[\![\varphi]\!]$ then there exists an invariant $S \subseteq X$ such that $S \subseteq \neg R^{-1}[\![\varphi]\!]$, with $a \in S$. But RS is an invariant such that $RS \subseteq \neg[\![\varphi]\!]$ and $b \in RS$, so we have proved the proposition. $\Box$

The proof of the corresponding proposition involving the *until* operator is a bit more difficult because it needs to resort to some more technical lemmas and auxiliary results. One of those will be Lemma 4, that, also, will be the base for proving the main proposition. To prove it we will need an auxiliary definition which will simplify the notation:

Definition 3. *Given a set Y and $P, Q \subseteq Y$, we define an operator $f^U_{(P,Q)}$ as:*

$$\begin{aligned} f^U_{(P,Q)} : \mathcal{P}(Y) &\longrightarrow \mathcal{P}(Y) \\ S \quad &\longmapsto Q \cup (P \cap \neg \bigcirc \neg S)\,. \end{aligned}$$

Notice that $y \in [\![\varphi_1]\!]\, \mathcal{U}\, [\![\varphi_2]\!]$ by definition is $y \in \mu S.([\![\varphi_2]\!] \vee ([\![\varphi_1]\!] \wedge \neg \bigcirc \neg S))$ that is,

$$y \in \mu S. f^U_{([\![\varphi_1]\!],[\![\varphi_2]\!])}(S)\,.$$

And, on the other hand, $x \in R^{-1}[\![\varphi_1]\!]\, \mathcal{U}\, R^{-1}[\![\varphi_2]\!]$ is equivalent to

$$x \in \mu S. f^U_{(R^{-1}[\![\varphi_1]\!],R^{-1}[\![\varphi_2]\!])}(S)\,.$$

We will also need two classical results, the first one is a definition of $\cup$-continuity (see for example [3]) and the second one says how to calculate least fixed points:

Definition 4. *Given a set S and a function $f : \mathcal{P}(S) \longrightarrow \mathcal{P}(S)$, we say that f is $\cup$-continuous if given an ascending chain of predicates $\{P_i \mid i \in \mathbb{N}\}$ with $P_i \subseteq P_{i+1}$ for each i, we have*

$$f(\bigcup_i P_i) = \bigcup_i f(P_i)\,.$$

Proposition 6. *If $f : \mathcal{P}(S) \longrightarrow \mathcal{P}(S)$ is monotonic and $\cup$-continuous, then*

$$\mu S.f(S) = \bigcup_{i=1}^{\infty} f^i(\emptyset)\,.$$

It is easy to prove that the function $f^U_{(P,Q)}$ is both monotonic and $\cup$-continuous. First we shall see monotony: Let $S_1 \subseteq S_2$ and we will see $f^U_{(P,Q)}(S_1) \subseteq f^U_{(P,Q)}(S_2)$. Since $S_1 \subseteq S_2$ then $\neg S_2 \subseteq \neg S_1$ which, by monotony of $\bigcirc$, implies $\bigcirc\neg S_2 \subseteq \bigcirc\neg S_1$. That is, $\neg \bigcirc \neg S_1 \subseteq \neg \bigcirc \neg S_2$; by definition of $f^U_{(P,Q)}$, that is enough to show that $f^U_{(P,Q)}(S_1) \subseteq f^U_{(P,Q)}(S_2)$.

On the other hand we have to prove that $f^U_{(P,Q)}$ satisfies the following equation

$$f^U_{(P,Q)}(\bigcup_i P_i) = \bigcup_i f^U_{(P,Q)}(P_i)\,. \tag{1}$$

Unfolding in both sides of (1) we get

$$Q \cup (P \cap \neg \bigcirc (\bigcap_i \neg P_i)) = Q \cup (P \cap (\bigcup_i \neg \bigcirc \neg P_i))\,,$$

so it is enough to prove

$$\neg \bigcirc (\bigcap_i \neg P_i) = \bigcup_i \neg \bigcirc \neg P_i\,.$$

This last equation is equivalent to

$$\bigcirc(\bigcap_i \neg P_i) = \bigcap_i \bigcirc\neg P_i\,,$$

that is,

$$\text{Pred}(F)(\bigcap_i \neg P_i) = \bigcap_i \text{Pred}(F)(\neg P_i)\,. \tag{2}$$

And this last equation trivially follows from the fact that $\text{Pred}(F)$ preserves intersections as it is shown in [8].

Before proving Lemma 4 it will be necessary to prove another technical lemma:

Lemma 7. *Given the function $f^U_{(P,Q)} : \mathcal{P}(Y) \longrightarrow \mathcal{P}(Y)$, the following is true*

$$R^{-1} f^U_{(\llbracket\varphi_1\rrbracket,\llbracket\varphi_2\rrbracket)}(S) \subseteq f^U_{(R^{-1}\llbracket\varphi_1\rrbracket,R^{-1}\llbracket\varphi_2\rrbracket)}(R^{-1}S)\,.$$

Proof. We introduce now the following simplified notation:

$$f_1 \quad \text{denotes} \quad f^U_{(\llbracket\varphi_1\rrbracket,\llbracket\varphi_2\rrbracket)}(S)\,,$$

$$f_2 \quad \text{denotes} \quad f^U_{(R^{-1}\llbracket\varphi_1\rrbracket,R^{-1}\llbracket\varphi_2\rrbracket)}(S)\,.$$

So, we have:

$$R^{-1}f_1(S) = R^{-1}[\![\varphi_2]\!] \cup (R^{-1}[\![\varphi_1]\!] \cap R^{-1}(\neg \bigcirc \neg S)))\,,$$

$$f_2(R^{-1}S) = R^{-1}[\![\varphi_2]\!] \cup (R^{-1}[\![\varphi_1]\!] \cap \neg \bigcirc \neg(R^{-1}S)))\,.$$

Indeed it will be enough to prove that $R^{-1}(\neg \bigcirc \neg S) \subseteq \neg \bigcirc \neg(R^{-1}S)$. So, if $x \in R^{-1}(\neg \bigcirc \neg S)$ then there exists some y such that xRy with $y \in \neg \bigcirc \neg S$; and we must prove that $x \in \neg \bigcirc \neg(R^{-1}S)$. Or, equivalently we can prove that given xRy, if $x \in \bigcirc\neg(R^{-1}S)$ then $y \in \bigcirc\neg S$.

We will prove this last result, by estructural induction on the functor F in the same way we did in Lemma 2.

1. $F = cnst$. In this case both $\mathrm{Pred}(F)(\neg R^{-1}S) = \top$ and $\mathrm{Pred}(F)(\neg S) = \top$, so we get the result.
2. $F = id$. In this case we have to show that if $c(x) \in \neg R^{-1}S$ then $d(y) \in \neg S$. Let us suppose that $d(y) \in S$. Then, since xRy, R is a bisimulation and we are working with the identity functor, we also have $c(x)Rd(y)$, and that way $c(x) \in R^{-1}S$; this contradicts the hypothesis, so it must be $d(y) \in \neg S$.
3. $F = F_1 \times F_2$. In this case we have $c_1(x) \in \mathrm{Pred}(F_1)(\neg R^{-1}S)$ and $c_2(x) \in \mathrm{Pred}(F_2)(\neg R^{-1}S)$, so by induction hypothesis we get both $d_1(y) \in \mathrm{Pred}(F_1)(\neg S)$ and $d_2(y) \in \mathrm{Pred}(F_2)(\neg S)$, and then $d(y) \in \mathrm{Pred}(F)(\neg S)$.
4. $F = F_1 + F_2$. Without loss of generallity, let us suppose that $c(x) = \kappa_1(c_1(x)) = (c_1(x), 1)$, so we have $c_1(x) \in \mathrm{Pred}(F_1)(\neg R^{-1}S)$. Let us consider the constant coalgebras:

$$\begin{array}{rl} c_X : X & \to F_1X \\ z & \mapsto c_1(x) \end{array} \qquad \begin{array}{rl} d_Y : Y & \to F_1Y \\ z & \mapsto d_1(y) \end{array}$$

   Trivially, R is a bisimulation between c_X and d_Y; applying the induction hypothesis we get $d_1(y) \in \mathrm{Pred}(F_1)(\neg S)$ and hence $d(y) \in \mathrm{Pred}(F)(\neg S)$. Reasoning in an analogous way we get that if $c(x) = \kappa_2(c_1(x))$, also $d(y) \in \mathrm{Pred}(F)(\neg S)$.
5. $F = G^A$.

$$\mathrm{Pred}(F)(\neg R^{-1}S) = \{f \mid \forall a \in A.\ \mathrm{Pred}(G)(\neg R^{-1}S)(f(a))\}\,.$$

   Now, for each $a \in A$ and any F-coalgebra $c : X \longrightarrow FX$, we can define a new G-coalgebra: $c^a : X \longrightarrow GX$ where, for each $x \in X$ we have $c^a(x) = c(x)(a)$; and analogously we define $d^a(y) = c(y)(a)$ for each $y \in Y$. This way we have xRy and $c^a(x) = c(x)(a) \in \mathrm{Pred}(G)(\neg R^{-1}S))$. Using the induction hypothesis we get $d^a(y) \in \mathrm{Pred}(G)(\neg S)$. This argument is valid for each $a \in A$, so we get $d(y) \in \mathrm{Pred}(F)(\neg S)$.
6. $F = \mathcal{P}(G)$. In this case

$$\mathrm{Pred}(F)(\neg R^{-1}S) = \{U \mid \forall u \in U.\ \mathrm{Pred}(G)(\neg R^{-1}S)(u)\}\,.$$

We have $c(x) \in \text{Pred}(F)(\neg R^{-1}S)$, so for all $a \in c(x)$ is true that $a \in \text{Pred}(G)(\neg R^{-1}S)$ and we want to prove that $d(y) \in \text{Pred}(F)(\neg S)$ or, equivalently, that for all $b \in d(y)$ we have $b \in \text{Pred}(G)(\neg S)$. Let us take one $b \in d(y)$ and we define the constant coalgebra:

$$\begin{aligned} d_Y^b : Y &\to GY \\ z &\mapsto b \end{aligned}$$

Now, from our hypothesis xRy, and R is a bisimulation, so we have $c(x)\text{Rel}(F)(R)d(y)$, and by definition of relation lifting it follows that there exists some $a \in c(x)$ such that $a\text{Rel}(G)(R)b$.
Now, we consider another constant coalgebra:

$$\begin{aligned} c_X^a : X &\to GX \\ z &\mapsto a \end{aligned}$$

Trivially R is a bisimulation between the coalgebras c_X^a and d_Y^b because $a\text{Rel}(G)(R)b$; also $c_X^a(x) = a \in \text{Pred}(G)(\neg R^{-1}S)$, so by induction hypothesis it follows that $d_Y^b(y) = b \in \text{Pred}(G)(\neg S)$. Since this argument is valid for all $b \in d(y)$ we have proved that $d(y) \in \text{Pred}(F)(\neg S)$.

7. $F = G^*$. As we have previously done, let us suppose that $c(x) = \langle c_1(x), \ldots, c_n(x)\rangle$ and $d(y) = \langle d_1(y), \ldots, d_n(y)\rangle$. Then, for each $i \leq n$ we have $c_i(x) \in \text{Pred}(G)(\neg R^{-1}S)$. Now, we define these two families of constant coalgebras:

$$\begin{aligned} c_i^x : X &\to GX \\ z &\mapsto c_i(x) \end{aligned} \qquad \begin{aligned} d_i^y : Y &\to GY \\ z &\mapsto d_i(y) \end{aligned}$$

Trivially, R is a bisimulation between c_i^x and d_i^y for each $i \leq n$. Therefore, using the induction hypothesis we get $d_i(y) \in \text{Pred}(G)(\neg S)$ for each $i \leq n$, that is, $d(y) \in \text{Pred}(F)(\neg S)$.

□

At last we can prove Lemma 4:

*Proof (***Lemma 4***).* Once again we use the following notation:

$$f_1 \quad \text{denotes} \quad f^U_{([\![\varphi_1]\!],[\![\varphi_2]\!])}(S)\,,$$

$$f_2 \quad \text{denotes} \quad f^U_{(R^{-1}[\![\varphi_1]\!],R^{-1}[\![\varphi_2]\!])}(S)\,.$$

Let us suppose $y \in [\![\varphi_1]\!]\mathcal{U}[\![\varphi_2]\!]$, that is, $y \in \mu S.f_1(S)$, and we have to show that $x \in \mu S.f_2(S)$. By monotonicity and continuity of $f^U_{(P,Q)}$, their least fixed points are given by

$$\mu S.f_1(S) = \bigcup_{i=0}^{\infty} f_1^i(\emptyset)\,,$$

$$\mu S.f_2(S) = \bigcup_{i=0}^{\infty} f_2^i(\emptyset)\,.$$

So, since $y \in \bigcup_i^\infty f_1^i(\emptyset)$ we have $y \in f_1^i(\emptyset)$ for some i. Henceforth, $x \in R^{-1} f_1^i(\emptyset)$. Also, by Lemma 7 we have

$$x \in R^{-1} f_1^i(\emptyset) \subseteq f_2(R^{-1} f_1^{i-1}(\emptyset))\,,$$

where by monotonicity of f_2,

$$f_2(R^{-1} f_1^{i-1}(\emptyset)) \subseteq f_2(f_2(R^{-1} f_1^{i-2}(\emptyset)))\,.$$

If we iterate this process we finally get

$$x \in f_2^i(R^{-1}\emptyset) = f_2^i(\emptyset)\,.$$

And, that way, we get $x \in \bigcup_i^\infty f_2^i(\emptyset) = \mu S.f_2(S)$. □

With this lemma, the proposition involving reflection and the *until* operator is immediate.

Proposition 7. *Let φ_1 and φ_2 be temporal formulas such that they are reflected by bisimulations, $\psi = \varphi_1 \;\mathcal{U}\; \varphi_2$ a temporal formula on Y and R a bisimulation between coalgebras $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$. Then, the property ψ is reflected by R.*

Proof. Let aRb and let us suppose $b \models \varphi_1 \;\mathcal{U}\; \varphi_2$, that is, $b \in [\![\varphi_1]\!] \;\mathcal{U}\; [\![\varphi_2]\!]$. By Lemma 4 we also get $a \in R^{-1}[\![\varphi_1]\!] \;\mathcal{U}\; R^{-1}[\![\varphi_2]\!]$, and from both monotony of the operator *until* and Lemma 5 we get $a \in [\![\varphi_1^{-1}]\!] \;\mathcal{U}\; [\![\varphi_2^{-1}]\!]$. □

Preservation of properties is a consequence of the reflection of properties together with the fact that if R is a bisimulation then R^{-1} is also a bisimulation. We have thus proved the following theorem.

Theorem 1. *Let ψ and φ be formulas over sets Y and X, respectively, which do not use the* negation *operator and let R be a bisimulation between coalgebras $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$. Then ψ is reflected by R and φ is preserved by R.*

4 Reflection and preservation in simulations

In [3, 15] it is proved not only that bisimulations reflect and preserve properties but also that simulations reflect them: it turns out that this result does not generalize straightforwardly to the coalgebraic setting.

The main problem that we have found concerning this is that the coalgebraic definition of simulation uses an arbitrary functorial order $\sqsubseteq$, and in general reflection of properties will not hold for all orders.

Let us show a counterexample that will convince us that simulations may not reflect properties without restricting the orders. Let us take $F = \mathcal{P}(id)$, $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and the coalgebras c and d defined as $c(x_1) = \{x_1, x_2\}$, $c(x_2) = \{x_2\}$, $d(y_1) = y_2$ and $d(y_2) = y_2$. We define $u \sqsubseteq v$ whenever $v \subseteq u$ and consider the formula $\varphi = \bigcirc P$, where $P = \{y_2\}$, and the simulation $R = \{(x_1, y_2)\}$. It is immediate to check that R is a simulation and $y_2 \in [\![\varphi]\!]$, but $x_1 \notin [\![\varphi^{-1}]\!]$.

- $y_2 \in [\![\varphi]\!]$. Indeed, since $d(y_2) = y_2$ then $y_2 \in [\![\varphi]\!] = \bigcirc P$ is equivalent to $y_2 \in P = \{y_2\}$, which is trivially true.
- $x_1 \notin [\![\varphi^{-1}]\!]$. By definition, $\varphi^{-1} = \bigcirc R^{-1} P = \bigcirc \{x_1\}$. Since $c(x_1) = \{x_1, x_2\}$, it is enough to see that $x_2 \notin \{x_1\}$, which is also true.

As a consequence, we will need to restrict the functorial orders that are involved in the definition of simulation. In a first approach we will impose an extra requirement that the order must fulfill, and later we will not only restrict the orders but also the functors that are involved.

4.1 Restricting the orders

The idea is that we are going to require an extra property for each pair of elements which are related by the order. In particular, we are particularly interested in the following property (which is defined in [7]):

Definition 5. *Given a functor F :* **Sets** $\longrightarrow$ **Sets***, we say that an order $\sqsubseteq$ associated to it is "down-closed" whenever $a \sqsubseteq b$, with $a, b \in FX$, implies that*

$$b \in \mathrm{Pred}(F)(P) \implies a \in \mathrm{Pred}(F)(P), \quad \textit{for all predicates } P \subseteq X\,.$$

We can show some examples of down-closed orders:

Example 2. 1. Kripke structures are defined by the functor $F = \mathcal{P}(AP) \times \mathcal{P}(id)$, so a down-closed order must fulfill that if $(u, v) \sqsubseteq (u', v')$, then $(u', v') \in \mathrm{Pred}(F)(P)$ implies $(u, v) \in \mathrm{Pred}(F)(P)$; that is, by definition of $\mathrm{Pred}(\mathcal{P}(AP) \times \mathcal{P}(id))$, $u, u' \subseteq \mathcal{P}(AP)$ and, if $v' \in \mathrm{Pred}(\mathcal{P}(id))(P) = \{U \mid \forall u \in U.\, u \in P\}$ then $v \in \mathrm{Pred}(\mathcal{P}(id))(P)$. In other words, for all $b \in v$ and $b' \in v'$, if $b' \in P$ then $b \in P$. Therefore, what is needed in this case is that the set of successors v of the smaller pair is contained in the set of successors v' of the bigger pair, that is,

$$\text{if} \quad (u, v) \sqsubseteq (u', v') \quad \text{then} \quad v \subseteq v'\,.$$

2. Labelled transition systems are defined by the functor $F = \mathcal{P}(id)^A$, so the order must fulfill the following:

$$\text{if} \quad u \sqsubseteq v \quad \text{then} \quad \forall a \in A.\, u(a) \subseteq u'(a)\,.$$

These examples show that there are not many down-closed orders, but it does not seem clear how to further extend this class in such a way that we could still prove the reflection of properties by simulations. Unfortunately, even under this restriction we can only prove reflection (or preservation) of formulas that only use the operators $\vee$, $\wedge$, $\bigcirc$ and $\Box$.

To convince us of this fact, we present a counterexample with operator $\Diamond$. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and the functor $F = \mathcal{P}(id)$. We consider the following down-closed order: $u \sqsubseteq v$ if $u \subseteq v$. We also define the coalgebras $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$ as $c(x_1) = \{x_1\}$, $c(x_2) = \{x_2\}$, $d(y_1) = \{y_1, y_2\}$ and $d(y_2) =$

$\{y_2\}$. Obviously $R = \{(x_1, y_1)\}$ is a simulation since $c(x_1) = \{x_1\} \sqsubseteq \{x_1\}$ and $\{y_1\} \sqsubseteq \{y_1, y_2\} = d(y_1)$ and, also, $\{x_1\}\text{Rel}(F)(R)\{y_1\}$. We have $y_1 \in \Diamond\{y_2\}$, since we can reach y_2 from y_1, but $x_1 \notin \Diamond R^{-1}\{y_2\} = \Diamond\emptyset$. Indeed, $x_1 \notin \Diamond\emptyset$ is equivalent to $x_1 \in \Box\neg\emptyset$ and this is true since $\{x_1\}$ is an invariant such that $x_1 \in \{x_1\}$, with $\{x_1\} \subseteq \neg\emptyset$.

In order to prove reflection of properties that only use the operators $\vee$, $\wedge$, $\bigcirc$ and $\Box$, we will need a previous elementary result involving binary relations.

Proposition 8. *Let $R \subseteq X \times Y$ be a binary relation and $P \subseteq Y$ a predicate. Let us suppose that $u\text{Rel}(F)(R)v$; then if $v \in \text{Pred}(F)(P)$ it is also true that $u \in \text{Pred}(F)(R^{-1}P)$.*

Proof. Once again the proof will proceed by structural induction on the functor F.

1. If F is constant, then the result follows trivially.
2. Let us suppose that $F = id$, then we have uRv and also $v \in P$ and therefore, by definition of $R^{-1}P$, we get $u \in R^{-1}P$.
3. Let us now suppose that $F = F_1 \times F_2$ and let $u = (u_1, u_2)$ and $v = (v_1, v_2)$. By definition of the relation lifting we have both $u_1\text{Rel}(F_1)(R)v_1$ and $u_2\text{Rel}(F_2)(R)v_2$; whereas by definition of predicate lifting, since $v \in \text{Pred}(F)(P)$ we have $v_1 \in \text{Pred}(F_1)(P)$ and $v_2 \in \text{Pred}(F_2)(P)$. So, applying the induction hypothesis we get $u_1 \in \text{Pred}(F_1)(R^{-1}P)$ and $u_2 \in \text{Pred}(F_2)(R^{-1}P)$, henceforth $u \in \text{Pred}(F)(R^{-1}P)$.
4. If $F = F_1 + F_2$, without loss of generality let us suppose $v = \kappa_1(v_0)$ and $u = \kappa_1(u_0)$. Then, by the definition of predicate lifting we have $v_0 \in \text{Pred}(F_1)(P)$. Also, by the definition of the relation lifting $u_0\text{Rel}(F_1)(R)v_0$, so by induction hypothesis we obtain $u_0 \in \text{Pred}(F_1)(R^{-1}P)$, that is, $u \in \text{Pred}(F)(R^{-1}P)$.
5. Let us suppose $F = G^A$. If $v \in \text{Pred}(F)(P)$ then for all $a \in A$ we will have $v(a) \in \text{Pred}(G)(P)$. But, on the other hand, since $u\text{Rel}(F)(R)v$ then for all $a \in A$ it is also true that $u(a)\text{Rel}(G)(R)\ v(a)$. Let us consider any $a_0 \in A$; then $v(a_0) \in \text{Pred}(G)(P)$ and $u(a_0)\ \text{Rel}(G)(R)v(a_0)$, so by induction hypothesis we get $u(a_0) \in \text{Pred}(G)(R^{-1}P)$. This is valid for any $a_0 \in A$, so it proves that $u \in \text{Pred}(F)(R^{-1}P)$.
6. Let us suppose $F = \mathcal{P}(G)$. In this case, since $v \in \text{Pred}(F)(P)$ we have that for each $b \in v$, then $b \in \text{Pred}(G)(P)$. Our goal is to show that $u \in \text{Pred}(F)(R^{-1}P)$, that is, for all $a \in u$ it must be $a \in \text{Pred}(G)(R^{-1}P)$. Let us take any $a \in u$, since $u\text{Rel}(F)(R)v$ there exists $b \in v$ such that $a\text{Rel}(G)(R)b$. By induction hypothesis we get $a \in \text{Pred}(G)(R^{-1}P)$; since this is a valid argument for all $a \in u$, it follows that $u \in \text{Pred}(F)(R^{-1}P)$.
7. Let us suppose $F = G^*$, $v = \langle v_1, \ldots, v_n\rangle$ and $u = \langle u_1, \ldots, u_n\rangle$. Then, since $v \in \text{Pred}(F)(P)$ for each $i \leq n$ we have that $v_i \in \text{Pred}(G)(P)$. By the definition of the relation lifting we have that for each $i \leq n$ then $u_i\text{Rel}(G)(R)v_i$, hence by induction hypothesis, for all $i \leq n$ it follows that $u_i \in \text{Pred}(G)(R^{-1}P)$ and therefore $u \in \text{Pred}(F)(R^{-1}P)$.

□

We will also need a subtle adaptation of Lemmas 3 and 5 from the framework of bisimulations to the framework of simulations. In particular, we can adapt Lemma 3 to prove that if Q is an invariant and R a simulation, $R^{-1}Q$ is still an invariant, whereas Lemma 5 will also be true in the framework of simulations for formulas that only use the operators $\vee$, $\wedge$, $\bigcirc$ and $\Box$.

Lemma 8. *Let R be a simulation between coalgebras $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$, with a down-closed order, and let $Q \subseteq Y$ be an invariant. Then $R^{-1}Q$ is also an invariant.*

Proof. We are going to show that for all $x \in R^{-1}Q$ we have $c(x) \in \mathrm{Pred}(F)(R^{-1}Q)$. Let us take an arbitrary $x \in R^{-1}Q$; then, by definition there exists $y \in Q$ such that xRy and, since Q is an invariant, $d(y) \in \mathrm{Pred}(F)(Q)$. On the other hand, since R is a simulation, $c(x) \sqsubseteq u\mathrm{Rel}(F)(R)v \sqsubseteq d(y)$. Henceforth, since we are working with a down-closed order and $d(y) \in \mathrm{Pred}(F)(Q)$, then $v \in \mathrm{Pred}(F)(Q)$. Also, by Proposition 8 we have $u \in \mathrm{Pred}(F)(R^{-1}Q)$ and, using again that the order is down-closed, it follows that $c(x) \in \mathrm{Pred}(F)(R^{-1}Q)$.

Lemma 9. *Let R be a simulation between coalgebras $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$, with a down-closed order. If φ is a temporal formula constructed only with operators $\vee$, $\wedge$, $\bigcirc$ and $\Box$, then*

$$R^{-1}[\![\varphi]\!]_Y \subseteq [\![\varphi^{-1}]\!]_X \,.$$

Proof. This time the proof will proceed by structural induction on the formula φ.

1. $\varphi = P$, where $P \subseteq Y$ is a predicate. Clearly, by definition we have that $R^{-1}[\![\varphi]\!] = R^{-1}P = [\![\varphi^{-1}]\!]$.
2. $\varphi = \varphi_1 \vee \varphi_2$, or $\varphi = \varphi_1 \wedge \varphi_2$. In both cases we trivially have the result.
3. $\varphi = \bigcirc\varphi_1$. Let us suppose that $x \in R^{-1}[\![\bigcirc\varphi_1]\!]$; henceforth there exists some y such that xRy with $y \in [\![\bigcirc\varphi_1]\!] = \bigcirc[\![\varphi_1]\!]$. Equivalently we have xRy with $d(y) \in \mathrm{Pred}(F)([\![\varphi_1]\!])$ and since R is a simulation we also get that $c(x) \sqsubseteq u\mathrm{Rel}(F)(R)v \sqsubseteq d(y)$. Using the fact that $\sqsubseteq$ is down-closed and Proposition 8 we obtain $c(x) \in \mathrm{Pred}(F)(R^{-1}[\![\varphi_1]\!])$. Now, by induction hypothesis we know that $R^{-1}[\![\varphi_1]\!] \sqsubseteq [\![\varphi_1^{-1}]\!]$; this, together with the monotonicity of the operator $\mathrm{Pred}(F)$ leads us to $c(x) \in \mathrm{Pred}(F)([\![\varphi_1^{-1}]\!])$, that is, $x \in [\![\varphi^{-1}]\!]$.
4. $\varphi = \Box\varphi_1$. By definition, $\Box[\![\varphi_1]\!]$ is the greatest invariant contained in $[\![\varphi_1]\!]$ henceforth, $R^{-1}[\![\varphi_1]\!]$ is also an invariant. Trivially, since $\Box[\![\varphi_1]\!] \subseteq [\![\varphi_1]\!]$ we have $R^{-1}\Box[\![\varphi_1]\!] \subseteq R^{-1}[\![\varphi_1]\!]$. By induction hypothesis, $R^{-1}[\![\varphi_1]\!] \subseteq [\![\varphi_1^{-1}]\!]$, so $R^{-1}\Box[\![\varphi_1]\!]$ is an invariant contained in $[\![\varphi_1^{-1}]\!]$ and hence it must be contained in the greatest invariant contained in $[\![\varphi_1^{-1}]\!]$, that is, it must be contained in $\Box[\![\varphi_1^{-1}]\!]$, as we wanted to prove.

□

Now we can state and prove the corresponding theorem:

Theorem 2. *Let R be a simulation between coalgebras $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$ with a down-closed order. If φ is a temporal formula constructed only with operators $\vee$, $\wedge$, $\bigcirc$ and $\Box$, then the property φ is reflected by the simulation.*

Proof. Let us suppose xRy; it will be enough to prove that $y \in [\![\varphi]\!]$ implies $x \in R^{-1}[\![\varphi]\!]$. As in the previous proofs, we use structural induction on the formula φ.

1. If $\varphi = P \subseteq Y$ is an arbitrary predicate then we must prove that $y \in P$ implies $x \in R^{-1}P$. Since xRy, the result follows.
2. Let us suppose $\varphi = \varphi_1 \vee \varphi_2$ and let $y \in [\![\varphi_1]\!] \cup [\![\varphi_2]\!]$. Without loss of generality we take $y \in [\![\varphi_1]\!]$. Then, since xRy we have $x \in R^{-1}[\![\varphi_1]\!] \subseteq R^{-1}[\![\varphi_1]\!] \cup R^{-1}[\![\varphi_2]\!]$, as required.
3. The case $\varphi = \varphi_1 \wedge \varphi_2$ is similar to the previous case.
4. Let $\varphi = \bigcirc\varphi_1$. If $y \in \bigcirc[\![\varphi_1]\!]$, by definition we have $d(y) \in \mathrm{Pred}(F)([\![\varphi_1]\!])$. Since xRy and R is a simulation, $c(x) \sqsubseteq u\mathrm{Rel}(F)(R)v \sqsubseteq d(y)$ and using that $\sqsubseteq$ is down-closed and Proposition 8 it follows that $c(x) \in \mathrm{Pred}(F)(R^{-1}[\![\varphi_1]\!])$.
5. Let us suppose $\varphi = \Box\varphi_1$: by definition, there is an invariant Q for the coalgebra d such that $y \in Q$ and $Q \subseteq [\![\varphi_1]\!]$. We must prove that there exists an invariant S for c such that $x \in S$ with $S \subseteq R^{-1}[\![\varphi_1]\!]$. Let us take the invariant S defined as $S = R^{-1}Q$ which trivially contains x. Now, since $Q \subseteq [\![\varphi_1]\!]$, we have $S = R^{-1}Q \subseteq R^{-1}[\![\varphi_1]\!]$, as required.

□

Instead of considering down-closed orders, we could have imposed the converse implication, that is, those orders that satisfy that if $a \in \mathrm{Pred}(F)(P)$ then $b \in \mathrm{Pred}(F)(P)$.

Definition 6. *Given a functor F : **Sets** $\longrightarrow$ **Sets** we say that an order $\sqsubseteq$ is* up-closed *if whenever $a \sqsubseteq b$ then*

$$a \in \mathrm{Pred}(F)(P) \implies b \in \mathrm{Pred}(F)(P), \quad \textit{for all predicates } P\,.$$

Obviously up-closed is symmetrical to down-closed, that is, it is equivalent to taking $\sqsubseteq^{op}$ instead of $\sqsubseteq$ in Definition 5. So, for example, in the case of Kripke structures an up-closed order would satisfy

$$(u, v) \sqsubseteq (u', v') \quad \text{if} \quad v' \subseteq v\,.$$

The interesting thing about up-closed orders is that they allow us to prove *preservation* of properties; again, this result will hold only for formulas constructed with the operators $\vee$, $\wedge$, $\bigcirc$ and $\Box$. We need the following auxiliary result whose proof is analogous to the case of down-closed orders.

It is well-known that if R is a simulation for the order $\sqsubseteq$, then R^{-1} is a simulation for the oposite order $\sqsubseteq^{op}$. Using this property we get the following:

Theorem 3. *Let R be a simulation between coalgebras $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$ carrying an up-closed order. If φ is a temporal formula constructed only with the operators $\vee$, $\wedge$, $\bigcirc$ and $\Box$, then R preserves the property φ.*

Proof. Let us suppose xRy and $x \in [\![\varphi]\!]$. Let us consider $S = R^{-1}$. We know that S is a simulation between $d : Y \longrightarrow FY$ and $c : X \longrightarrow FX$ with the down-closed order $\sqsubseteq^{op}$ and since $(x, y) \in R$ then, $(y, x) \in S$. Hence, we can apply Theorem 2 and since $x \in [\![\varphi]\!]$ then, $y \in [\![\varphi^{-1}]\!]$. But when considering $S^{-1} = R$ the latter is the same as $y \in [\![\varphi^{*}]\!]$ (remember Remark 1). Hence we have proved that if $x \in [\![\varphi]\!]$ then, $y \in [\![\varphi^{*}]\!]$, that is, the preservation of the property. □

4.2 Restricting the class of functors

As we have just seen, it is not enough to restrict ourselves to down-closed (or up-closed) orders to get a valid result for all properties. What we want is a necessary and sufficient condition over functorial orders that implies reflection (or preservation) of properties by simulations. So far we have not found such a condition, but we have a sufficient one for simulations to reflect properties (and, in fact, also so that they preserve properties).

Recalling the structure of lemmas and propositions used to prove reflection and preservation of properties by bisimulations, we notice that the key ingredients were Lemmas 2 and 1. With these lemmas we were able to prove directly preservation of invariants (Lemma 3) and the relation between R^{-1} (respectively R) of a formula and the inverse of a formula (respectively direct image of a formula). Also, Lemmas 2 and 1 were essential to prove directly reflection and preservation of formulas built with the *nexttime* operator and the rest of temporal operators.

In the previous section the problem we faced was that either Lemma 2 (for down-closed orders) or Lemma 1 (for up-closed orders) held, but not both simultaneously. As a consequence, the results for the operators *eventually* and *until* did not hold. So, if we were capable of finding a subclass of functors and orders such that they fulfill results analogous to Lemmas 2 and 1 then, translating those proofs, we would get reflection and preservation of arbitrary properties.

We are going to define a subclass of functors and orders in the way that Hughes and Jacobs did in [7] for the subclass **Poly**.

Definition 7. *The class* **Order** *is the least class of functors closed under the following:*

1. *For every preorder $(A, \leq)$, the constant functor $X \mapsto A$ with the order given by $\sqsubseteq_X = \leq_A$ is in* **Order**.
2. *The identity functor with equality order is in* **Order**.
3. *Given two polynomial functors F_1 and F_2 with orders $\sqsubseteq^1$ and $\sqsubseteq^2$, the product functor $F_1 \times F_2$ with order $\sqsubseteq_X$ given by*

$$(u, v) \sqsubseteq_X (u', v') \quad \textit{if} \quad u \sqsubseteq^1 u' \quad \textit{and} \quad v \sqsubseteq^2 v',$$

   is in **Order**.
4. *Given the polynomial functor F with order $\sqsubseteq^F$ and the set A, the functor F^A with order $\sqsubseteq_X$ given by*

$$u \sqsubseteq_X v \textit{ if } u(a) \sqsubseteq^F v(a) \textit{ for all } a \in A,$$

is in **Order**.

5. *Given two polynomial functors* F_1 *and* F_2 *with orders* $\sqsubseteq^1$ *and* $\sqsubseteq^2$*, the coproduct functor* $F_1 + F_2$ *with order* $\sqsubseteq_X$ *given by*

$$\begin{aligned} u \sqsubseteq_X v \ \ &\textit{if } u = \kappa_1(u_0) \textit{ and } v = \kappa_1(v_0) \textit{ with } u_0 \sqsubseteq^1 v_0 \\ &\textit{or } u = \kappa_2(u_0) \textit{ and } v = \kappa_2(v_0) \textit{ with } u_0 \sqsubseteq^2 v_0 \ , \end{aligned}$$

is in **Order**.

6. *Given the polynomial functor* F *with order* $\sqsubseteq^F$*, the powerset functor* $\mathcal{P}(F)$ *with order* $\sqsubseteq_X$ *given by*

$$\begin{aligned} u \sqsubseteq_X v \qquad \textit{if} \qquad &\forall a \in u\ \exists b \in v \quad \textit{such that} \quad a \sqsubseteq^F b \\ \textit{and also } &\forall b \in v\ \exists a \in u \quad \textit{such that} \quad a \sqsubseteq^F b\,, \end{aligned}$$

is in **Order**.

For example the usual order for Kripke structures is not in the class **Order**. Besides, in the definition of **Poly** in [7] the authors did not consider the powerset functor but we do, although we are not using the *usual* order for this functor.

At first, to obtain that simulations not only reflect but also preserve properties may seem a little surprising. If we think about the elements in the subclass **Order** we notice that we have restricted the orders to equality-like orders, that is, almost all possible orders in **Order** are the equality. However, since the class **Order** is very similar to the class **Poly**, it has the same good properties shown in [7] (like the stability of the orders and functors). Let us see some orders and functors that belong to **Order**:

Example 3. 1. If we consider the functor $\mathcal{P}(id)$, then the order $\sqsubseteq$ defined in Definition 7 says that $u \sqsubseteq v$ if and only if for each $a \in u$ there exists $b \in v$ such that $a = b$, and if for each $b \in v$ there exists $a \in u$ such that $a = b$. This means that $\sqsubseteq$ is the identity relation. As an immediate consequence for transition systems the only possible **Order** simulations are bisimulation.

2. If we consider the functor $A \times id$ where A has a preorder $\leq_A$ different from the identity, the order $\sqsubseteq$ from Definition 7 is the following: $(u, v) \sqsubseteq (u', v')$ iff $v = v'$ and $u \leq_A u'$. So, if $\leq_A$ is not the identity, neither is $\sqsubseteq$. For example, let us take $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2\}$, $AP = \{p_1, p_2, p_3\}$ and consider the functor $F = \mathcal{P}(id) \times \mathcal{P}(AP)$ and the coalgebras $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$ defined by $c(x_1) = (\{x_2, x_3\}, \{p_1\})$, $c(x_2) = (\{x_3\}, \{p_2\})$, $c(x_3) = (\{x_2\}, \{p_3\})$, $d(y_1) = (\{y_2\}, \{p_2\})$ and $d(y_2) = (\{y_2\}, \{p_1\})$. Obviously there is no bisimulation between x_1 and y_1 since this atomic propositions are not the same, but taking the order $\sqsubseteq$ defined as $(u, v) \sqsubseteq (u', v')$ iff $u = u'$ (that is, taking as the preorder $\leq_{AP}$ the total relation) we have that there exists a simulation R in **Order** between x_1 and y_1.

Lemma 10. *Let* $R \subseteq X \times Y$ *be a simulation between coalgebras* $c : X \longrightarrow FX$ *and* $d : Y \longrightarrow FY$*, such that the functor* F *is in the class* **Order**. *Let us also suppose that* $P \subseteq Y$ *and* xRy*; then, if* $d(y) \in \mathrm{Pred}(F)(P)$ *we have* $c(x) \in \mathrm{Pred}(F)(R^{-1}P)$.

Proof. Again, the proof will be done by structural induction on the functor F.

1. $F = cnst$. In this case $\text{Pred}(F)(P) = \top$ and also $\text{Pred}(F)(R^{-1}P) = \top$, and $c(x) \in \text{Pred}(F)(R^{-1}P)$ trivially.
2. $F = id$. In this case we have that $\sqsubseteq$ coincides with the equality and $\text{Pred}(F)(P) = P$ for each P. Henceforth, we must check that $c(x) \in R^{-1}P$. Since R is a simulation we have $c(x) \sqsubseteq uRv \sqsubseteq d(y)$ and this is equivalent to $c(x)Rd(y)$, because the order is the equality.
3. Let F_1 and F_2 have orders $\sqsubseteq^1$ and $\sqsubseteq^2$ and consider $F = F_1 \times F_2$ with the order defined in Def. 7. We have

$$\text{Pred}(F)(P) = \{(u, v) \mid \text{Pred}(F_1)(P)(u) \wedge \text{Pred}(F_2)(P)(v)\}\,.$$

Let us suppose that $d(y) = (d_1(y), d_2(y))$ and analogously $c(x) = (c_1(x), c_2(x))$. Then, if $d(y) \in \text{Pred}(F)(P)$ we have $d_1(y) \in \text{Pred}(F_1)(P)$ and $d_2(y) \in \text{Pred}(F_2)(P)$. Now, as R is a simulation between c and d, from xRy it follows the existence of $u = (u_1, u_2)$ and $v = (v_1, v_2)$ such that $c(x) \sqsubseteq (u_1, u_2)\text{Rel}(F)(R)(v_1, v_2) \sqsubseteq d(y)$. By definition of $\sqsubseteq$, in particular we have both $c_1(x) \sqsubseteq^1 u_1\text{Rel}(F_1)(R)v_1 \sqsubseteq^1 d_1(y)$ and $c_2(x) \sqsubseteq^2 u_2\text{Rel}(F_2)(R)v_2 \sqsubseteq^2 d_2(y)$. That is, R is also a simulation between c_1 and d_1, and c_2 and d_2. Thus, we can use our induction hypothesis and since $d_1(y) \in \text{Pred}(F_1)(P)$, we get $c_1(x) \in \text{Pred}(F_1)(R^{-1}P)$ and, analogously, $c_2(x) \in \text{Pred}(F_2)(R^{-1}P)$, so $c(x) \in \text{Pred}(F)(R^{-1}P)$, as we wanted to prove.
4. Let F_1 and F_2 have orders $\sqsubseteq^1$ and $\sqsubseteq^2$ and consider $F = F_1 + F_2$ with the order given by Def. 7. In this case, we have

$$\text{Pred}(F)(P) = \{\kappa_1(u) \mid \text{Pred}(F_1)(P)(u)\} \cup \{\kappa_2(v) \mid \text{Pred}(F_2)(P)(v)\}.$$

Without loss of generality we suppose $d(y) = \kappa_1(d_1(y)) = (d_1(y), 1)$; we must have $d_1(y) \in \text{Pred}(F_1)(P)$. Let us consider the following constant coalgebras:

$$\begin{array}{llll} c_X : X \to F_1X & \qquad & d_Y : Y \to F_1Y \\ \quad z \;\mapsto c_1(x) & & \quad z \;\mapsto d_1(y) \end{array}$$

Since R is a simulation and the order is the disjoint sum, R is also a simulation between c_X and d_Y. Applying the induction hypothesis, we have that $c_1(x) \in \text{Pred}(F_1)(R^{-1}P)$ and hence $c(x) \in \text{Pred}(F)(R^{-1}P)$.
5. Let F be a functor with orden $\sqsubseteq^F$ and consider the functor F^A with the order given by Def. 7. In this case,

$$\text{Pred}(F^A)(P) = \{f \mid \forall a \in A.\ \text{Pred}(F)(P)(f(a))\}\,,$$

$$\text{Rel}(F^A)(R) = \{(f, g) \mid \forall a \in A.\ \text{Rel}(F)(R)(f(a), g(a))\}\,.$$

Hence, there exists u and v such that $c(x) \sqsubseteq u\text{Rel}(F^A)(R)v \sqsubseteq d(y)$. Now, for each $a \in A$ and F^A-coalgebra $d : Y \longrightarrow F^A(Y)$ we can define a coalgebra on F: $d^a : Y \longrightarrow F(Y)$ where, for each $y \in Y$, $d^a(y) = d(y)(a)$; analogously we define $c^a(x) = c(x)(a)$ for each $x \in X$. In this way we have that xRy and $d^a(y) = d(y)(a) \in \text{Pred}(F)(P)$.

Now, using the definition of the order $\sqsubseteq$ we have that $c^a(x) \sqsubseteq^F u(a)\text{Rel}(F)(R)v(a) \sqsubseteq^F d^a(y)$, that is, R is also a simulation between c^a and d^a. Applying the induction hypothesis we get $c^a(x) \in \text{Pred}(F)(R^{-1}P)$. Since this argument is valid for all $a \in A$, we finally get $c(x) \in \text{Pred}(F^A)(R^{-1}P)$.

6. Let F be a functor with order $\sqsubseteq^F$ and let us consider the functor $\mathcal{P}(F)$ with the order given by Def. 7. In this case

$$\text{Pred}(\mathcal{P}(F))(P) = \{U \mid \forall u \in U.\ \text{Pred}(F)(P)(u)\}\,.$$

We have $d(y) \in \text{Pred}(\mathcal{P}(F))(P)$ so for each $b \in d(y)$ we have that $b \in \text{Pred}(F)(P)$, and we must prove that $c(x) \in \text{Pred}(\mathcal{P}(F))(R^{-1}P)$, or equivalently, that for all $a \in c(x)$ also $a \in \text{Pred}(F)(R^{-1}P)$. Let us take an arbitrary $a \in c(x)$, and we define the following constant coalgebra:

$$\begin{aligned} c^a_X : X &\to FX \\ z &\mapsto a \end{aligned}$$

Now, since xRy and R is a simulation, $c(x) \sqsubseteq u\text{Rel}(\mathcal{P}(F))(R)v \sqsubseteq d(y)$. By definition of $\sqsubseteq$, from $c(x) \sqsubseteq u$ it follows that for each $a \in c(x)$ there exists some $a_1 \in u$ such that $a \sqsubseteq^F a_1$. Also, by the definition of the relation lifting we have that for each element $a_1 \in u$ there exists $b_1 \in v$ such that $a_1\text{Rel}(F)(R)b_1$. Again by the definition of the order, for each $b_1 \in V$ there exists a $b \in d(y)$ such that $b_1 \sqsubseteq^F b$.
Now we consider:

$$\begin{aligned} d^b_Y : Y &\to FY \\ z &\mapsto b \end{aligned}$$

Trivially, R is a simulation between the coalgebras c^a_X and d^b_Y, because $a \sqsubseteq^F a_1\text{Rel}(G)(R)b_1 \sqsubseteq^F b$; also $d^b_Y = b \in \text{Pred}(F)(P)$, so by induction hypothesis it follows that $c^a_X = a \in \text{Pred}(F)(R^{-1}P)$. Since this argument is valid for each $a \in c(x)$ we get $c(x) \in \text{Pred}(\mathcal{P}(F))(R^{-1}P)$.

□

In a similar way we have the corresponding lemma involving direct predicates.

Lemma 11. *Let $R \subseteq X \times Y$ be a simulation between coalgebras $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$, such that the functor F is in* **Order**. *Let us suppose also that $P \subseteq X$ and xRy. Then, if $c(x) \in \text{Pred}(F)(P)$, $d(y) \in \text{Pred}(F)(RP)$.*

Now we can conclude that under these hypothesis simulations reflect and preserve properties, simultaneously! This fact is a straightforward result from Lemmas 10 and 11.

Theorem 4. *Let R be a simulation between coalgebras $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$, with F a polynomial functor in the class* **Order**. *Then, the simulation R reflects and preserves properties.*

5 Including atomic propositions

A consequence of the fact that the logic proposed by Jacobs does not introduce atomic propositions was the need of giving non-standard definitions of reflection and preservation of properties. Kurz, in his work [12] includes atomic propositions in a temporal logic for coalgebras by means of natural transformations.

Definition 8. *Given a set AP of atomic propositions, the formulas of the temporal logic associated to a coalgebra $c : X \longrightarrow FX$ are given by the BNF expression:*

$$\varphi = p \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \Rightarrow \varphi \mid \bigcirc\varphi \mid \Diamond\varphi \mid \Box\varphi \mid \varphi\,\mathcal{U}\,\varphi$$

where $p \in AP$ is an atomic proposition.

Kurz also defines when a state x satisfies an atomic proposition p, that is, he defines the semantics of an atomic proposition.

Definition 9. *Let F : **Sets** $\longrightarrow$ **Sets** be a functor and AP a set of atomic propositions. Let $\nu : F \Rightarrow \mathcal{P}(AP)$ be a natural transformation and $c : X \longrightarrow FX$ a coalgebra. We say that x satisfies an atomic proposition $p \in AP$, and denote it $x \models p$, when*

$$p \in (\nu_X \circ c)(x)\,.$$

This way $[\![p]\!] = \{x \mid p \in (\nu_X \circ c)(x)\}$.

Notice that in fact this defines not only a semantics but a family of possible semantics that depends on the natural transformation. For example, we can define a natural transformation for the functor for Kripke structures in this way:

$$\begin{array}{rcl} \nu_X : \mathcal{P}(AP) \times \mathcal{P}(X) & \longrightarrow & \mathcal{P}(AP) \\ (P, Q) & \longmapsto & P \end{array}$$

With ν_X we have characterized the standard semantics of LTL for Kripke structures. Analogously, we could define the following interpretation: $\nu'_X(P,Q) = \mathcal{P}(AP) \setminus P$.

Introducing in our temporal logic the semantics of the atomic propositions, we can state the following theorem involving bisimulations:

Theorem 5. *Let R be a bisimulation between coalgebras $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$. Let φ be a temporal formula; then, the following is true for all $x \in X$ and $y \in Y$ such that xRy:*

$$x \in [\![\varphi]\!]_X \quad \Longleftrightarrow \quad y \in [\![\varphi]\!]_Y\,.$$

Here we have captured in the same theorem the classical ideas of reflection and preservation of properties: we have some property in the lefthand side of a bisimulation if and only if we have the property in its righthand side. In this case the theorem is true also for the negation operator thanks to the atomic propositions. Intuitively, this is because now we have an "if and only if" theorem, whereas in Theorem 1 we needed to reason separately for each implication using monotonicity, and negation lacks it. Also notice that even though we could think that in Theorem 1 our predicates played the role of atomic propositions, there are some essential differences: first, predicates are not independent of each other, unlike atomic propositions, and secondly, while atomic propositions stay the same predicates vary with each set of states.

Now we prove the theorem:

Proof. Once again the proof will proceed by structural induction on the formula φ.

1. Let $\varphi = p$ where p is an arbitrary atomic proposition. This way we have the following diagram, for ν an arbitrary natural trasformation:

$$
\begin{array}{ccccc}
X & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & Y \\
{\scriptstyle c}\downarrow & & {\scriptstyle [c,d]}\downarrow & & {\scriptstyle d}\downarrow \\
FX & \xleftarrow{F\pi_1} & FR & \xrightarrow{F\pi_2} & FY \\
{\scriptstyle \nu_X}\downarrow & & {\scriptstyle \nu_R}\downarrow & & {\scriptstyle \nu_Y}\downarrow \\
\mathcal{P}(AP) & \xleftarrow{id} & \mathcal{P}(AP) & \xrightarrow{id} & \mathcal{P}(AP)
\end{array}
$$

   This diagram is commutative. Indeed, since R is a bisimulation the upper side commutes, while the lower side commutes because ν is a natural transformation.
   So, $x \in [\![\varphi]\!]_X$ means by definition that $p \in (\nu_X \circ c)(x)$. Since the diagram commutes then $p \in (\nu_R \circ [c,d])(x,y) \Leftrightarrow p \in (\nu_Y \circ d)(y)$, that is, $y \in [\![\varphi]\!]_Y$.
2. Let us suppose $\varphi = \neg\varphi_0$. In this case we must show that $x \in \neg[\![\varphi_0]\!]_X$ if and only if $y \in \neg[\![\varphi_0]\!]_Y$, that is, we must see that $x \notin [\![\varphi_0]\!]_X$ if and only if $y \notin [\![\varphi_0]\!]_Y$. By induction hypothesis we have $x \in [\![\varphi_0]\!]_X$ if and only if $y \in [\![\varphi_0]\!]_Y$.
3. The cases $\varphi = \varphi_1 \wedge \varphi_2$ and $\varphi = \varphi_1 \vee \varphi_2$ are analogous to the previous case.
4. Let us suppose now that $\varphi = \bigcirc\varphi_0$. We must prove that $x \in \bigcirc[\![\varphi_0]\!]_X$ is equivalent to $y \in \bigcirc[\![\varphi_0]\!]_Y$, that is, $c(x) \in \mathrm{Pred}(F)([\![\varphi_0]\!]_X)$ is equivalent to $d(y) \in \mathrm{Pred}(F)([\![\varphi_0]\!]_Y)$. The latter will be proved by structural induction on the functor F.
   (a) $F = cnst$. In this case, both $\mathrm{Pred}(F)([\![\varphi_0]\!]_X)$ and $\mathrm{Pred}(F)([\![\varphi_0]\!]_Y)$ are equal to $\top$, so we trivally get the result.
   (b) $F = id$. In this case we must see that $c(x) \in [\![\varphi_0]\!]_X$ is equivalent to $d(y) \in [\![\varphi_0]\!]_Y$. Now, since we have xRy then, $c(x)Rd(y)$ and by induction hypothesis on φ, we know that, if aRb then $a \in [\![\varphi_0]\!]_X$ if and only if $b \in [\![\varphi_0]\!]_Y$.
   (c) $F = F_1 \times F_2$. Let us suppose that $c(x) = (c_1(x), c_2(x))$ and $d(y) = (d_1(y), d_2(y))$. Then, if $c(x) \in \mathrm{Pred}(F)([\![\varphi_0]\!]_X)$ we have

$$c_1(x) \in \mathrm{Pred}(F_1)([\![\varphi_0]\!]_X) \quad \text{and} \quad c_2(x) \in \mathrm{Pred}(F_2)([\![\varphi_0]\!]_X)\,.$$

By induction hypothesis on F_1 and F_2 we get both $d_1(y) \in \text{Pred}(F_1)(\llbracket\varphi_0\rrbracket_Y)$ and $d_2(y) \in \text{Pred}(F_2)(\llbracket\varphi_0\rrbracket_Y)$, so we get $y \in \llbracket\varphi\rrbracket_Y$. The other implication is analogous.

(d) $F = F_1 + F_2$. Let us suppose that $c(x) = \kappa_1(c_1(x)) = (c_1(x), 1)$; in this case we have $c_1(x) \in \text{Pred}(F_1)(\llbracket\varphi_0\rrbracket_X)$. Let us define:

$$\begin{aligned} c_X : X &\to F_1X \\ z &\mapsto c_1(x) \end{aligned} \qquad\qquad \begin{aligned} d_Y : Y &\to F_1Y \\ z &\mapsto d_1(y) \end{aligned}$$

Trivially, R is a bisimulation between c_X and d_Y; then, if we apply the induction hypothesis on the functor we get $d_1(y) \in \text{Pred}(F_1)(\llbracket\varphi_0\rrbracket_Y)$ and hence $d(y) \in \text{Pred}(F)(\llbracket\varphi_0\rrbracket_Y)$. Analogously, if we suppose $c(x) = \kappa_2(c_1(x))$ we also get $d(y) \in \text{Pred}(F)(\llbracket\varphi_0\rrbracket_Y)$.
The other implication is similar.

(e) $F = G^A$. Let us prove only one implication since the other one is almost identical. We have

$$\text{Pred}(F)(\llbracket\varphi_0\rrbracket_X) = \{f \mid \forall a \in A.\ \text{Pred}(G)(\llbracket\varphi_0\rrbracket_X)(f(a))\}\,.$$

Once again, as we have shown in other proofs, we define for each $a \in A$ and each F-coalgebra $c : X \longrightarrow F(X)$ a G-coalgebra, $c^a : X \longrightarrow G(X)$ where for each $x \in X$ we have $c^a(x) = c(x)(a)$. In this way, we have xRy and $c^a(x) = c(x)(a) \in \text{Pred}(()G)(\llbracket\varphi_0\rrbracket_X)$. By induction hypothesis we have that $d^a(y) \in \text{Pred}(G)(\llbracket\varphi_0\rrbracket_Y)$. Since this is a valid argument for all $a \in A$, we obtain $d(y) \in \text{Pred}(F)(\llbracket\varphi_0\rrbracket_Y)$.

(f) $F = \mathcal{P}(G)$. Let us show only one of the implications. Let us suppose $d(y) \in \{U \mid \forall u \in U.\ \text{Pred}(G)(\llbracket\varphi_0\rrbracket_Y)(u)\}$; then, for all $b \in d(y)$, we have $b \in \text{Pred}(G)(\llbracket\varphi_0\rrbracket_Y)$. Let us show that $c(x) \in \text{Pred}(F)(\llbracket\varphi_0\rrbracket_X)$, or equivalently, that for all $a \in c(x)$, $a \in \text{Pred}(G)(\llbracket\varphi_0\rrbracket_X)$. Let us take an arbitrary $a \in c(x)$ and we define the constant coalgebra:

$$\begin{aligned} c^a_X : X &\to GX \\ z &\mapsto a \end{aligned}$$

Now, since xRy and R is a bisimulation, then $c(x)\text{Rel}(F)(R)d(y)$, so there must exist some $b \in d(y)$ such that $a\text{Rel}(G)(R)b$.
So we define:

$$\begin{aligned} d^b_Y : Y &\to GY \\ z &\mapsto b \end{aligned}$$

Trivially R is a bisimulation between the coalgebras c^a_X and d^b_Y and also $d^b_Y(y) = b \in \text{Pred}(G)(\llbracket\varphi_0\rrbracket_Y)$, hence by induction hypothesis it follows that $c^a_X(x) = a \in \text{Pred}(G)(\llbracket\varphi_0\rrbracket_X)$. This argument is valid for all $a \in c(x)$, therefore, $c(x) \in \text{Pred}(F)(\llbracket\varphi_0\rrbracket_X)$.

(g) $F = G^*$. Applying an analogous reasoning we get the result.

5. $\varphi = \Box\varphi_0$. Assuming that $x \in \llbracket\varphi\rrbracket_X$ we get that there exists

$$Q \subseteq X \text{ an invariant for } c \text{ with } Q \subseteq \llbracket\varphi_0\rrbracket_X \text{ and } x \in Q.$$

Now, RQ is a invariant for d and, also, such that $RQ \subseteq [\![\varphi_0]\!]_Y$ with $y \in RQ$. Indeed, if $x \in Q$ then $y \in RQ$ and if $b \in RQ$ there must exists some $a \in Q \subseteq [\![\varphi_0]\!]_X$ such that aRb. So, by induction hypothesis we get that $b \in [\![\varphi_0]\!]_Y$

On the other hand, if $y \in [\![\varphi]\!]_Y$ there must exists some invariant T on Y, such that $T \subseteq [\![\varphi_0]\!]_Y$ with $y \in T$, hence for proving $x \in [\![\varphi]\!]_X$ it is enough to consider the invariant $R^{-1}T$.

6. $\varphi = \Diamond\varphi_0$. We must prove $x \in \neg\Box\neg[\![\varphi_0]\!]_X$ if and only if $y \in \neg\Box\neg[\![\varphi_0]\!]_Y$. Or equivalently, $x \in \Box\neg[\![\varphi_0]\!]_X$ if and only if $y \in \Box\neg[\![\varphi_0]\!]_Y$. Let us show only one of the implications. If $y \in \Box\neg[\![\varphi_0]\!]_Y$ then, by definition, there exists an invariant $T \subseteq Y$ such that $T \subseteq \neg[\![\varphi_0]\!]_Y$ with $y \in T$. Once again, taking the invariant $R^{-1}T$ we get by induction hypothesis that $R^{-1}T \subseteq \neg[\![\varphi_0]\!]_X$ and $x \in R^{-1}T$, as required.
7. $\varphi = \varphi_1\mathcal{U}\varphi_2$. We are going to proceed in a similar way as we did in Prop. 4. We must prove $y \in [\![\varphi_1]\!]_Y \,\mathcal{U}\, [\![\varphi_2]\!]_Y$ if and only if $x \in [\![\varphi_1]\!]_X \,\mathcal{U}\, [\![\varphi_2]\!]_X$.
The induction hypothesis give us the next property: if xRy then,

$$y \in [\![\varphi_i]\!]_Y \Leftrightarrow x \in [\![\varphi_i]\!]_X \quad \forall i \in \{1,2\}\,.$$

Hence, we have both $R^{-1}[\![\varphi_i]\!]_Y \subseteq [\![\varphi_i]\!]_X$ and $R[\![\varphi_i]\!]_X \subseteq [\![\varphi_i]\!]_Y$, for $i \in \{1,2\}$. This way, we can equivalently prove the following: $y \in [\![\varphi_1]\!]_Y \,\mathcal{U}\, [\![\varphi_2]\!]_Y$ implies $x \in R^{-1}[\![\varphi_1]\!]_Y \,\mathcal{U}\, R^{-1}[\![\varphi_2]\!]_Y$ and $x \in [\![\varphi_1]\!]_X \,\mathcal{U}\, [\![\varphi_2]\!]_X$ implies $y \in R[\![\varphi_1]\!]_X \,\mathcal{U}\, R[\![\varphi_2]\!]_X$.
Once again, we just show one of the implications. Let us show that

$$x \in [\![\varphi_1]\!]_X \,\mathcal{U}\, [\![\varphi_2]\!]_X \Longrightarrow y \in R[\![\varphi_1]\!]_Y \,\mathcal{U}\, R[\![\varphi_2]\!]_Y\,.$$

Also, we consider the auxiliary function

$$\begin{aligned} f^U_{(P,Q)} : \mathcal{P}(Y) &\longrightarrow \mathcal{P}(Y) \\ S \quad &\longmapsto Q \cup (P \cap \neg \bigcirc \neg S)\,, \end{aligned}$$

with the notation

$$f_1 \quad \text{denotes} \quad f^U_{([\![\varphi_1]\!]_X, [\![\varphi_2]\!]_X)}(S)$$

$$f_2 \quad \text{denotes} \quad f^U_{(R[\![\varphi_1]\!]_Y, R[\![\varphi_2]\!]_Y)}(S)\,.$$

Recall that f_1 and f_2 satisfy $Rf_1(S) \subseteq f_2(RS)$ and that by Prop. 6 $f^U_{(P,Q)}$ is monotonic and $\cup$-continuous for any pair of predicates P and Q. Also, by Prop. 6 we have that the least fixed points are given by

$$\mu S.f_1(S) = \bigcup_{i=1}^{\infty} f_1^i(\emptyset)$$

$$\mu S.f_2(S) = \bigcup_{i=1}^{\infty} f_2^i(\emptyset)\,.$$

Hence, since $x \in \bigcup_i^\infty f_1^i(\emptyset)$ then, for some i we have $x \in f_1^i(\emptyset)$, so $y \in Rf_1^i(\emptyset)$. If we consider in Lemma 7 the bisimulation R^{-1} we get

$$y \in Rf_1^i(\emptyset) \subseteq f_2(R{f_1}^{i-1}(\emptyset))\,,$$

and by monotonicity of f_2,

$$f_2(R{f_1}^{i-1}(\emptyset)) \subseteq f_2(f_2(R{f_1}^{i-2}(\emptyset)))\,.$$

If we iterate this process we finally get

$$y \in {f_2}^i(R\emptyset) = {f_2}^i(\emptyset)\,.$$

And, that way, we get $y \in \bigcup_i^\infty f_2^i(\emptyset) = \mu S.f_2(S)$.

□

To obtain a similar result for simulations, we will need again to restrict the class of functors and orders as we did in Sections 4.1 and 4.2. In particular we are interested in the following antimonotonicity property: if $u \sqsubseteq u'$ then $\nu(u') \subseteq \nu(u)$.

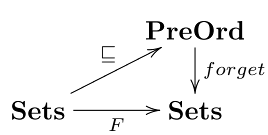
Definition 10. *Let* $F : \mathbf{Sets} \longrightarrow \mathbf{Sets}$ *be a functor,* AP *a set of atomic propositions and* $\nu : F \Rightarrow \mathcal{P}(AP)$ *a natural transformation. We say that* $\sqsubseteq$ *is a down-natural* ν*-order if, whenever* $u \sqsubseteq u'$ *then* $\nu(u') \subseteq \nu(u)$.

Obviously this definition depends on the natural transformation that we consider in each case. For example, for Kripke structures we have the following natural transformation: $\nu_X((A_X, B_X)) = A_X \subseteq AP$. To obtain a down-natural ν-order the following must hold: $(u,v) \sqsubseteq (u',v')$ then $\nu((u',v')) \subseteq \nu((u,v))$, that is, it will be enough to require $(u,v) \sqsubseteq (u',v')$ iff $u' \subseteq u$.

This way, if we combine the down-closed and the down-natural orders we get:

$$\text{If} \quad (u,v) \sqsubseteq (u',v') \quad \text{then} \quad u' \subseteq u \text{ and } v \subseteq v'\,.$$

This characterization is not as restrictive as one could think. Indeed, if we recall the definition of functorial order we had:



This diagram means that the functor F and the order $\sqsubseteq$ almost have the same structure and indeed, we could use a natural transformation between $\sqsubseteq$ and $\mathcal{P}(AP)$ in Definition 9 instead of a natural transformation between F and $\mathcal{P}(AP)$, that is, $\nu : \sqsubseteq \Rightarrow \mathcal{P}(AP)$. Considering ν in this way, an immediate consequence is that if we take as order in $\mathcal{P}(AP)$ the relation $\supseteq$ (as is done in [15]), then $u \sqsubseteq v$ implies $\nu(u) \sqsubseteq \nu(v)$.

We can tackle the proof of reflection of properties (with atomic propositions) by simulations as we did in Section 4.1, imposing to the order not only to be down-natural but also down-closed. But, if we do that we will find the same difficulties we faced in Section 4.1 (that is, we would not be able to prove reflection of formulas built with the operators *until* and *eventually*). Therefore, we must restrict the class of functors and orders, as we did with the class **Order** in Section 4.2, but imposing also that the orders must be down-natural.

Definition 11. *The class* **Down-Natural ν-Order** *is the subclass of* **Order** *where all orders are down-natural.*

Notice that we are defining a different class for each natural transformation ν.

Under this condition we state and prove the corresponding theorem involving simulations and the reflection of properties (with atomic propositions).

Theorem 6. *Let R be a simulation between coalgebras $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$ on the same polynomial functor F from* **Sets** *to* **Sets** *belonging to the class* **Down-Natural ν-Order** *and let φ be a temporal formula. Then, for each $x \in X$ and $y \in Y$ such that xRy:*

$$y \in [\![\varphi]\!]_Y \quad \Longrightarrow \quad x \in [\![\varphi]\!]_X \, .$$

Proof. We will prove the theorem by structural induction on φ.

1. Let $\varphi = p$ where p is an atomic proposition. Let us suppose that $y \in [\![p]\!]_Y$, so $p \in (\nu_Y \circ d)(y)$. Since R is a simulation there must exists u and v such that $c(x) \sqsubseteq u\mathrm{Rel}(F)(R)v \sqsubseteq d(y)$. We must prove that $x \models p$, that is, $p \in \nu_X(c(x))$. Since we are supposing that $z \sqsubseteq z'$ implies $\nu(z') \subseteq \nu(z)$, we have that $p \in \nu_Y(v)$; so it will be enough to show that $p \in \nu_X(u)$ and use that we are dealing with a down-natural order.
   Let us show that $\nu_X(u) = \nu_Y(v)$. By definition, $\mathrm{Rel}(F)(R)$ is the image of $\langle Fr_1, Fr_2\rangle : FR \longrightarrow FX \times FY$. Hence, since $u\mathrm{Rel}(F)(R)v$, there exists some $w \in FR$ such that $(u, v) = (Fr_1(w), Fr_2(w))$; so $p \in \nu_Y(v) = \nu_R(w) = \nu_X(u)$, as the following diagram shows:

$$\begin{array}{ccccc} FX & \xleftarrow{Fr_1} & FR & \xrightarrow{Fr_2} & FY \\ \downarrow{\scriptstyle \nu_X} & & \downarrow{\scriptstyle \nu_R} & & \downarrow{\scriptstyle \nu_Y} \\ \mathcal{P}(AP) & \xleftarrow{id} & \mathcal{P}(AP) & \xrightarrow{id} & \mathcal{P}(AP) \end{array}$$

2. The cases $\varphi = \varphi_1 \wedge \varphi_2$ are $\varphi = \varphi_1 \vee \varphi_2$ trivial.
3. Let us suppose that $\varphi = \bigcirc\varphi_0$. We must prove that $y \in \bigcirc[\![\varphi_0]\!]_Y$ implies $x \in \bigcirc[\![\varphi_0]\!]_X$, that is, that $d(y) \in \mathrm{Pred}(F)([\![\varphi_0]\!]_Y)$ implies $c(x) \in \mathrm{Pred}(F)([\![\varphi_0]\!]_X)$. Once again this will be proved applying structural induction over the functor F.
   (a) $F = cnst$. In this case both $\mathrm{Pred}(F)([\![\varphi_0]\!]_Y) = \top$ and $\mathrm{Pred}(F)([\![\varphi_0]\!]_X) = \top$, so we trivially get the result.

(b) $F = id$. In this case we have that both $\sqsubseteq$ and $\text{Pred}(F)$ are the identity. Hence, we must show that $c(x) \in [\![\varphi_0]\!]_X$. Since R is a simulation we have $c(x) \sqsubseteq uRv \sqsubseteq d(y)$, that is, $c(x)Rd(y)$. By induction hypothesis over φ we have that, if xRy, $y \in [\![\varphi_0]\!]_Y$ implies $x \in [\![\varphi_0]\!]_X$, and that way we obtain $c(x) \in \text{Pred}(F)([\![\varphi_0]\!]_X)$.

(c) Let F_1 and F_2 be two functors with down-natural orders $\sqsubseteq^1$ and $\sqsubseteq^2$ and let us consider $F = F_1 \times F_2$ with the down-closed order given by Def. 7. In this case we have

$$\text{Pred}(F)(P) = \{(u, v) \mid \text{Pred}(F_1)(P)(u) \wedge \text{Pred}(F_2)(P)(v)\}\,.$$

Let us suppose $d(y) = (d_1(y), d_2(y))$ and $c(x) = (c_1(x), c_2(x))$. Henceforth, if $d(y) \in \text{Pred}(F)([\![\varphi_0]\!]_Y)$ then $d_1(y) \in \text{Pred}(F_1)([\![\varphi_0]\!]_Y)$ and $d_2(y) \in \text{Pred}(F_2)([\![\varphi_0]\!]_Y)$. Now, since R is a simulation between c and d then, given xRy, it follows the existence of $u = (u_1, u_2)$ and $v = (v_1, v_2)$ such that $c(x) \sqsubseteq (u_1, u_2)\text{Rel}(F)(R)(v_1, v_2) \sqsubseteq d(y)$. By definition of $\sqsubseteq$ we have $c_1(x) \sqsubseteq^1 u_1\text{Rel}(F_1)(R)v_1 \sqsubseteq^1 d_1(y)$ and $c_2(x) \sqsubseteq^2 u_2\text{Rel}(F_2)(R)v_2 \sqsubseteq^2 d_2(y)$. That is, R is also a simulation between c_1 and d_1, and c_2 and d_2. Since both orders $\sqsubseteq^1$ and $\sqsubseteq^2$ are down-natural, we can apply the induction hypothesis over F and obtain that $c_1(x) \in \text{Pred}(F_1)([\![\varphi_0]\!]_X)$ and $c_2(x) \in \text{Pred}(F_2)([\![\varphi_0]\!]_X)$, hence $c(x) \in \text{Pred}(F)([\![\varphi_0]\!]_X)$, as we wanted to prove.

(d) Let F_1 and F_2 be two functors with down-natural orders $\sqsubseteq^1$ and $\sqsubseteq^2$ and let us consider $F = F_1 + F_2$ with the order given by Definition 7.
Without loss of generality let us suppose that $d(y) = \kappa_1(d_1(y)) = (d_1(y), 1)$; we have $d_1(y) \in \text{Pred}(F_1)([\![\varphi_0]\!]_Y)$. Let us consider now the following constant coalgebras:

$$\begin{array}{rl} c_X : X \to F_1 X & \qquad d_Y : Y \to F_1 Y \\ z \;\mapsto c_1(x) & \qquad z \;\mapsto d_1(y) \end{array}$$

Since R is a simulation and the order is the disjoint sum, we also have that R is a simulation between c_X and d_Y with down-natural orders; hence, by induction hypothesis we have $c_1(x) \in \text{Pred}(F_1)([\![\varphi_0]\!]_X)$ and in this way, $c(x) \in \text{Pred}(F)([\![\varphi_0]\!]_X)$.

(e) Let F be a functor with a down-natural order $\sqsubseteq^F$ and let us consider the functor F^A with the order given by Def. 7. Since R is a simulation, there exists u and v such that $c(x) \sqsubseteq u\text{Rel}(F^A)(R)v \sqsubseteq d(y)$. Now, for each $a \in A$ and each F^A-coalgebra $d : Y \longrightarrow F^A(Y)$ we define a coalgebra over F this way: $d^a : Y \longrightarrow F(Y)$ where, for each $y \in Y$, $d^a(y) = d(y)(a)$; analogously we define $c^a(x) = c(x)(a)$ for all $x \in X$. In this way we have xRy and $d^a(y) = d(y)(a) \in \text{Pred}(F)([\![\varphi_0]\!]_Y)$.
Now, by definition of $\sqsubseteq$ we have $c^a(x) \sqsubseteq^F u(a)\ \text{Rel}(F)(R)v(a) \sqsubseteq^F d^a(y)$, that is, R is a simulation between c^a and d^a. By induction hypothesis we have $c^a(x) \in \text{Pred}(F)([\![\varphi_0]\!]_X)$. Since this argument is valid for all $a \in A$, we have that $c(x) \in \text{Pred}(F^A)([\![\varphi_0]\!]_X)$.

(f) Let F be a functor with a down-natural order $\sqsubseteq^F$ and let us consider the functor $\mathcal{P}(F)$ with the order given by Def. 7. In this case we have $d(y) \in \text{Pred}(\mathcal{P}(F))(\llbracket\varphi_0\rrbracket_Y)$, so for all $b \in d(y)$ we will have that $b \in \text{Pred}(F)(\llbracket\varphi_0\rrbracket_Y)$, and we have to prove that $c(x) \in \text{Pred}(\mathcal{P}(F))(\llbracket\varphi_0\rrbracket_X)$, or equivalently, that for all $a \in c(x)$ we have $a \in \text{Pred}(F)(\llbracket\varphi_0\rrbracket_X)$. Let us take an arbitrary $a \in c(x)$ and define the following constant coalgebra:

$$\begin{aligned} c_X^a : X &\to FX \\ z &\mapsto a \end{aligned}$$

Since xRy and R is a simulation we have $c(x) \sqsubseteq u\text{Rel}(\mathcal{P}(F))(R)v \sqsubseteq d(y)$. By definition of $\sqsubseteq$, it follows that since $c(x) \sqsubseteq u$, then for each $a \in c(x)$ there exists $a_1 \in u$ such that $a \sqsubseteq^F a_1$. Also, by the definition of relation lifting it follows that for each element $a_1 \in u$ there exists an element $b_1 \in v$ such that $a_1\text{Rel}(F)(R)b_1$. Again, by the definition of the order it follows that for each $b_1 \in V$ there exists a $b \in d(y)$ such that $b_1 \sqsubseteq^F b$. Now, we define the following:

$$\begin{aligned} d_Y^b : Y &\to FY \\ z &\mapsto b \end{aligned}$$

Trivially, R is a simulation between c_X^a and d_Y^b, and also $d_Y^b = b \in \text{Pred}(F)(\llbracket\varphi_0\rrbracket_Y)$; hence by induction hypothesis over F it follows $c_X^a = a \in \text{Pred}(F)(\llbracket\varphi_0\rrbracket_X)$. Since this is a valid argument for each $a \in c(x)$, it follows that $c(x) \in \text{Pred}(\mathcal{P}(F))(\llbracket\varphi_0\rrbracket_X)$.

4. $\varphi = \Box\varphi_0$. Suppose that $y \in \llbracket\varphi\rrbracket_Y$: there exists an invariant $Q \subseteq Y$ such that $Q \subseteq \llbracket\varphi_0\rrbracket_Y$ and $y \in Q$. Now, recall that the functors of the class **Order** satisfied Lemmas 10 and 11; hence, $R^{-1}Q$ is an invariant too. Furthermore $R^{-1}Q \subseteq \llbracket\varphi_0\rrbracket_X$ with $x \in R^{-1}Q$. Indeed, since $y \in Q$ then $x \in R^{-1}Q$; on the other hand, if $a \in R^{-1}Q$ there must exist some $b \in Q \subseteq \llbracket\varphi_0\rrbracket_Y$ such that aRb. Hence, by induction hypothesis, $a \in \llbracket\varphi_0\rrbracket_X$ so $x \in \llbracket\varphi_0\rrbracket_X$ as requested.
5. $\varphi = \Diamond\varphi_0$. We must prove that $y \in \neg\Box\neg\llbracket\varphi_0\rrbracket_Y$ implies $x \in \neg\Box\neg\llbracket\varphi_0\rrbracket_X$, or equivalently, that $x \in \Box\neg\llbracket\varphi_0\rrbracket_X$ implies $y \in \Box\neg\llbracket\varphi_0\rrbracket_Y$. Indeed, as in the previous case there exists an invariant $T \subseteq X$ such that $T \subseteq \neg\llbracket\varphi_0\rrbracket_X$ with $x \in T$. Once again, RT is an invariant such that $RT \subseteq Y$, $RT \subseteq \neg\llbracket\varphi_0\rrbracket_Y$ and $y \in RT$, as required.
6. $\varphi = \varphi_1\mathcal{U}\varphi_2$. We are going to prove that $y \in \llbracket\varphi_1\rrbracket_Y \ \mathcal{U}\ \llbracket\varphi_2\rrbracket_Y$ implies $x \in \llbracket\varphi_1\rrbracket_X \ \mathcal{U}\ \llbracket\varphi_2\rrbracket_X$.
As we showed in the proof of Theorem 5, the induction hypothesis provides us with the following property: if xRy then

$$y \in \llbracket\varphi_i\rrbracket_Y \Rightarrow x \in \llbracket\varphi_i\rrbracket_X \quad \forall i \in \{1,2\}\,.$$

Hence, we have that $R^{-1}\llbracket\varphi_i\rrbracket_Y \subseteq \llbracket\varphi_i\rrbracket_X$, for $i \in \{1,2\}$. So we must prove that $y \in \llbracket\varphi_1\rrbracket_Y \ \mathcal{U}\ \llbracket\varphi_2\rrbracket_Y$ implies $x \in R^{-1}\llbracket\varphi_1\rrbracket_Y \ \mathcal{U}\ R^{-1}\llbracket\varphi_2\rrbracket_Y$.
Once again we define

$$\begin{aligned} f^U_{(P,Q)} : \mathcal{P}(Y) &\longrightarrow \mathcal{P}(Y) \\ S &\longmapsto Q \cup (P \cap \neg \bigcirc \neg S)\,, \end{aligned}$$

with the following notation:

$$f_1 \quad \text{denotes} \quad f^U_{(\llbracket\varphi_1\rrbracket_Y, \llbracket\varphi_2\rrbracket_Y)}(S)$$

$$f_2 \quad \text{denotes} \quad f^U_{(R^{-1}\llbracket\varphi_1\rrbracket_Y, R^{-1}\llbracket\varphi_2\rrbracket_Y)}(S)\,.$$

Recall that since Lemmas 11 and 10 are satisfied, we can guarantee that f_1 and f_2 satisfy the relation $R^{-1}f_1(S) \subseteq f_2(R^{-1}S)$. On the other hand, since $f^U_{(P,Q)}$ is monotonic and $\cup$-continuos we have that

$$\mu S.f_1(S) = \bigcup_{i=1}^{\infty} f_1^i(\emptyset)$$

$$\mu S.f_2(S) = \bigcup_{i=1}^{\infty} f_2^i(\emptyset)\,.$$

Hence, since $y \in \bigcup_i^\infty f_1^i(\emptyset)$ then, for some i we have $y \in f_1^i(\emptyset)$, so, $y \in Rf_1^i(\emptyset)$, and also

$$x \in R^{-1}f_1^i(\emptyset) \subseteq f_2(R^{-1}{f_1}^{i-1}(\emptyset))\,.$$

By monotonicity of f_2,

$$f_2(R^{-1}{f_1}^{i-1}(\emptyset)) \subseteq f_2(f_2(R^{-1}{f_1}^{i-2}(\emptyset)))\,.$$

If we iterate this process we finally get

$$x \in {f_2}^i(R\emptyset) = {f_2}^i(\emptyset)\,.$$

And that way $x \in \bigcup_i^\infty f_2^i(\emptyset) = \mu S.f_2(S)$, as required.

□

We showed above that simulations for functors in the class **Order** reflected and preserved all kinds of properties. Instead, now we can only prove one implication, that corresponding to the reflection of properties. This is so because down-natural ν-orders have a natural direction.

Exactly in the same way as we did with down-natural ν-orders, we can define the corresponding class of up-natural ν-orders:

Definition 12. *Let F : **Sets** $\longrightarrow$ **Sets** be a functor, AP a set of atomic propositions and $\nu : F \Rightarrow \mathcal{P}(AP)$ a natural transformation. We say that $\sqsubseteq$ is an up-natural ν-order if $u \sqsubseteq u'$ implies $\nu(u) \subseteq \nu(u')$.*

Also, as we did for down-natural ν-orders, we must define a subclass of **Order**:

Definition 13. *The class* **Up-Natural ν-Order** *is the subclass of* **Order** *where all orders are up-natural.*

Theorem 7. *Let R be a simulation between coalgebras $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$ on the same polynomial functor F in the class* **Up-Natural ν-Order***, and let φ be a temporal formula. Then, for all $x \in X$ and $y \in Y$ such that xRy:*

$$x \in \llbracket\varphi\rrbracket_X \quad \Longrightarrow \quad y \in \llbracket\varphi\rrbracket_Y\,.$$

6 Conclusions

The main goal of this paper was to study under what assumptions coalgebraic simulations reflect properties. In our way towards the proof of this result, we were also able to prove reflection and preservation of properties by coalgebraic bisimulations. For expressing the properties we used Jacobs' temporal logic [8], later extended with atomic propositions using the idea presented in [12].

That coalgebraic bisimulations reflect and preserve properties expressed in modal logic is a well-known topic (e.g, [9, 12, 16]), but not so the corresponding results for simulations. The main difficulty is that Hughes and Jacobs' notion of simulation is defined by means of an arbitrary functorial order which bestows them with a high degree of freedom. We have dealt with this by restricting the class of functorial orders (although even so we are not able of obtaining a general result) and by restricting also the class of allowed functors.

In order to get more general results on the subject, an interesting path that we intend to explore is the search for a canonical notion of simulation. This definition would provide us, not only with a "natural" way to understand simulations but, hopefully, would also give rise to "natural" general results about reflection of properties.

Another promising direction of research is the study of reflection and preservation of properties in probabilistic systems following our results of [4] in combination with the ideas presented in [6, 5, 2].

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